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Note

Lattice polytopes of degree 2

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ABSTRACT

A theorem of Scott gives an upper bound for the normalized volume of lattice polygons with exactly $i > 0$ interior lattice points. We will show that the same bound is true for the normalized volume of lattice polytopes of degree 2 even in higher dimensions. In particular, there is only a finite number of quadratic polynomials with fixed leading coefficient being the h^*-polynomial of a lattice polytope.

1. Introduction

An n-dimensional lattice polytope $P \subset \mathbb{R}^n$ is the convex hull of a finite number of elements of $\mathbb{Z}^n$. In the following, we denote by $\mathrm{Vol}(P) = n!\mathrm{vol}(P)$ the normalized volume of P and call it the volume of P. By $\Pi^{(1)} := \Pi(P) \subset \mathbb{R}^{n+1}$, we denote the convex hull of $(P, 0) \subset \mathbb{R}^{n+1}$ and $(0, \dots, 0, 1) \in \mathbb{R}^{n+1}$, which we will call the standard pyramid over P. Recursively we define $\Pi^{(k)}(P) = \Pi(\Pi^{(k-1)}(P))$ for all $k > 0$. Δ_n will denote the n-dimensional basic lattice simplex throughout, i.e. $\mathrm{Vol}(\Delta_n) = 1$. If two lattice polytopes P and Q of the same dimension are equivalent via some affine unimodular transformation, we will write $P \cong Q$. The k-fold of a polytope P will be the convex hull of the k-fold vertices of P for every $k \geqslant 0$.

Pick's formula gives a relation between the normalized volume, the number of interior lattice points and the number of lattice points of a lattice polygon, i.e. of a two-dimensional lattice polytope: $\mathrm{Vol}(P) = |P \cap \mathbb{Z}^2| + |P^\circ \cap \mathbb{Z}^2| - 2$. Here P° means the interior of the polytope P.

In 1976 Paul Scott [10] proved that the volume of a lattice polygon with exactly $i \geqslant 1$ interior lattice points is constrained by i:

Theorem 1 *(Scott). Let $P \subset \mathbb{R}^2$ be a lattice polygon such that $|P^\circ \cap \mathbb{Z}^2| = i \geqslant 1$. If $P \cong 3\Delta_2$, then $\mathrm{Vol}(P) = 9$ and $i = 1$. Otherwise the normalized volume is bounded by $\mathrm{Vol}(P) \leqslant 4(i+1)$. According to Pick's formula, this implies $|P \cap \mathbb{Z}^2| \leqslant 3i + 6$ and $|P \cap \mathbb{Z}^2| \leqslant \frac{3}{4}\mathrm{Vol}(P) + 3$.*

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Besides Scott's proof, there are two proofs by Christian Haase and Joseph Schicho [5]. Another proof is given in [15].

Our aim is to generalize Scott's theorem. Therefore we need to introduce another invariant, the degree of a lattice polytope: It is known from [4,11,12] that $h^*_P(t) := (1-t)^{n+1} \sum_{k \geqslant 0} |kP \cap \mathbb{Z}^n| t^k \in \mathbb{Z}[t]$ is a polynomial of degree $d \leqslant n$. This number is described as the degree of P and is the largest number $k \in \mathbb{N}$ such that there is an interior lattice point in $(n+1-k)P$ (cf. [2]). The leading coefficient of h^*_P is the number of interior lattice points in $(n+1-d)P$ and the constant coefficient is $h^*_P(0) = 1$. Moreover the sum of all coefficients is the normalized volume of P and all coefficients are non-negative integers by the non-negativity theorem of Richard P. Stanley [11].

It is easy to show that the h^*-polynomial of P and $\Pi(P)$ are equal. So P and $\Pi(P)$ have the same degree and the same normalized volume, which is the sum of all coefficients of the h^*-polynomial. Moreover

$$\left|\big((n+2-d)\Pi(P)\big)^\circ \cap \mathbb{Z}^{n+1}\right| = \left|\big((n+1-d)P\big)^\circ \cap \mathbb{Z}^n\right|.$$

Scott's theorem shows that the normalized volume of a two-dimensional lattice polytope of degree 2 with exactly $i > 0$ interior lattice points is bounded by $4(i+1)$, except for one single polytope: $3\Delta_2$. We generalize this result to the case of n-dimensional lattice polytopes of degree 2.

Theorem 2. *Let $P \subset \mathbb{R}^n$ be an n-dimensional lattice polytope of degree 2. If $P \cong \Pi^{(n-2)}(3\Delta_2)$, then* $\mathrm{Vol}(P) = 9$, $|P \cap \mathbb{Z}^n| = 8 + n$ *and* $|((n-1)P)^\circ \cap \mathbb{Z}^n| = 1$. *Otherwise the following equivalent statements hold*:

(1) $\mathrm{Vol}(P) \leqslant 4(i+1)$,
(2) $b \leqslant 3i + n + 4$,
(3) $b \leqslant \frac{3}{4}\mathrm{Vol}(P) + n + 1$,

where $b := |P \cap \mathbb{Z}^n|$ *and* $i := |((n-1)P)^\circ \cap \mathbb{Z}^n| \geqslant 1$.

The following theorem of Victor Batyrev [1] motivates our estimation of the normalized volume of a lattice polytope of degree d:

Theorem 3 *(Batyrev). Let $P \subset \mathbb{R}^n$ be an n-dimensional lattice polytope of degree d. If*

$$n \geqslant 4d\binom{2d + \mathrm{Vol}(P) - 1}{2d},$$

then P is a standard pyramid over an $(n-1)$-dimensional lattice polytope.

There is a recent result by Benjamin Nill [8] which even strengthens this bound:

Theorem 4 *(Nill). Let $P \subset \mathbb{R}^n$ be an n-dimensional lattice polytope of degree d. If*

$$n \geqslant \big(\mathrm{Vol}(P) - 1\big)(2d+1),$$

then P is a standard pyramid over an $(n-1)$-dimensional lattice polytope.

Jeffrey C. Lagarias and Günter M. Ziegler showed in [7] that up to unimodular transformation there is only a finite number of n-dimensional lattice polytopes having a fixed volume. From Theorem 3 or Theorem 4 follows

Corollary 5 *(Batyrev). For a family $\mathcal{F}$ of lattice polytopes of degree d, the following is equivalent*:

(1) *$\mathcal{F}$ is finite modulo standard pyramids and affine unimodular transformation.*
(2) *There is a constant $C_d > 0$ such that* $\mathrm{Vol}(P) \leqslant C_d$ *for all $P \in \mathcal{F}$.*

Conjecture 6 *(Batyrev). Let P be a lattice polytope of degree d with exactly $i \geqslant 1$ interior lattice points in its $(\dim(P) + 1 - d)$-fold. Its normalized volume $\mathrm{Vol}(P)$ can then be bounded by a constant $C_{d,i}$, only depending on d and i. The finiteness of lattice polytopes of degree d with this property up to standard pyramids and affine unimodular transformation follows from Theorem 3.*

Theorem 2 proves Conjecture 6 in the case $d = 2$.

Corollary 7. *Up to affine unimodular transformations and standard pyramids there is only a finite number of lattice polytopes of degree 2 having exactly $i \geqslant 1$ interior lattice points in their adequate multiple.*

This follows from Theorems 2 and 3.

Corollary 8. *There is only a finite number of quadratic polynomials $h \in \mathbb{Z}[t]$ with leading coefficient $i \in \mathbb{N}$, such that h is the h^*-polynomial of a lattice polytope.*

This follows from Theorem 2 and the fact that all coefficients of h^*_P are positive integers summing up to $\mathrm{Vol}(P)$.

In the remaining part of the paper we prove Theorem 2.

2. Preparations

The formula of Pick can be easily generalized for higher dimensional polytopes of degree 2 using their h^*-polynomial. This shows that statements (1)–(3) in Theorem 2 are equivalent.

Lemma 9. *An n-dimensional lattice polytope of degree 2 has normalized volume $\mathrm{Vol}(P) = b + i - n$, where $b := |P \cap \mathbb{Z}^n|$ and $i := |((n-1)P)^\circ \cap \mathbb{Z}^n|$.*

Proof. The normalized volume of P can be computed by adding the coefficients of the h^*-polynomial of P. Recall that $h^*_1 = b - n - 1$. Consequently $\mathrm{Vol}(P) = 1 + (b - n - 1) + i$. □

Let $s \subset P$ be a face of P. By $\mathrm{st}(s) = \bigcup F$, we denote the star of s in P, where the union is over all faces $F \subset P$ of P containing s.

Lemma 10. *Let P be an n-dimensional lattice polytope of degree 2 and $s \subset P$ a face of P having exactly $j > 0$ interior lattice points in its $(n-2)$-fold:*

$$\big((n-2)s\big)^\circ \cap \mathbb{Z}^n = \{x_1, \dots, x_j\}.$$

Moreover, we suppose

$$z := \big|P \setminus \mathrm{st}(s) \cap \mathbb{Z}^n\big| \geqslant 1.$$

Then $0 < j + z - 1 \leqslant |((n-1)P)^\circ \cap \mathbb{Z}^n|$.

Proof. Given non-empty finite sets $A, B \subset \mathbb{Z}^n$, there is the following well-known inequality:

$$|A + B| \geqslant |A| + |B| - 1.$$

This is a special case of Kneser's addition theorem [6] or Theorem 5.5 in [14].

The claim follows by applying this inequality to $A := \{x_1, \dots, x_j\}$ and $B := P \setminus \mathrm{st}(s) \cap \mathbb{Z}^n$, since $A + B \subseteq ((n-1)P)^\circ \cap \mathbb{Z}^n$. □

3. Proof of the main theorem

For the proof recall that a Lawrence polytope L is a lattice polytope projecting along an edge onto an $(n-1)$-dimensional basic simplex, i.e.

$$L \cong \operatorname{conv}(0, h_1 e_1, e_l, e_l + h_l e_1\colon\ 2 \leqslant l \leqslant n),$$

where $\{e_1, \ldots, e_n\}$ should denote a lattice basis of $\mathbb{Z}^n$. The numbers $h_1, \ldots, h_n \in \mathbb{N}$ are called the heights of L.

If $n = 2$, then Theorem 2 is equal to Scott's Theorem 1. So let $n > 2$.

The monotonicity theorem of Stanley [13] says that the degree of every face of a polytope is not greater than the degree of the polytope itself. In particular this is true for every facet. So we will distinguish the two cases that there is a facet of P having degree 2 or there is not.

For the second case we need a result of Victor Batyrev and Benjamin Nill. They proved in [2] that every n-dimensional lattice polytope of degree less than 2 either is equivalent to a pyramid over the exceptional lattice simplex $2\Delta_2$ or it is a Lawrence polytope.

Case 1. There is a facet $F \subset P$ of P having degree two, i.e.

$$\left|\left((n-2)F\right)^\circ \cap \mathbb{Z}^n\right| = j \geqslant 1.$$

Define $z := |P \setminus F \cap \mathbb{Z}^n|$. From Lemma 10 we get $z + j - 1 \leqslant i$. Thus, by induction, we get, if $F \ncong \Pi^{(n-3)}(3\Delta_2)$,

$$\begin{aligned}\left|P \cap \mathbb{Z}^n\right| &= \left|F \cap \mathbb{Z}^n\right| + \left|(P \setminus F) \cap \mathbb{Z}^n\right| \leqslant 3j + n - 1 + 4 + z \\ &= 3(j + z - 1) - 2z + 2 + n + 4 \overset{z \geqslant 1}{\leqslant} 3i + n + 4.\end{aligned}$$

Otherwise $F \cong \Pi^{(n-3)}(3\Delta_2)$ and again by induction and Lemma 10: $|F \cap \mathbb{Z}^n| = (n-1) + 8$, $z \leqslant i$ and so $|P \cap \mathbb{Z}^n| = n - 1 + 8 + z \leqslant i + 7 + n$. This term is smaller than $3i + n + 4$ if $i \geqslant 2$. If $i = 1$ however, we get

$$n + 8 \leqslant \left|P \cap \mathbb{Z}^n\right| = n + 7 + z \leqslant i + 7 + n = 8 + n,$$

so $|P \cap \mathbb{Z}^n| = 8 + n$ and $\operatorname{Vol}(P) = 9$ by Lemma 9. In this case $P \cong \Pi^{(n-2)}(3\Delta_2)$ because $\operatorname{Vol}(F) = 9$ and $F \cong \Pi^{(n-3)}(3\Delta_2)$.

Case 2. Every facet F of P has degree $\deg(F) \leqslant 1$.

Let y be an edge of P having the maximal number of lattice points; its length will be denoted by h_1, i.e. $h_1 = |y \cap \mathbb{Z}^n| - 1$. Among all 2-codimensional faces of P containing y, s should be the face having the maximal number of lattice points. We will denote by F_1 and F_2 the two facets of P containing s.

Again the monotonicity theorem of Stanley [13] implies $\deg(s) \leqslant \deg(F_1) = 1$. Similarly to Case 1, we will denote by $z := |P \setminus \{F_1 \cup F_2\} \cap \mathbb{Z}^n|$ the number of lattice points of P not in F_1 and F_2.

By the result of Victor Batyrev and Benjamin Nill [2] we find that the facets F_1 and F_2 are either $(n-1)$-dimensional Lawrence polytopes or pyramids over $2\Delta_2$.

(A) F_1 and F_2 are Lawrence polytopes with heights $h_1^{(k)}, h_2^{(k)}, \ldots, h_{n-1}^{(k)}$ $\forall k \in \{1, 2\}$, where we assume that $h_l^{(1)} = h_l^{(2)} = h_l$ $\forall l \in \{1, \ldots, n-2\}$,

$$s = \operatorname{conv}(0, h_1 e_1, e_l, e_l + h_l e_1\colon\ 2 \leqslant l \leqslant n-2),$$

where $\{e_1, \ldots, e_{n-2}, e_{n-1}^{(k)}\}$ should denote a lattice basis of $\operatorname{lin}(F_k) \cap \mathbb{Z}^n$ such that $F_k = \operatorname{conv}(s, e_{n-1}^{(k)}, e_{n-1}^{(k)} + h_{n-1}^{(k)} e_1)$ for $k \in \{1, 2\}$. Since the degree of the Lawrence prism s is at most one, we obtain

$$\left|\left((n-2)s\right)^\circ \cap \mathbb{Z}^n\right| = \operatorname{Vol}(s) - 1 = \left(\sum_{l=1}^{n-2} h_l\right) - 1.$$

We may assume $z = |(P \setminus \{F_1 \cup F_2\}) \cap \mathbb{Z}^n| \neq 0$ because otherwise P would be a prism over the face $P \cap \{X_1 = 0\}$, which is an $(n-1)$-dimensional lattice simplex of degree at most 1, whose only lattice points are vertices. By [2] this is a basic simplex and hence P is a Lawrence polytope. Consequently $\deg(P) < 2$, a contradiction. We have to distinguish the following two cases:

(i) $|((n-2)s)^\circ \cap \mathbb{Z}^n| \geqslant 1$.

Because of Lemma 10, we get the estimation

$$z + \left(\left(\sum_{l=1}^{n-2} h_l\right) - 1\right) - 1 \leqslant i.$$

In particular, $h_1 \leqslant i + 1$. So we can bound the number of lattice points of P:

$$\begin{aligned} \left|P \cap \mathbb{Z}^n\right| &= \left|(F_1 \cup F_2) \cap \mathbb{Z}^n\right| + z = \left|s \cap \mathbb{Z}^n\right| + h^{(1)}_{n-1} + 1 + h^{(2)}_{n-1} + 1 + z \\ &= \sum_{l=1}^{n-2} h_l + (n-2) + h^{(1)}_{n-1} + h^{(2)}_{n-1} + 2 + z \leqslant i + n + 2h_1 + 2 \\ &\overset{h_1 \leqslant i+1}{\leqslant} i + n + 2(i+1) + 2 = 3i + n + 4. \end{aligned}$$

(ii) $|((n-2)s)^\circ \cap \mathbb{Z}^n| = 0$.

In this case, s has degree zero, so it is a basic simplex. Our assumption on s implies that every lattice point of P is a vertex. If $n = 3$, then Howe's theorem [9] yields that P has at most 8 vertices, therefore $|P \cap \mathbb{Z}^n| \leqslant 8 < n + 4 + 3i$. So let $n \geqslant 4$.

In that case, since every 2-codimensional face is a simplex and every facet is a Lawrence prism, we see that P is simplicial, i.e. every facet is a simplex. We may suppose that P is not a simplex. Let S be a subset of the vertices of P such that the convex hull of S is not a face of P. Then the sum over the vertices of S is a lattice point in the interior of $|S| \cdot P$. Since the degree of P is two, this implies $|S| \geqslant n - 1$. In other words, every subset of the vertices of P that has cardinality at most $n - 2$ forms the vertex set of a face of P, i.e. P is $(n-2)$-neighbourly. As is known from [3], a polytope of dimension n that is not a simplex is at most $\lfloor \frac{n}{2} \rfloor$-neighbourly. Therefore $n - 2 \leqslant \frac{n}{2}$. This shows $n = 4$.

Let $f_j \geqslant 0$ be the number of j-dimensional faces of P. Since P is a 2-neighbourly simplicial 4-dimensional polytope we get $f_1 = \binom{f_0}{2}$ and $f_2 = 2f_3$. Since the Euler characteristic of the boundary of P vanishes, i.e. $f_0 - f_1 + f_2 - f_3 = 0$, we deduce $f_3 = \frac{f_0(f_0-3)}{2}$. Let $\mathcal{D}$ denote the set of subsets Δ of the vertices of P such that Δ has cardinality three but Δ is not the vertex set of a face of P. Therefore, $|\mathcal{D}| = \binom{f_0}{3} - f_2 = f_0(\frac{(f_0-1)(f_0-2)}{6} - (f_0 - 3))$. Since $|\{(e, \Delta)\colon e \text{ is an edge of } P,\ \Delta \in \mathcal{D},\ e \subset \mathcal{D}\}| = 3|\mathcal{D}|$, double counting yields that there exists an edge e of P that is contained in at least $\frac{3|\mathcal{D}|}{f_1}$ many elements $\Delta \in \mathcal{D}$. Therefore, any such Δ contains one vertex that is not in the star of e, and hence Lemma 10 yields

$$i \geqslant \frac{3|\mathcal{D}|}{f_1} = f_0 - 2 - 6\frac{f_0 - 3}{f_0 - 1} \geqslant f_0 - 8.$$

Thus, $|P \cap \mathbb{Z}^n| = f_0 \leqslant 8 + i < n + 4 + 3i$.

(A′) F_1, F_2 and s are Lawrence polytopes that have no common projection direction.

Without loss of generality let F_1 and s have two different projection directions. If s contains an edge of length at least 2, then this has to be a common projection direction with F_1, because s and F_1 are Lawrence prisms. But this is a contradiction. Hence, all lattice points in s are vertices. In particular, y has length one, so also all lattice points of P are vertices.

Since any of the two different projection directions of the Lawrence prism s maps a four-gon face onto the edge of an unimodular base simplex and two edges of the four-gon give the projection direction, we see that there is at most one four-gon face in s. Therefore, s contains at most $(n-2)+2=n$ lattice points.

Since F_k contains at most two vertices not in s for $k \in \{1,2\}$, we get $|(F_1 \cup F_2) \cap \mathbb{Z}^n| \leqslant n+4 < n+4+3i$. Therefore we may assume $z := |P \setminus (F_1 \cup F_2) \cap \mathbb{Z}^n| \neq 0$.

If $|((n-2)s)^\circ \cap \mathbb{Z}^n| = 0$, then we will proceed exactly like in case (ii) from (A). So let $j := |((n-2)s)^\circ \cap \mathbb{Z}^n| \geqslant 1$.

Because of Lemma 10, we get the estimation $z + j - 1 \leqslant i$, in particular $z \leqslant i$. Hence we can bound the number of lattice points of P:

$$\left|P \cap \mathbb{Z}^n\right| = \left|(F_1 \cup F_2) \cap \mathbb{Z}^n\right| + z \leqslant n+4+i < 3i+n+4.$$

(B) F_1 is a Lawrence polytope with the heights $h_1 \geqslant h_2 \geqslant \cdots \geqslant h_{n-1}$, $F_2 \cong \Pi^{(n-3)}(2\Delta_2)$.
Here

$$s = \operatorname{conv}(0, h_1 e_1, e_l,\ 2 \leqslant l \leqslant n-2)$$

and $h_1 = 2$, $h_2 = \cdots = h_{n-2} = 0$, because s is contained in the simplex F_2. If $z = |P \setminus \{F_1 \cup F_2\} \cap \mathbb{Z}^n| = 0$, then

$$\begin{aligned}\left|P \cap \mathbb{Z}^n\right| &= \left|F_2 \cap \mathbb{Z}^n\right| + \left|F_1 \setminus F_2 \cap \mathbb{Z}^n\right| = 6 + (n-3) + h_{n-1} + 1\\ &\overset{h_{n-1} \leqslant h_1 = 2}{\leqslant} 4 + n + 2 < 3i + n + 4.\end{aligned}$$

Otherwise if $z \geqslant 1$, we obtain just like in (A) $0 < z + (h_1 - 1) - 1 \leqslant i$. Therefore

$$\begin{aligned}\left|P \cap \mathbb{Z}^n\right| &= \left|(F_1 \cup F_2) \cap \mathbb{Z}^n\right| + z = \left|s \cap \mathbb{Z}^n\right| + (h_{n-1}+1) + 3 + z\\ &= h_1 + (n-2) + (h_{n-1}+1) + 3 + z \leqslant i + 4 + h_{n-1} + n\\ &\overset{h_{n-1} \leqslant h_1 = 2}{\leqslant} 3i + n + 4.\end{aligned}$$

(C) $F_1 \cong F_2 \cong \Pi^{(n-3)}(2\Delta_2)$.
Here either s is a pyramid over $2\Delta_1$ or $s \cong \Pi^{(n-4)}(2\Delta_2)$. Again $h_1 = 2$.
If $z = |P \setminus \{F_1 \cup F_2\} \cap \mathbb{Z}^n| = 0$, then

$$\left|P \cap \mathbb{Z}^n\right| = \left|F_2 \cap \mathbb{Z}^n\right| + \left|F_1 \setminus F_2 \cap \mathbb{Z}^n\right| \leqslant 6 + (n-3) + 3 < 3i + n + 4.$$

Otherwise if $z \geqslant 1$, we obtain $z \leqslant i$ because of $|((n-2)s)^\circ \cap \mathbb{Z}^n| \geqslant 1$ and Lemma 10. So as a result

$$\begin{aligned}\left|P \cap \mathbb{Z}^n\right| &= \left|F_1 \cap \mathbb{Z}^n\right| + \left|F_2 \setminus F_1 \cap \mathbb{Z}^n\right| + z \leqslant (6 + n - 3) + 3 + z = n + z + 6\\ &\leqslant n + i + 6 \leqslant n + 3i + 4.\end{aligned}$$

This completes the proof. □

Remark 11. In [12], Stanley shows that the coefficients of h_P^* also appear in the polynomial $(1-t)^{n+1} \sum_{k \geqslant 0} |(kP)^\circ \cap \mathbb{Z}^n| t^k \in \mathbb{Z}[t]$. So we can also compute the coefficients of h_P^* in a different way than in Lemma 9. Then it is easy to show that the bounds of Theorem 2 are also equivalent to the following estimations:

$$\begin{aligned}&\left|(nP)^\circ \cap \mathbb{Z}^n\right| \leqslant (n+4)i + 3,\\ &\left|2P \cap \mathbb{Z}^n\right| \leqslant (4+3n)(i+1) + \frac{n(n+3)}{2}.\end{aligned}$$

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