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Metric Spaces in Fuzzy Set Theory

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1. INTRODUCTION

It is shown that a pseudo-quasi-metric (p.q. metric) on a set X may be equivalently regarded as a distance function function between subsets of X. This equivalent definition is generalized to fuzzy set theory where points need not have Boolean properties and hence in vhich a naive generalization of a p.q. metric is unsatisfactory. Additional axioms are introduced which correspond to pseudo-metrics (p. metrics) and metrics in fuzzy set theory.

We define a uniformity for a metric space on a fuzzy set, using the definition of uniformity given by Hutton [1]. Complementing the results of Hutton [1], we obtain results on the generation of topologies on fuzzy sets by p.q. metrics. Conjugate pseudo-metrics are defined and a pseudo-metric for the fuzzy unit interval is given.

2. PRELIMINARIES

In the usual set theory, by a set X we mean the quadruple $\langle \mathscr{P}(X), \cap, \cup, ' \rangle$ where $\mathscr{P}(X)$ is the power set of X and the operations are those of intersection, union and complementation. This is lattice isomorphic to $\langle 2^X, \wedge, \vee, ' \rangle$ where the operations $\wedge$, $\vee$, $'$ are defined as follows:

$$\left(\bigwedge_i \lambda_i\right)(x) = \bigwedge_i \lambda_i(x) \qquad \text{for } x \in X$$

$$\left(\bigvee_i \lambda_i\right)(x) = \bigvee_i \lambda_i(x) \qquad \text{for } x \in X$$

$$\lambda'(x) = \lambda(x)' \qquad \text{for } x \in X.$$

The lattice isomorphism is the one which associates a set with its characteristic function. The theory of fuzzy sets considers lattices more general than $2 = \{0, 1\}$, which is a lattice with complement under the usual operations.

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Throughout this paper $\langle L, \wedge, \vee, ' \rangle$ will be a completely distributive lattice with order reversing involution $'$ (see Birkhoff [7]), and following the terminology of [6], it will be called a fuzz. We make L^X into a fuzz by giving it the product operations, also denoted by $\wedge$, $\vee$, $'$, of $\langle L, \wedge, \vee, ' \rangle$. For example, if $\lambda_1, \lambda_2 \in L^X$ then

$$(\lambda_1 \wedge \lambda_2)(x) = \lambda_1(x) \wedge \lambda_2(x) \qquad \text{for } x \in X.$$

$\langle L^X, \wedge, \vee, ' \rangle$ will be called the fuzz X or the fuzzy set X. If $\lambda \in L^X$, then $\lambda(x)$ may be regarded as the degree of membership of x in λ. A fuller description of fuzzy set theory may be found in Zadeh [2] and Goguen [13].

Since $\langle L, \wedge, \vee, ' \rangle$ is a complete lattice, it has least and greatest elements, say 0 and 1 respectively. If $\underline{0} \in L^X$ denotes the map which is everywhere 0 and if $\underline{1}$ is similarly defined, then these are the least and greatest element of $\langle L^X, \wedge, \vee, ' \rangle$.

DEFINITION 2.1. A fuzzy topology for X, or a topology for the fuzz X, is a pair $\langle L^X, \mathscr{T} \rangle$ (strictly speaking $\langle L^X, \wedge, \vee', \mathscr{T} \rangle$) where $\mathscr{T} \subset L^X$ and

(1) $\underline{0}, \underline{1} \in \mathscr{T}$
(2) $\mathscr{T}$ is closed under arbitrary supremums
(3) $\mathscr{T}$ is closed under finite infimums.

We define open and closed sets and interior and closure operators in the usual way. For example, if $\lambda \in L^X$ then interior λ is: $\lambda^0 = \bigvee\{\mu \in \mathscr{T} \mid \mu \leqslant \lambda\}$. The pair $\langle L^X, \mathscr{T} \rangle$ will be called a fuzzy topological space (see Chang [3]).

3. P. Q. METRICS AS DISTANCE FUNCTIONS BETWEEN SETS

Consider a p.q. metric space $\langle X, p \rangle$. Let d be a function which assigns to each ordered pair $(A, B) \in \mathscr{P}(X) \times \mathscr{P}(X)$ a value in $[0, \infty]$. We would like d to give exactly the same information about the p.q. metric topology of p as does p itself.

For all real $r > 0$ and for all $A \in \mathscr{P}(X)$, we define $D_r^p(A)$ by

$$D_r^p(A) = \left\{ y \mid \exists a \in A \text{ such that } p(a, y) < r \right\}$$
$$= \left\{ y \mid \bigwedge_{a \in A} p(a, y) < r \right\}.$$

Then $\{D_r^p(A) \mid r > 0\}$ is a basis for the neighborhood system of A. It is clear that $p(x, y) = \bigwedge\{r \mid y \in D_r^p(\{x\})\}$. Since it is desirable that d and p agree on ordered pairs of singletons, we make the following definition:

$$d(A, B) = \bigwedge\{r \mid B \subset D_r^p(A)\}.$$

Then if $A, B \in \mathscr{P}(X)\backslash\{\varnothing\}$, we have:

$$d(A, B) = \bigwedge \left\{r \mid B \subset \left\{y \mid \bigwedge_{a\in A} p(a, y) < r\right\}\right\}$$

$$= \bigwedge \left\{r \mid \forall b \in B, \bigwedge_{a\in A} p(a, b) < r\right\}.$$

Hence

$$\bigwedge \left\{r \mid \bigvee_{b\in B} \bigwedge_{a\in A} p(a, b) \leqslant r\right\} \leqslant d(A, B) \leqslant \bigwedge \left\{r \mid \bigvee_{b\in B} \bigwedge_{a\in A} p(a, b) < r\right\}.$$

Thus

$$d(A, B) = \bigvee_{b\in B} \bigwedge_{a\in A} p(a, b).$$

This function appears in Hausdorff [8] as an intermediate stage in the evolution of a metric for spaces of closed bounded sets. It will henceforth be called a Hausdorff p.q. metric.

PROPOSITION 3.1. *If $d(A, B) = \bigvee_{r\in B} \bigwedge_{a\in A} p(a, b)$ for $A, B \in \mathscr{P}(X)\backslash\{\varnothing\}$ and if we define*

$$d(A, \varnothing) = 0 \qquad \text{for } A \in \mathscr{P}(X)$$

and

$$d(\varnothing, A) = \infty \qquad \text{for } A \in \mathscr{P}(X)\backslash\{\varnothing\}$$

then the following statements are valid:

(M1) $d(\varnothing, A) = \infty \qquad \forall A \in \mathscr{P}(X)\backslash\{\varnothing\}$

$d(A, A) = 0 \qquad \forall A \in \mathscr{P}(X)$

$d(A, \varnothing) = 0 \qquad \forall A \in \mathscr{P}(X).$

(M2) $d(A, C) \leqslant d(A, B) + d(B, C) \qquad \forall A, B, C \in \mathscr{P}(X).$

(M3) (i) $A \subset B \Rightarrow d(A, C) \geqslant d(B, C) \qquad \forall C \in \mathscr{P}(X).$

(ii) $d(A, \bigcup_\alpha B_\alpha) = \bigvee_\alpha d(A, B_\alpha) \qquad$ *for* $B_\alpha \in \mathscr{P}(X).$

(M4) *Suppose $B, C_\alpha \in \mathscr{P}(X)$ for all α in some index set Δ.*

If $d(C_\alpha, D) < r \Rightarrow D \subset B$ *for* $D \in \mathscr{P}(X), \alpha \in \Delta$ *then* $d(\bigcup_{\alpha\in\Delta} C_\alpha, E) < r \Rightarrow E \subset B$ *for* $E \in \mathscr{P}(X)$.

Proof. (M1) It is enough to show that for $A \in \mathscr{P}(X)\backslash\{\varnothing\}$, $d(A, A) = 0$. Fix $b \in A$. Then $\bigwedge_{a\in A} d(a, b) \leqslant d(b, b) = 0$. Hence

$$d(A, A) = \bigvee_{b\in A} \bigwedge_{a\in A} d(a, b) = 0.$$

(M2) Let A, B, $C \in \mathscr{P}(X)$. If A, B or C is empty, (M2) follows using (M1). Suppose now that A, B, $C \in \mathscr{P}(X)\backslash\{\varnothing\}$. Then for all $a \in A$, $b \in B$, $c \in C$

$$p(a, c) \leqslant p(a, b) + p(b, c)$$

$$\therefore \quad \bigwedge_{c\in C} p(a, c) \leqslant p(a, b) + \bigwedge_{c\in C} p(b, c)$$

$$\leqslant p(a, b) + d(B, C).$$

This is valid for all $b \in B$, giving

$$\bigwedge_{c\in C} p(a, c) \leqslant \bigwedge_{b\in B} p(a, b) + d(B, C),$$

so that $d(A, C) \leqslant d(A, B) + d(B, C)$.

(M3) Follows immediately from the definition of d.

(M4) Let $d(\bigcup_{\alpha\in\Delta} C_\alpha, E) < r$ and suppose $e \in E$.

Since $\bigvee_{f\in E} d(\bigcup_{\alpha\in\Delta} C_\alpha, f) < r$ it follows that $d(\bigcup_{\alpha\in\Delta} C_\alpha, e) = \bigwedge_{a\in\bigcup_\alpha C_\alpha} p(a, e) < r$.

Hence there exists $a \in \bigcup_{\alpha\in\Delta} C_\alpha$ such that $p(a, e) < r$.

Suppose $a \in C_\beta$ where $\beta \in \Delta$.

Then $d(C_\beta, e) < r$, so by hypothesis, $e \in B$. This is true for arbitrary $e \in E$, which gives $E \subset B$ as required.

Remarks. 1. From (M1) and (M2) it is clear that d is a p.q. metric on $\mathscr{P}(X)$.

2. In view of (M3), d is contravariant in its first variable and covariant in its second.

3. (M4) is equivalent to:

$$D_r^p\left(\bigcup_\alpha C_\alpha\right) \subseteq \bigcup_\alpha D_r^p(C_\alpha) \qquad \text{(see Theorem 4.3).}$$

4. (M3) (i) may be written in the form of (M3)(ii) as:

$$d\left(\bigcup_\alpha B_\alpha, A\right) \leqslant \bigwedge_\alpha d(B_\alpha, A).$$

The reverse inequality is not true in general.

We now show that (M1)–(M4) completely characterize a Hausdorff p.q. metric.

THEOREM 3.2. *If* $d: \mathscr{P}(X) \times \mathscr{P}(X) \to [0, \infty]$ *then*

(1) $d(A, B) = \bigvee_{b\in B} \bigwedge_{a\in A} d(\{a\}, \{b\})\ \forall A, B \in \mathscr{P}(X) \Leftrightarrow d$ *satisfies* (M3) *and* (M4). *In which case,*

(2) *if* $p(x, y) = d(\{x\}, \{y\})\ \forall x, y \in X$ *then* p *is a* p.q. *metric on* $X \Leftrightarrow d$ *satisfies* (M1) *and* (M2).

Proof.

(1) ($\Rightarrow$) By Proposition 3.1. ($\Leftarrow$) In defining $D_r^p(A)$ as before and $D_r^d(A) = \cup\{B \mid d(A, B) < r\}$ then

$$\begin{aligned} D_r^p(\{x\}) &= \{y \mid p(x, y) < r\} \\ &= \{y \mid d(\{x\}, \{y\}) < r\} \\ &= \bigcup \{B \mid d(\{x\}, B) < r\} \end{aligned}$$

since

$$d(\{x\}, B) = \bigvee_{b \in B} d(\{x\}, \{b\}) \qquad \text{by (M3) (ii).}$$

Hence

$$D_r^p(\{x\}) = D_r^d(\{x\}).$$

In Theorem 4.3, using (M3)(i) and (M4) it is shown that

$$D_r^d(A) = \bigcup_{a \in A} D_r^d(\{a\}).$$

Hence

$$D_r^d(A) = D_r^p(A).$$

In view of (M3)(ii) we have:

$$d(A, B) = \bigwedge \{r \mid B \subset D_r^d(A)\}.$$

The proof is given in Theorem 4.5 for a more general case. By the above, $d(A, B) = \wedge\{r \mid B \subset D_r^p(A)\}$. Thus by the reasoning preceding Proposition 3.1,

$$d(A, B) = \bigvee_{b \in B} \bigwedge_{a \in A} p(\{a\}, \{b\}).$$

(2) The proof involves Proposition 3.1.

COROLLARY 3.3. *A map* $d: \mathscr{P}(X) \times \mathscr{P}(X) \to [0, \infty]$ *is a Hausdorff* p.q. *metric iff d satisfies* (M1)–(M4).

We now consider a topology for such maps. If d is a map satisfying (M1)–(M4) then $\{D_r^d(A) \mid r \in (0, \infty), A \in \mathscr{P}(X)\}$ is a base for a topology on X. The proof is given for a more general case in Theorem 4.8.

As observed in Theorem 3.2, restricting d to ordered pairs of singletons gives a p.q. metric, p say. From the proof of Theorem 3.2, we have

$$\begin{aligned} D_r^d(A) &= D_r^p(A) \\ &= \bigcup_{a \in A} D_r^d(\{a\}). \end{aligned}$$

The topologies associated with the map d and its restriction p are therefore the same.

COROLLARY 3.4. *A topological space is* p.q. *metrizable iff its topology is that associated with a* map *satisfying* (M1)–(M4).

Remark. A map satisfying (M1)–(M4) is an alternative definition of a p.q. metric. It is topologically equivalent to the usual definition in the sense of Corollary 3.4. We use this equivalence to make the following definition.

4. P. Q. METRICS ON FUZZY SETS

DEFINITION 4.1. A p.q. metric on the fuzz X, or a fuzzy p.q. metric on X, is a map $p: L^X \times L^X \to [0, \infty]$ satisfying

(M1) $p(\underset{\sim}{0}, \lambda) = \infty \quad \forall \lambda \in L^X, \quad \lambda \neq \underset{\sim}{0},$
$p(\lambda, \lambda) = 0 \quad \forall \lambda \in L^X,$
$p(\lambda, \underset{\sim}{0}) = 0 \quad \forall \lambda \in L^X.$

(M2) $p(\lambda, \mu) \leqslant p(\lambda, \chi) + p(\chi, \mu) \quad \forall \lambda, \chi, \mu \in L^X.$

(M3) (i) $\lambda \leqslant \mu \Rightarrow p(\lambda, \chi) \geqslant p(\mu, \chi) \quad \forall \chi \in L^X.$
(ii) $p(\chi, \bigvee_\alpha \lambda_\alpha) = \bigvee_\alpha p(\chi, \lambda_\alpha) \quad \forall \chi, \lambda_\alpha \in L^X.$

(M4) Suppose $\mu, \chi_\alpha \in L^X \; \forall \alpha \in \Delta$.

If $p(\chi_\alpha, \beta) < r \Rightarrow \beta \leqslant \mu$ for $\beta \in L^X$, $\alpha \in \Delta$, then $p(\bigvee_\alpha \chi_\alpha, \gamma) < r \Rightarrow \gamma \leqslant \mu$ for $\gamma \in L^X$.

Remark. Kramosil and Michalek [9] have defined a fuzzy metric which is not related to the above. They consider a metric as a subset of $X \times X \times [0, \infty)$ and in making this into a fuzzy set they obtain their fuzzy metric.

DEFINITION 4.2. For all $r \in (0, \infty)$ let

$$D_r: L^X \to L^X \text{ be defined by}$$

$$D_r(\lambda) = \bigvee \{\mu \mid p(\lambda, \mu) < r\}.$$

Then $\{D_r \mid r > 0\}$ will be called the associated neighbourhood maps of p.

THEOREM 4.3. *The following statements are valid for all* $r \in (0, \infty)$.

(A1) $D_r(\underset{\sim}{0}) = \underset{\sim}{0}.$

(A2) $\lambda \leqslant D_r(\lambda).$

(A3) $D_r(\bigvee_\alpha \lambda_\alpha) = \bigvee_\alpha D_r(\lambda_\alpha).$

Proof. (A1) and (A2) follow from (M1) and the definition of D_r. Now

$$p(\lambda_\beta, \mu) < r \Rightarrow p\left(\bigvee_\alpha \lambda_\alpha, \mu\right) \leqslant p(\lambda_\beta, \mu) \quad \text{by (M3) (i),}$$
$$< r$$

so that

$$\mu \leqslant D_r\left(\bigvee_\alpha \lambda_\alpha\right).$$

Hence

$$D_r(\lambda_\beta) = \bigvee \{\mu \mid p(\lambda_\beta, \mu) < r\}$$
$$\leqslant D_r\left(\bigvee_\alpha \lambda_\alpha\right),$$

that is

$$\bigvee_\alpha D_r(\lambda_\alpha) \leqslant D_r\left(\bigvee_\alpha \lambda_\alpha\right).$$

By (M4),

$$p\left(\bigvee_\alpha \lambda_\alpha, \gamma\right) < r \Rightarrow \gamma \leqslant \bigvee_\alpha D_r(\lambda_\alpha).$$

Hence

$$D_r\left(\bigvee_\alpha \lambda_\alpha\right) \leqslant \bigvee_\alpha D_r(\lambda_\alpha).$$

Maps satisfying (A2) and (A3) were introduced in [1] and were further considered in [6]. If $f: L^X \to L^X$ satisfies (A1)–(A3) we define its inverse to be

$$f^{-1}: L^X \to L^X \qquad \text{where} \qquad f^{-1}(\lambda) = \bigwedge \{\mu \mid f(\mu') \leqslant \lambda'\}.$$

It is clear that (A1) is necessary in order that f^{-1} be well defined. (Consider, for example, $f^{-1}(\underline{1})$).

Hutton [1] proved the following result.

PROPOSITION 4.4. *If $f: L^X \to L^X$ satisfies* (A1)–(A3) *then so does f^{-1}. Further if f and g satisfy* (A1)–(A3) *then:*

$$f(\lambda) \leqslant \mu \Leftrightarrow f^{-1}(\mu') \leqslant \lambda'. \tag{1}$$

$$(f^{-1})^{-1} = f. \tag{2}$$

$$f \leqslant g \Leftrightarrow f^{-1} \leqslant g^{-1}. \tag{3}$$

$$(f \circ g)^{-1} = g^{-1} \circ f^{-1}. \tag{4}$$

Unless otherwise mentioned, if f satisfies (A1)–(A3), f^{-1} will be used to denote the inverse as defined above, rather than the usual function inverse.

The following theorem indicates the importance of the neighbourhood maps.

THEOREM 4.5. *If p is a fuzzy* p.q. *metric on X with associated neighbourhood maps $\{D_r \mid r > 0\}$ then $\forall \lambda, \mu \in L^X$,*

$$p(\lambda, \mu) = \bigwedge \{r \mid \mu \leqslant D_r(\lambda)\}.$$

Proof. Let $f(\lambda, \mu) = \wedge \{r \mid \mu \leqslant D_r(\lambda)\}$. Then $\forall r > p(\lambda, \mu)$, $\mu \leqslant D_r(\lambda)$, so that $f(\lambda, \mu) \leqslant r$. Hence

$$f(\lambda, \mu) \leqslant p(\lambda, \mu).$$

Now $\forall r$ such that $\mu \leqslant D_r(\lambda)$,

$$\begin{aligned} p(\lambda, \mu) &\leqslant p(\lambda, D_r(\lambda)) \\ &= p\left(\lambda, \bigvee \{\chi \mid p(\lambda, \chi) < r\}\right) \\ &= \bigvee \{p(\lambda, \chi) \mid p(\lambda, \chi) < r\} \qquad \text{by (M3) (ii)} \\ &\leqslant r. \end{aligned}$$

Hence

$$\begin{aligned} p(\lambda, \mu) &\leqslant \bigwedge \{r \mid \mu \leqslant D_r(\lambda)\} \\ &= f(\lambda, \mu). \end{aligned}$$

Remark. At the beginning of Section 3 we considered a distance function d on a p.q. metric space $\langle X, p\rangle$ in defining

$$d(A, B) = \wedge \{r \mid B \subset D_r^{\,p}(A)\}.$$

In the proof of Proposition 3.2 it was shown that $D_r^{\,d}(A) = D_r^{\,p}(A)$. Thus Theorem 4.5 shows the consistency with our motivating example.

PROPOSITION 4.6. *If p is a fuzzy* p.q. *metric with neighbourhood maps D_r, then*

$$D_r \circ D_s \leqslant D_{r+s} \qquad \forall r, s > 0.$$

Proof. Directly from (M2).

We now give a partial converse to Theorem 4.5.

PROPOSITION 4.7. *If $\mathscr{D} = \{D_r \mid r \in (0, \infty)\}$ is a family of maps, $D_r: L^X \to L^X$, satisfying* (A1)–(A3) *and such that $\forall r, s \in (0, \infty)$ $D_r \circ D_s \leqslant D_{r+s}$, then $p: L^X \times L^X \to [0, \infty]$ defined by*

$$p(\lambda, \mu) = \bigwedge \{r \mid \mu \leqslant D_r(\lambda)\}$$

is a fuzzy p.q. *metric on X.*

Further its associated neighbourhood maps, E_r say, are given by

$$E_r = \bigvee_{s<r} D_s\,,$$

(i.e. $E_r(\lambda) = \bigvee_{s<r} D_s(\lambda)$ for $\lambda \in L^X$).

Proof. (M1) follows by definition of p and properties (A1) and (A2) of D_r.

(M2) Let $\lambda, \chi, \mu \in L^X$.

Now $\forall r > p(\lambda, \chi)\ \forall s > p(\chi, \mu)$ we have

$$\chi \leqslant D_r(\lambda) \qquad \text{and} \qquad \mu \leqslant D_s(\chi).$$

Hence

$$\mu \leqslant D_s \circ D_r(\lambda) \leqslant D_{r+s}(\lambda).$$

So by definition of p, $p(\lambda, \mu) \leqslant r + s$, i.e. $p(\lambda, \mu) \leqslant p(\lambda, \chi) + p(\chi, \mu)$.

(M3) (i) Let $\lambda \leqslant \mu$. Then $\forall r > p(\lambda, \chi)$, $\chi \leqslant D_r(\lambda)$, hence $\chi \leqslant D_r(\mu)$ by (A3). Therefore $p(\mu, \chi) \leqslant r$ which gives

$$p(\mu, \chi) \leqslant p(\lambda, \chi).$$

(M3) (ii) Let $r > p(\mu, \bigvee_\alpha \lambda_\alpha)$. Then

$$\begin{aligned} & \bigvee_\alpha \lambda_\alpha \leqslant D_r(\mu) \\ \Rightarrow\quad & \lambda_\alpha \leqslant D_r(\mu) \qquad \forall\alpha \\ \Rightarrow\quad & p(\mu, \lambda_\alpha) \leqslant r \qquad \forall\alpha \\ \Rightarrow\quad & \bigvee_\alpha p(\mu, \lambda_\alpha) \leqslant r. \end{aligned}$$

Hence

$$\bigvee_\alpha p(\mu, \lambda_\alpha) \leqslant p\left(\mu, \bigvee_\alpha \lambda_\alpha\right).$$

Let $r > \bigvee_\alpha p(\mu, \lambda_\alpha)$. Then

$$\begin{aligned} & r > p(\mu, \lambda_\alpha) \qquad \forall\alpha \\ \Rightarrow\quad & \lambda_\alpha \leqslant D_r(\mu) \qquad \forall\alpha \\ \Rightarrow\quad & \bigvee_\alpha \lambda_\alpha \leqslant D_r(\mu) \\ \Rightarrow\quad & p\left(\mu, \bigvee_\alpha \lambda_\alpha\right) \leqslant r. \end{aligned}$$

Hence

$$p\left(\mu, \bigvee_\alpha \lambda_\alpha\right) \leqslant \bigvee_\alpha p(\mu, \lambda_\alpha).$$

To show $E_r = \bigvee_{s<r} D_s$:

$$\forall \mu \leqslant D_r(\lambda), \qquad p(\lambda, \mu) \leqslant r$$

$$< s \qquad \forall s > r,$$

$$\Rightarrow \mu \leqslant E_s(\lambda) \qquad \forall s > r.$$

Hence

$$D_r(\lambda) \leqslant E_s(\lambda) \qquad \forall s > r,$$

or

$$D_s(\lambda) \leqslant E_r(\lambda) \qquad \forall s < r,$$

so that

$$\bigvee_{s<r} D_s(\lambda) \leqslant E_r(\lambda).$$

Now $\forall \mu$ such that $p(\lambda, \mu) < r$, we have $p(\lambda, \mu) < t$ for some $t < r$.

Hence

$$\mu \leqslant D_t(\lambda)$$

$$\leqslant \bigvee_{s<r} D_s(\lambda),$$

that is

$$E_r(\lambda) = \bigvee \{\mu: p(\lambda, \mu) < r\}$$

$$\leqslant \bigvee_{s<r} D_s(\lambda).$$

To prove (M4):

Suppose $\mu, \chi_\alpha \in L^X \ \forall \alpha \in \Delta$ satisfy

$$p(\chi_\alpha, \beta) < r \Rightarrow \beta \leqslant \mu \qquad \text{for } \beta \in L^X, \quad \alpha \in \Delta.$$

Now if $\gamma \in L^X$ satisfies $p(\bigvee_\alpha \chi_\alpha, \gamma) < r$, then

$$\gamma \leqslant E_r\left(\bigvee_\alpha \chi_\alpha\right)$$

$$= \bigvee_{s<r} D_s\left(\bigvee_\alpha \chi_\alpha\right)$$

$$= \bigvee_{s<r} \bigvee_\alpha D_s(\chi_\alpha)$$

$$= \bigvee_\alpha E_r(\chi_\alpha)$$

$$\leqslant \mu \qquad \text{by hypothesis.}$$

Hutton [1] defined p.q. metrizability in terms of the neighbourhood maps and it was further considered in Hutton and Reilly [4].

We now consider the topology of the fuzzy p.q. metric space. We have anticipated the following result in a comment before Corollary 3.4.

THEOREM 4.8. *If p is a fuzzy* p.q. *metric on X with associated neighbourhood maps D_r, then $\{D_r(\lambda) \mid \lambda \in L^X, r \in (0, \infty)\}$ is a base for a topology on the fuzz X.*

It will be called the topology of the fuzzy p.q. metric p.

Proof. It must be shown that the arbitrary supremums of this set, together with $\underline{0}$ and $\underline{1}$, form a fuzzy topology. It is enough to prove that for all $\lambda, \mu \in L^X$, $r, s \in (0, \infty)$ there exist $K_\alpha \in L^X$, $t_\alpha \in (0, \infty)$ $\alpha \in \Delta$ for some index set Δ, such that

$$D_r(\lambda) \wedge D_s(\mu) = \bigvee_{\alpha \in \Delta} D_{t_\alpha}(K_\alpha).$$

Let $K = D_r(\lambda) \wedge D_s(\mu)$. If $K = \underline{0}$, then $K = D_r(\underline{0})$. If $K \neq \underline{0}$, then

$$\begin{aligned} K &= \bigvee \left\{\chi_1 \mid p(\lambda, \chi_1) < r\right\} \wedge \bigvee \{\chi_2 \mid p(\mu, \chi_2) < s\} \\ &= \bigvee_{\chi_1} \bigvee_{\chi_2} \{\chi_1 \wedge \chi_2 \mid p(\lambda, \chi_1) < r, p(\mu, \chi_2) < s\} \\ &= \bigvee \{\chi_\alpha \mid \alpha \in \Delta\} \end{aligned}$$

say, where

$$\chi_\alpha = \chi_{\alpha_1} \wedge \chi_{\alpha_2}$$

and

$$p(\lambda, \chi_{\alpha_1}) < r, \qquad p(\mu, \chi_{\alpha_2}) < s.$$

Now $\forall \alpha \in \Delta$ let $t_\alpha = [r - p(\lambda, \chi_{\alpha_1})] \wedge [s - p(\mu, \chi_{\alpha_2})]$ and $K_\alpha = \chi_\alpha$. If $p(K_\alpha, \eta) < t_\alpha$ then

$$p(K_\alpha, \eta) < r - p(\lambda, \chi_{\alpha_1}).$$

Now

$$\chi_{\alpha_1} \geqslant K_\alpha \Rightarrow p(\chi_{\alpha_1}, \eta) \leqslant p(K_\alpha, \eta).$$

Hence

$$p(\chi_{\alpha_1}, \eta) + p(\lambda, \chi_{\alpha_1}) < r.$$

By (M2)

$$p(\lambda, \eta) < r.$$

Hence

$$\eta \leqslant D_r(\lambda).$$

Similarly

$$\eta \leqslant D_s(\mu).$$

Hence

$$D_{t_\alpha}(K_\alpha) \leqslant K \qquad \text{for all } \alpha \in \Delta.$$

Thus $\bigvee_{\alpha\in\Delta} D_{t_\alpha}(K_\alpha) \leqslant K$. Now

$$\chi_\alpha = K_\alpha \leqslant D_{t_\alpha}(K_\alpha)$$

$$\Rightarrow \quad K = \bigvee_\alpha \chi_\alpha \leqslant \bigvee_\alpha D_{t_\alpha}(K_\alpha).$$

Remark. Since $\lambda \leqslant D_r(\lambda)$ for $r \in (0, \infty)$, $D_r(\lambda)$ is indeed a neighbourhood of $\lambda \in L^X$.

PROPOSITION 4.9. *In the topology of the fuzzy* p.q. *metric p, with neighbourhood maps D_r,*

$$\mathring{\lambda} = \bigvee \{\chi \mid D_r(\chi) \leqslant \lambda \text{ for some } r > 0\}.$$

Proof. Let $T = \bigvee \{\chi \mid D_r(\chi) \leqslant \lambda \text{ for some } r > 0\}$. Now

$$\begin{aligned}\mathring{\lambda} &= \bigvee \{\chi \mid \chi \leqslant \lambda, \chi \text{ open}\}\\ &= \bigvee \{D_r(\kappa) \mid \kappa \in L^X, r > 0 \text{ and } D_r(K) \leqslant \chi\}.\end{aligned}$$

If $\chi \in L^X$ satisfies $D_r(\chi) \leqslant \lambda$ for some $r > 0$ then $\chi \leqslant D_r(\chi) \leqslant \mathring{\lambda}$. Hence $T \leqslant \mathring{\lambda}$. If $D_r(\kappa) \leqslant \lambda$ for some $K \in L^X$, $r > 0$ and if μ satisfies $p(K, \mu) < r$ then there exists $s \in (0, r)$ such that $p(K, \mu) < s$. Hence $\mu \leqslant D_s(\kappa)$. Thus

$$\begin{aligned}D_{r-s}(\mu) &\leqslant D_{r-s} \circ D_s(\kappa)\\ &\leqslant D_r(\kappa) \qquad \text{by Proposition 4.6}\\ &\leqslant \lambda.\end{aligned}$$

Thus

$$\begin{aligned}&\mu \leqslant T\\ \Rightarrow\ &D_r(K) \leqslant T\\ \Rightarrow\ &\mathring{\lambda} \leqslant T.\end{aligned}$$

LEMMA 4.10. *If $\langle L^X, \mathcal{T}\rangle$ is a fuzzy topological space then*

$$\bar{\lambda} = [\mathrm{Int}(\lambda')]',$$

where

$$\mathrm{Int}(\chi) = \chi^0.$$

Proof. $\text{Int}(\lambda') = \bigvee \{\chi \mid \chi \leqslant \lambda', \chi \in \mathscr{T}\}$. Thus

$$\begin{aligned}[\text{Int}(\lambda')]' &= \bigwedge \{\chi' \mid \chi' \geqslant \lambda, \chi \in \mathscr{T}\} \\ &= \bigwedge \{\kappa \mid \kappa \geqslant \lambda, \kappa \text{ closed}\} \\ &= \bar{\lambda}.\end{aligned}$$

THEOREM 4.11. *In the fuzzy* p.q. *metric space* $\langle L^X, p, D_r \rangle$ (*where all symbols have an obvious meaning*),

$$\bar{\lambda} = \bigwedge_{r>0} D_r^{-1}(\lambda).$$

Recall that D_r^{-1} *is the inverse in Proposition* 4.4).

Proof. $\text{Int}(\lambda) = \bigvee \{\kappa \mid D_r(\kappa) \leqslant \lambda$ for some $r > 0\}$ by Proposition 4.9. Thus $\text{Int}(\lambda) = \bigvee \{\kappa \mid D_r^{-1}(\lambda') \leqslant \kappa'$ some $r > 0\}$ by Proposition 4.4. So $\text{Int}(\lambda) = \bigvee \{\kappa \mid \kappa \leqslant [D_r^{-1}(\lambda')]'$ some $r > 0\} = \bigvee_{r>0} [D_r^{-1}(\lambda')]'$. By Lemma 4.10,

$$\begin{aligned}\bar{\lambda} &= \left\{\bigvee_{r>0} [D_r^{-1}(\lambda)]'\right\}', \\ &= \bigwedge_{r>0} D_r^{-1}(\lambda).\end{aligned}$$

5. FUZZY PSEUDO-METRICS

DEFINITION 5.1. A fuzzy pseudo-metric (p. metric) on X, or p. metric on the fuzz X, is a fuzzy p.q. metric d, with neighbourhood maps D_r, satisfying

$$D_r = D_r^{-1} \qquad \forall r \in (0, \infty). \tag{A4}$$

Remark. This is equivalent to the usual definition when $L = 2$.

Indeed if e is a p.q. metric on X in the usual sense with neighbourhood maps D_r, i.e. $D_r(A) = \{y \mid \bigwedge_{a \in A} e(a, y) < r\}$ (see Section 3), then it is enough to prove $e(x, y) = e(y, x)\ \forall x, y \in X$

$$\Leftrightarrow \quad D_r = D_r^{-1} \qquad \forall r \in (0, \infty).$$

The proof is as follows:

$$\begin{aligned}D_r^{-1}(\{x\}) &= \bigcap \{A \subset X \mid D_r(A') \subset X \backslash \{x\}\} \\ &= \left[\bigcup \{A \mid x \notin D_r(A)\}\right]'\end{aligned}$$

$$= \{z \mid x \notin D_r(z)\}' \qquad \text{since} \qquad D_r(A) = \bigcup_{a \in A} D_r(\{a\})$$

$$= \{z \mid x \in D_r(z) = \{y \mid e(z, y) < r\}\}$$

$$= \{z \mid e(z, x) < r\}.$$

($\Rightarrow$) If $e(x, y) = e(y, x)$ then

$$D_r^{-1}(\{x\}) = \{z \mid e(x, z) < r\}$$
$$= D_r(\{x\}).$$

($\Leftarrow$) If $D_r = D_r^{-1}$ $\forall r \in (0, \infty)$, then

$$y \in D_r(\{x\}) \Leftrightarrow y \in D_r^{-1}(\{x\})$$
$$\Leftrightarrow e(y, x) < r.$$

Now

$$e(x, y) = \bigwedge \{r \mid y \in D_r(\{x\})$$
$$= \bigwedge \{r \mid e(y, x) < r\}$$
$$= e(y, x).$$

PROPOSITION 5.2. *In a* p. *metric space* $\langle L^X, d, D_r \rangle$

$$\bar{\lambda} = \bigwedge_{r>0} D_r(\lambda) = \bigvee \{\mu \mid d(\lambda, \mu) = 0\}.$$

Proof.

$$\bar{\lambda} = \bigwedge_{r>0} D_r(\lambda) \qquad \text{by Theorem 4.11.}$$

Let

$$T = \bigvee \{\mu \mid d(\lambda, \mu) = 0\}.$$

clearly

$$T \leqslant \bigwedge_{r>0} D_r(\lambda).$$

Now suppose

$$\mu \leqslant \bigwedge_{r>0} D_r(\lambda).$$

Then $\forall r > 0$ $\mu \leqslant D_r(\lambda)$, so that

$$d(\lambda, \mu) \leqslant d(\lambda, D_r(\lambda)) \qquad \forall r > 0$$
$$= r \qquad \forall r > 0.$$

Hence

$$d(\lambda, \mu) = 0.$$

Thus

$$\bigwedge_{r>0} D_r(\lambda) \leqslant T.$$

DEFINITION 5.3. Hutton and Reilly [4] have defined a fuzzy topological space to be R_0 iff every open set is a supremum of closed sets.

COROLLARY 5.4. *Every fuzzy* p. *metric space is* R_0.

COROLLARY 5.5. $\overline{D_r(\lambda)} \leqslant \bigvee \{\mu \mid d(\lambda, \mu) \leqslant r\}$ *in a fuzzy* p. *metric space* $\langle L^X, d, D_r\rangle$. *Hence*

$$\overline{D_r(\lambda)} \leqslant D_s(\lambda) \qquad \forall s > r.$$

Proof.

$$\begin{aligned}\overline{D_r(\lambda)} &= \bigwedge_{s>0} D_s(D_r(\lambda)) \\ &\leqslant \bigwedge_{s>0} D_{s+r}(\lambda) \\ &= \bigwedge_{t>r} D_t(\lambda) \\ &= \bigwedge \{\mu \mid d(\lambda, \mu) \leqslant r\}.\end{aligned}$$

Remark. The reverse inequality need not hold, even in the usual set theory. Consider, for example, a pseudo-metric on a two element set giving the discrete topology.

DEFINITION 5.6. A fuzzy topological space $\langle L^X, \mathscr{T}\rangle$ is normal iff $\forall \lambda, \mu \in L^X$ such that $\lambda', \mu \in \mathscr{T}$ $\exists \chi \in L^X$ satisfying $\lambda \leqslant \mathring{\chi} \leqslant \bar{\chi} \leqslant \mu$.

THEOREM 5.7. *Every fuzzy* p. *metric space* $\langle L^X, d, D_r\rangle$ *is normal.*

Proof. Let $\lambda, \mu \in L^X$ where $\lambda = \bar{\lambda}$ and $\mu = \mathring{\mu}$ in the pseudo-metric topology. $\bar{\lambda} = \bigwedge_{r>0} D_r(\lambda)$ and $\lambda \leqslant \mu$ give

$$\begin{aligned}\mu &= \lambda \vee \mu \\ &= \bigwedge_{r>0} [D_r(\lambda) \vee \mu].\end{aligned}$$

$$\begin{aligned}\mathring{\mu} &= \bigvee \{\kappa \mid Dr(\kappa) \leqslant \mu \text{ for some } r > 0\} \\ &= \bigvee \{\kappa_\alpha \mid \alpha \in \Delta\} \quad \text{say,}\end{aligned}$$

where $\forall \alpha \in \Delta$ $\exists r_\alpha \in (0, \infty)$ so that

$$D_{r_\alpha}(\kappa_\alpha) \leqslant \mu.$$

Hence

$$\lambda = \lambda \wedge \mu$$
$$= \bigvee_{\alpha \in \Delta} [\lambda \wedge \kappa_\alpha].$$

Let

$$\chi = \bigvee_{\alpha \in \Delta} D_{r_\alpha/2}(\lambda \wedge \kappa_\alpha).$$

Then

$$\chi = \mathring{\chi} \qquad \text{and} \qquad \lambda \leqslant \chi \leqslant \bar{\chi}.$$

Now

$$\begin{aligned}
\bar{\chi} &= \bigwedge_{s>0} D_s(\chi) \\
&= \bigwedge_{s>0} D_{s/2}(\chi) \\
&= \bigwedge_{s>0} \bigvee_{\alpha \in \Delta} D_{s/2}(D_{r_\alpha/2}(\lambda \wedge \kappa_\alpha)) \\
&\leqslant \bigwedge_{s>0} \bigvee_{\alpha \in \Delta} D_{(s+r_\alpha)/2}(\lambda \wedge \kappa_\alpha) \\
&\leqslant \bigwedge_{s>0} \bigvee_{\alpha \in \Delta} D_{s \vee r_\alpha}(\lambda \wedge \kappa_\alpha) \\
&= \bigwedge_{s>0} \bigvee_{\alpha \in \Delta} [D_s(\lambda \wedge \kappa_\alpha) \vee D_{r_\alpha}(\lambda \wedge \kappa_\alpha)] \\
&\leqslant \bigwedge_{s>0} \bigvee_{\alpha \in \Delta} [D_s(\lambda) \vee \mu] \\
&= \bigwedge_{s>0} [D_s(\lambda) \vee \mu] \\
&= \mu.
\end{aligned}$$

Remark. The proof of this theorem is based on one given by Hutton and Reilly [4] where pseudo-metrizability is defined in terms of neighbourhood maps (see Remark 1 after Theorem 6.5).

6. Quasi-Uniformities on Fuzzy Sets

Hutton [1] has defined quasi-uniformities on fuzzy sets by considering entourages as maps $D: L^X \to L^X$ satisfying (A1)–(A3), (see Theorem 4.3). For quasi-uniformities in the standard case, i.e. $L = 2$, see Murdeshwar and Naimpally [10].

If D and E are entourages we require that their "intersection", written $D \wedge E$, is also an entourage. We define

$$D \wedge E: L^X \to L^X$$

by

$$(D \wedge E)(\lambda) = \bigwedge_{\sup \Gamma = \lambda} \left(\bigvee_{\gamma \in \Gamma} [D(\gamma) \wedge E(\gamma)] \right) \qquad \text{where} \qquad \Gamma \subset L^X.$$

Hutton has shown that $D \wedge E$ is the largest of all maps satisfying (A1)–(A3) which are bounded above by both D and E. He has further shown that

$$(D \wedge E)(\lambda) = \bigwedge_{\lambda_1 \vee \lambda_2 = \lambda} [D(\lambda_1) \vee E(\lambda_2)]$$

and

$$(D \wedge E)^{-1} = D^{-1} \wedge E^{-1}.$$

DEFINITION 6.1. Let φ be the set of all maps $E: L^X \to L^X$ satisfying (A1)–(A3).

A uniformity on the fuzz X is a set $\mathscr{D} \subset \varphi$ satisfying

(Q1) $\mathscr{D} \neq \phi$

(Q2) $D \in \mathscr{D} \Rightarrow D^{-1} \in \mathscr{D}$

(Q3) $D, E \in \Rightarrow D \wedge E \in \mathscr{D}$

(Q4) $D \in \mathscr{D} \Rightarrow \exists E \in \mathscr{D}$ such that $E \circ E \leqslant D$

(Q5) $D \in \mathscr{D},\ D \leqslant E \in \varphi \Rightarrow E \in \mathscr{D}$

$\mathscr{D}$ is a quasi-uniformity if (Q1), (Q3), (Q4) and (Q5) are true.

The usual definitions of basis and sub-basis will hold.

The fuzzy topology generated by a quasi-uniformity $\mathscr{D}$ is the fuzzy topology obtained by taking as interior operator the map $\text{Int}: L^X \to L^X$ defined by

$$\text{Int}(\lambda) = \bigvee \{\mu \in L^X \mid D(\mu) \leqslant \lambda \text{ for some } D \in \mathscr{D}\}.$$

THEOREM 6.2. *If* $\langle L^X, p, D_r \rangle$ *is a* p.q. *metric space then* $\mathscr{D} = \{D_r \mid r \in (0, \infty)\}$ *is a basis for a quasi-uniformity on the fuzz* X. *Further, the fuzzy topology of the quasi-uniformity is that of the fuzzy* p.q. *metric space.*

Proof. Clearly $\phi \neq \mathscr{D} \subset \varphi$. For (Q3) it is enough to prove that $\forall r, s \in (0, \infty)$ $\exists t \in (0, \infty)$ such that $D_t \leqslant D_r \wedge D_s$. Let $t = r \wedge s$. Suppose that $\lambda_1 \vee \lambda_2 = \lambda \in L^X$. Then

$$D_t(\lambda) = D_{r \wedge s}(\lambda_1) \vee D_{r \wedge s}(\lambda_2)$$

$$\leqslant D_r(\lambda_1) \vee D_s(\lambda_2).$$

However

$$(D_r \wedge D_s)(\lambda) = \bigwedge_{\lambda_1 \vee \lambda_2 = \lambda} [D_r(\lambda_1) \vee D_s(\lambda_2)],$$

which gives the required result.

$\mathscr{D}$ satisfies (Q4) since $D_{r/2} \circ D_{r/2} \leqslant D_r$.

The remainder of the theorem follows by Proposition 4.9.

We shall now prove results concerning the metrization of uniform spaces.

LEMMA 6.3. *Let $U = \{U_n \mid n \in \omega\} \subset \varphi$ be a sequence of maps in φ satisfying*

$$\begin{aligned} U_0(\lambda) &= \underline{1} \quad \text{if} \quad \lambda \neq \underline{0} \\ &= \underline{0} \quad \text{if} \quad \lambda = \underline{0}, \end{aligned}$$

and

$$U_{n+1} \circ U_{n+1} \circ U_{n+1} \leqslant U_n \qquad \forall n \in \omega.$$

Then there is a set $\mathscr{D} \subset \varphi$, $\mathscr{D} = \{D_r \mid r \in (0, \infty)\}$, satisfying

$$D_r \circ D_s \leqslant D_{r+s} \qquad \forall r, s \in (0, \infty)$$

and

$$U_n \leqslant \bigvee_{s < 2^{-n}} D_s \leqslant U_{n-1} \qquad \forall n \geqslant 1.$$

Remark. This theorem is just that of Hutton [1: Theorem 9]. The proof is included here for completeness.

Proof. $\forall r \in (0, \infty)$ define a map $\phi_r \in \varphi$ by

$$\phi_r = U_n \qquad \text{if} \qquad r \in [2^{-(n+1)}, 2^{-n})$$

or

$$\phi_r = U_0 \qquad \text{if} \qquad r \geqslant 1.$$

Then

$$\phi_r \circ \phi_r \circ \phi_r \leqslant \phi_{2r} \qquad \forall r \in (0, \infty).$$

Now $\forall r \in (0, \infty)$ define a map $D_r \in \varphi$ by

$$D_r = \bigvee \left\{ \phi_{r_1} \circ \cdots \circ \phi_{r_k} \,\middle|\, \sum_{i=1}^{k} r_i = r \right\}.$$

Clearly $\phi_r \leqslant D_r$ $\forall r \in (0, \infty)$.

Further $D_r \leqslant \phi_{2r}$ $\forall r \in (0, \infty)$, since

$$\forall k \geqslant 1 \quad \forall r_1, \ldots, r_k \qquad \text{such that} \qquad r = \sum_{i=1}^{k} r_i$$
$$\phi_{r_1} \circ \cdots \circ \phi_{r_k} \leqslant \phi_{2r}.$$

Indeed, if $k = 1$, the statement is trivial.

If $k > 1$, let j be the largest integer satisfying $\sum_{i=1}^{j} r_i \leqslant r/2$. Then

$$\sum_{i=j+2}^{k} r_i \leqslant r/2 \qquad \text{and} \qquad r_{j+1} \leqslant r/2.$$

By induction

$$\phi_{r_1} \circ \cdots \circ \phi_{r_j} \leqslant \phi_r ,$$
$$\phi_{r_{j+1}} \leqslant \phi_r ,$$

and

$$\phi_{r_{j+2}} \circ \cdots \circ \phi_{r_k} \leqslant \phi_r .$$

Hence

$$\phi_{r_1} \circ \cdots \circ \phi_{r_k} \leqslant \phi_r \circ \phi_r \circ \phi_r \leqslant \phi_{2r} .$$

Since

$$\phi_r \leqslant D_r \leqslant \phi_{2r} \qquad \forall r \in (0, \infty),$$
$$U_n \leqslant D_r \leqslant U_{n-1} \qquad \forall r \in [2^{-(n+1)}, 2^{-n}).$$

Thus

$$U_n \leqslant \bigvee_{r<2^{-n}} D_r \leqslant U_{n-1} .$$

It is clear that

$$D_r \circ D_s \leqslant D^t_{r \circ s} \qquad \forall r, s \in (0, \infty).$$

PROPOSITION 6.4. *If U is the sequence of maps in Lemma 6.3, then there exists a fuzzy* p.q. *metric p on X with associated neighbourhood maps E_r, satisfying*

$$U_{n+1} \leqslant E_{2^{-n}} \leqslant U_{n-1} \qquad \forall n \geqslant 1.$$

Proof. By Lemma 6.3 and Proposition 4.7.

THEOREM 6.5 (P.Q. Metrization Theorem). *A quasi-uniform space on the fuzz X, $\langle L^X, \mathscr{D} \rangle$, is fuzzy* p.q. *metrizable iff $\mathscr{D}$ has a countable base.*

Proof. ($\Rightarrow$) Trivial.

($\Leftarrow$) If $\mathscr{D}$ has a countable base, say $\mathscr{U} = \{U_n \mid n \in \omega\}$ we may rechoose $\mathscr{U}$ so as to satisfy the hypothesis of Lemma 6.3, (see for example Pervin [11]). The result follows by Proposition 6.4.

Remarks. 1. Hutton [1] obtained the following result. $\langle L^X, \mathscr{D} \rangle$ has a countable base iff $\mathscr{D}$ has a base $\{D_r \mid r \in (0, \infty)\}$ satisfying

$$D_r \circ D_s \leqslant D_{r+s} \qquad \forall r, s \in (0, \infty).$$

This condition was taken to mean fuzzy p.q. metrizability in [1] and [4]. Theorem 6.5 shows it to be the same as our own definition of p.q. metrizability.

2. When $L = 2$, i.e. in the usual set theory, we have the well-known result: A quasi-uniform space is p.q. metrizable iff the quasi-uniformity has a countable base.

In view of Theorem 6.5, we have for the case $L = 2$:

A topological space is p.q. metrizable iff it is fuzzy p.q. metrizable. This is just Corollary 3.4.

We now obtain a result on the generation of topologies by fuzzy p.q. metrics.

DEFINITION 6.6. *Let* $\langle L^X, p, D_r \rangle$ be a fuzzy p.q. metric space and let $\langle L^X, \mathscr{U} \rangle$ be a fuzzy quasi-uniform space.

Then p is quasi-uniformly lower semicontinuous (q.u.l.s.c.) on $\langle L^X, \mathscr{U} \rangle$ iff $\{D_r \mid r \in (0, \infty)\} \subset \mathscr{U}$.

Remark. We know this is equivalent to the usual definition when $L = 2$ (see for example Kelley [12]).

DEFINITION 6.7. If $P = \{p_\alpha\}_{\alpha \in \Delta}$ is a set of fuzzy p.q. metrics on the fuzz X, then P is said to generate the quasi-uniformity $\mathscr{U}$ whose sub-base is the set of neighbourhood maps of each p_α.

P is called the gage of $\langle L^X, \mathscr{U} \rangle$.

THEOREM 6.8. *Let $\langle L^X, \mathscr{U} \rangle$ be a quasi-uniform space and let P be the set of all fuzzy* p.q. *metrics which are* q.u.l.s.c. *on $\langle L^X, \mathscr{U} \rangle$. Then P generates $\mathscr{U}$.*

Proof. Let $\mathscr{D}$ be the quasi-uniformity generated by P. Then clearly $\mathscr{D} \subset \mathscr{U}$. To prove the converse, suppose $U \in \mathscr{U}$.

Let

$$\begin{aligned} U_0(\lambda) &= \underline{1} \quad \text{if} \quad \lambda \neq \underline{0}. \\ &= \underline{0} \quad \text{if} \quad \lambda = \underline{0}, \end{aligned}$$

and $U_1 = U$.

Define $U_n \in \mathscr{U}$ inductively so that $U_{n+1} \circ U_{n+1} \circ U_{n+1} \leqslant U_n \ \forall n \in \omega$. Then by Lemma 6.3, there exists a fuzzy p.q. metric p on X with neighbourhood maps D_r satisfying $U_n \leqslant D_{2^{-n}} \leqslant U_{n-1} \ \forall n \geqslant 1$. Hence $p \in P$ and $U \geqslant D_{1/4}$.

Thus $U \in \mathscr{D}$.

We now prove a symmetric version of Proposition 4.7.

PROPOSITION 6.9. *If $\mathscr{D} = \{D_r \mid r \in (0, \infty)\}$ is a family of maps $D_r \colon L^X \to L^X$ satisfying* (A1)–(A4) *such that $\forall r, s \in (0, \infty)$, $D_r \circ D_s \leqslant D_{r+s}$*, then *$d \colon L^X \times L^X \to [0, \infty]$ defined by*

$$d(\lambda, \mu) = \bigwedge \{r \mid \mu \leqslant D_r(\lambda)\}$$

is a fuzzy pseudo-metric on X, with neighbourhood maps $E_r = \bigvee_{s<r} D_s$.

Proof. Using Proposition 4.7 it remains to prove that $E_r = E_r^{-1}\ \forall r \in (0, \infty)$. Now

$$D_s^{-1} = D_s \leqslant \bigvee_{t<r} D_t \qquad \forall s < r$$

$$\Rightarrow \quad D_s = (D_s^{-1})^{-1} \leqslant \Big[\bigvee_{t<r} D_t\Big]^{-1} \qquad \forall s < r,$$

by Proposition 4.4.

Hence

$$\bigvee_{s<r} D_s \leqslant \Big[\bigvee_{t<r} D_t\Big]^{-1},$$

so that

$$E_r \leqslant E_r^{-1}.$$

By Proposition 4.4, this gives $E_r^{-1} \leqslant E_r$, and hence $E_r = E_r^{-1}$.

Using this result it is now easy to obtain symmetric versions of Theorems 6.5 and 6.8.

We note that if, in Proposition 6.9, $\mathscr{D}$ is the basis of a uniformity then the uniform topology is that of the pseudo-metric d. This follows since

$$D_{r/2} \leqslant E_r \leqslant D_r \qquad \forall r \in (0, \infty).$$

EXAMPLE. Hutton [1] has defined a fuzzy unit interval, [0, 1] (L), which reduces to the [0, 1] when $L = 2$. He has also defined a uniformity for [0, 1] (L). It satisfies the hypothesis of Proposition 6.9 and so a fuzzy pseudo-metric may be defined on [0, 1] (L). By the note above, the fuzzy p-metric topology is that of the uniformity on [0, 1) (L).

7. CONJUGATE FUZZY P. Q. METRICS

THEOREM 7.1. *Let $\langle L^X, p, D_r\rangle$ be a fuzzy* p.q. *metric space. Define $q: L^X \times L^X \to [0, \infty]$ by $q(\lambda, \mu) = \wedge\{r \mid \mu \leqslant D_r^{-1}(\lambda)\}$. Then q is a fuzzy* p.q. *metric on X with associated neighbourhood maps $\{D_r^{-1} \mid r \in (0, \infty)\}$.*

Proof. By Proposition 4.4, $\{D_r^{-1} \mid r \in (0, \infty)\}$ is a family of maps satisfying the hypothesis of Proposition 4.7. Hence q is a fuzzy p.q. metric on X with associated neighbourhood maps $E_r = \vee_{s<r} D_s^{-1}\ \forall r \in (0, \infty)$. Now $D_r \leqslant \vee_{s<r} D_s$, since

$$p(\lambda, \mu) < r \Rightarrow p(\lambda, \mu) < s \qquad \text{for some } s \in (0, r)$$

$$\Rightarrow \mu \leqslant D_s(\lambda) \leqslant \bigvee_{s<r} D_s(\lambda).$$

Hence

$$D_r \leqslant \bigvee_{s<r} D_s .$$

Now

$$\forall s < r \qquad D_s \leqslant D_r .$$

Hence $D_s^{-1} \leqslant D_r^{-1}$ by Proposition 4.4

$$\therefore \quad \bigvee_{s<r} D_s^{-1} \leqslant D_r^{-1}, \qquad \text{so that} \qquad E_r \leqslant D_r^{-1}.$$

The reverse inequality is also true since

$$D_s^{-1} \leqslant E_r \qquad \forall s < r.$$

Hence $D_s = (D_s^{-1})^{-1} \leqslant E_r^{-1}\ \forall s < r$. Thus $\bigvee_{s<r} D_s \leqslant E_r^{-1}$, so that $D_r \leqslant E_r^{-1}$ by the above. Hence $D_r^{-1} \leqslant E_r$ by Proposition 4.4.

DEFINITION 7.2. The fuzzy p.q. metric q is said to be the conjugate of p.

PROPOSITION 7.3. *In the fuzzy* p.q. *metric space* $\langle L^X, p, D_r\rangle$

$$\bar{\lambda} = \bigvee \{\mu \mid q(\lambda, \mu) = 0\}.$$

Proof. From Proposition 4.11,

$$\bar{\lambda} = \bigwedge_{r>0} D_r^{-1}(\lambda).$$

Now

$$D_r^{-1}(\lambda) = \bigvee \{\mu \mid q(\lambda, \mu) < r\}.$$

Thus

$$\bar{\lambda} = \bigwedge_{r>0} \bigvee \{\mu \mid q(\lambda, \mu) < r\}$$

$$\geqslant \bigvee \{\mu \mid q(\lambda, \mu) = 0\}.$$

Conversely, if

$$\chi \leqslant D_r^{-1}(\lambda) \qquad \forall r \in (0, \infty),$$

then

$$q(\lambda, \chi) \leqslant r \qquad \forall r \in (0, \infty),$$

so that

$$q(\lambda, \chi) = 0.$$

Thus

$$\chi \leqslant \bigvee \{\mu \mid q(\lambda, \mu) = 0\}.$$

Hence

$$\bar{\lambda} \leqslant \bigvee \{\mu \mid q(\lambda, \mu) = 0\}.$$

Proposition 5.2 is the special case of Proposition 7.3 when $p = q$.

If p and q are conjugate p.q. metrics in the usual sense then the map d defined by $d(x, y) = p(x, y) \vee q(x, y)$ is a pseudo-metric in the usual sense. The next result generalizes this method.

PROPOSITION 7.4. *If $\langle L^X, p, D_r\rangle$ and $\langle L^X, q, D_r^{-1}\rangle$ are conjugate fuzzy* p.q. *metrics then $\langle L^X, d, E_r\rangle$ is a fuzzy pseudo-metric where $E_r = \bigvee_{s<r} (D_s \wedge D_s^{-1})$ and*

$$d(\lambda, \mu) = \bigwedge \{r \mid \mu \leqslant (D_r \wedge D_r^{-1})(\lambda)\}.$$

Proof. Since $\{D_r \wedge D_r^{-1} \mid r \in (0, \infty)\}$ satisfy (A1)–(A4) it is enough to show that $(D_r \wedge D_r^{-1}) \circ (D_s \wedge D_s^{-1}) \leqslant D_{r+s} \wedge D_{r+s}^{-1}$ and then to apply Proposition 6.9.

Now

$$(D_r \wedge D_r^{-1}) \circ (D_s \wedge D_s^{-1}) \leqslant D_r \circ D_s \leqslant D_{r+s}.$$

Similarly

$$(D_r \wedge D_r^{-1}) \circ (D_s \wedge D_s^{-1}) \leqslant D_{r+s}^{-1}.$$

Since $D_{r+s} \wedge D_{r+s}^{-1}$ is the largest map satisfying (A1)–(A3) which is smaller than both D_{r+s} and D_{r+s}^{-1}, it is evident that

$$(D_r \wedge D_r^{-1}) \circ (D_s \wedge D_s^{-1}) \leqslant D_{r+s} \wedge D_{r+s}^{-1}.$$

Remark. When $L = 2$, the fuzzy pseudo-metric d satisfies

$$d(\{x\}, \{y\}) = p(\{x\}, \{y\}) \vee q(\{x\}, \{y\}) \qquad \forall x, y \in X.$$

Indeed, since $(D_r \wedge D_r^{-1})(A) = D_r(A) \wedge D_r^{-1}(A)$ when $L = 2$, we have

$$\begin{aligned} d(\{x\}, \{y\}) &= \bigwedge \{r \mid y \in D_r(\{x\}) \cap D_r^{-1}(\{x\})\} \\ &= \bigwedge \{r \mid p(\{x\}, \{y\}) < r \text{ and } q(\{x\}, \{y\}) < r\} \\ &= p(\{x\}, \{y\}) \vee q(\{x\}, \{y\}). \end{aligned}$$

8. FUZZY METRICS

DEFINITION 8.1. A fuzzy pseudo-metric $\langle L^X, p, D_r\rangle$ is a fuzzy metric if it satisfies

$$\forall \lambda \in L^X, \qquad \Big(\bigwedge_{r>0} D_r\Big)(\lambda) = \lambda, \tag{A5}$$

where $\bigwedge_{r>0} D_r: L^X \to L^X$ is the largest map satisfying (A1)–(A3) and

$$\Big(\bigwedge_{r>0} D_r\Big)(\lambda) \leqslant D_s(\lambda) \qquad \forall s \in (0, \infty).$$

From Hutton [1],

$$\Big(\bigwedge_{r>0} D_r\Big)(\lambda) = \bigwedge_{\sup\Gamma=\lambda} \bigvee_{\gamma\in\Gamma} \bigwedge_{r>0} D_r(\gamma) \qquad \text{where} \qquad \Gamma \subset L^X$$

$$= \bigwedge_{\sup\Gamma=\lambda} \bigvee_{\gamma\in\Gamma} \bar{\gamma} \qquad \text{by Proposition 5.2.}$$

Hence (A5) becomes

$$\lambda = \bigwedge_{\sup\Gamma=\lambda} \bigvee_{\gamma\in\Gamma} \bar{\gamma} \qquad \forall \lambda \in L^X. \tag{A5$'$}$$

Remarks. 1. When $L = 2$, this is equivalent to

$$A = \bigcup_{x\in A} \overline{\{x\}} \qquad \forall A \subset X,$$

or

$$\{x\} = \overline{\{x\}} \qquad \forall x \in X,$$

which is the usual condition for a pseudo-metric, d, to be a metric since

$$\overline{\{x\}} = \{y \mid d(x, y) = 0\}.$$

2. Murdeshwar and Naimpally [10] give the condition for a quasi-uniform space (in the usual sense), $\langle X, \mathscr{U}\rangle$, to be T_1 as:

$$\Delta = \cap\{U \mid U \in \mathscr{U}\}$$

where Δ is the diagonal in $X \times X$.

3. (A5) is equivalent to:
The identity on L^X is the largest map f satisfying (A1)–(A3) such that

$$\lambda \leqslant f(\lambda) \leqslant \bar{\lambda} \qquad \forall \lambda \in L^X.$$

This follows by Proposition 5.2.

Definition 8.2. A fuzzy topological space is T_0, according to [4], if every fuzzy set $\lambda \in L^X$ may be written as the union and intersection of closed and open sets.

I.e. $\lambda = \bigwedge_{i\in I} \bigvee_{j\in J_i} \mu_{ij}$ where μ_{ij} is either open or closed, for all i, j.

PROPOSITION 8.3. *Every fuzzy metric space is T_0.*

Proof. From (A5)′.

Remark. I have not been able to show that every T_0 fuzzy pseudo metric space is a fuzzy metric space. It is possible that (A5) should be weakened. I have also been unable to show that the fuzzy pseudo-metric on [0, 1] (L) in the example of Section 6 is a fuzzy metric, or even that it is T_0.

9. FUZZY POINTS

Wong [5] has introduced the concept of fuzzy points.

DEFINITION 9.1. A fuzzy point in the fuzz X is a fuzzy set $u \in L^X$ such that there exist $y_0 \in L$, $x_0 \in X$ satisfying

$$\begin{aligned} u(x) &= y_0 \quad \text{if} \quad x = x_0 \\ &= 0 \quad \text{if} \quad x \neq x_0 . \end{aligned}$$

This definition differs from Wong's in that he required $y_0 \in L \backslash \{0, 1\}$.

If u is a fuzzy point, we shall write $u = (x_0, y_0)$. If $\lambda \in L^X$ then $\lambda = \bigvee_{x \in X} \lambda_x$ where $\lambda_x = (x, \lambda(x))$.

A naive generalization of a p.q. metric is as follows, where $\mathscr{P}$ denotes the fuzzy points of the fuzz X.

DEFINITION 9.2. A naive p.q. metric on the fuzz X is a map

$$p: \mathscr{P} \times \mathscr{P} \to [0, \infty] \text{ satisfying}$$

(M1)* $p(u, u) = 0 \ \forall u \in \mathscr{P}$.

(M2)* $p(u, w) \leqslant p(u, v) + p(v, w) \ \forall u, v, w \in \mathscr{P}$.

We may extend p to $L^X \times L^X$ by defining

$$p(\underline{0}, \lambda) = \infty \qquad \forall \lambda \in L^X, \quad \lambda \neq \underline{0},$$

$$p(\lambda, \underline{0}) = 0 \qquad \forall \lambda \in L^X,$$

and

$$p(\lambda, \mu) = \bigvee_{y \in X} \bigwedge_{x \in X} p(\lambda_x, \mu_y) \qquad \forall \lambda, \quad \mu \in L^X, \quad \lambda \neq \underline{0}, \quad \mu \neq \underline{0}.$$

As in Proposition 3.1, (M1)* $\Rightarrow$ (M1) and (M2)* $\Rightarrow$ (M2). In general (M3) and (M4) do not hold. The reasoning in Proposition 3.1 which gives (M3) and (M4) when $L = 2$, does not hold since there may exist fuzzy points (x_0, λ_1) and (x_0, λ_2) with $\lambda_1, \lambda_2, 0$ all distinct.

Now (M3)(i) and (M4) are used to prove condition (A3) of Theorem 4.3 and (M3)(ii) gives Theorem 4.5. These two fundamental results have allowed us to parallel the usual theory of metrics. In particular we are able to consider a metric space as a special uniform space.

We conclude that Definition 9.2 is too general for the development of a satisfactory theory of fuzzy metric spaces.

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