Completeness of Products of Solutions and Some Inverse Problems for PDE

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In this paper we give conditions which guarantee that products of solutions of partial differential equations \( Pu + au = 0 \) are complete in \( L^2(Q) \). Here \( P \) is a linear partial differential operator with constant coefficients and \( a \) is a function in \( L^1(Q) \). We check these conditions for elliptic, parabolic, and hyperbolic equations of second order and give applications to inverse problems for related equations where \( a \) is to be found. © 1991 Academic Press, Inc.

INTRODUCTION

Recently, J. Sylvester and G. Uhlmann [11] proved uniqueness of so-called conductivity coefficient of a second order elliptic equation with the given Neumann-to Dirichlet map. In fact they reduced this inverse problem to the problem of completeness of products of solutions of the Schrödinger equation and they proved completeness by constructing perturbed complex exponential solutions suggested by A. Calderon in 1980. First they considered smooth coefficients and then they and A. Nachman [9] reduced the regularity assumption to \( C^{1,1} \). The problem of recovering a conductivity from many boundary measurements was first considered by R. Kohn and M. Vogelius [5] in the analytic case. G. Alessandrini [1] and Sylvester and Uhlmann [12] also gave stability estimates and in the paper of Nachman [8] constructive algorithms and applications to the inverse problem of scattering theory were obtained. The author [4] obtained uniqueness results for piecewise smooth conductivity coefficient \( 1 + \chi(D) b \) where \( \chi(D) \) is the characteristic function of an unknown domain \( D \) and \( b \) is an unknown function. In the Sylvester–Uhlmann proof a study of an elliptic differential operator with constant coefficients and a (large) complex parameter is crucial. In this paper we suggest the use of well-known regular fundamental solutions for such equations; it simplifies proofs and
enables us to consider higher order elliptic equations and second order parabolic and hyperbolic equations with lower order terms with variable coefficients. We give related applications to identification problems for such equations when a natural analog of the Neumann-to-Dirichlet map is given.

Notation:

\( \alpha = (\alpha_1, ..., \alpha_n) \) is a multiindex with natural components,

\( \partial_\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}, \)

\( D^\alpha = (-i)^{\left|\alpha\right|} \partial^\alpha, \)

\( \xi = (\xi_1, ..., \xi_n), \xi_\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}, \)

\( P(D) \) is a linear differential operator with constant coefficients and \( P(\xi) \) is its symbol,

\( P^{(\alpha)}(\xi) = \partial^\alpha \xi P(\xi), \bar{P}(\xi) = (\Sigma_\alpha |P^{(\alpha)}(\xi)|^2)^{1/2}, \)

\( \|u\|_p(\Omega) = (\int_\Omega |u|^p)^{1/p}. \)

**Completeness of Products of Solutions**

Consider the partial differential equation

\[ P_j u_j + a_j u_j = 0 \quad \text{in} \quad \Omega, j = 1, 2, \quad (1.1) \]

where \( P_j \) is a linear partial differential operator with constant coefficients, \( a_j \) is a function from \( L_\infty(\Omega) \), and \( \Omega \) is an bounded open set in \( \mathbb{R}^n \). We extend \( a_j \) to the complement of \( \Omega \) as zero.

Let \( \Sigma_0 \) be a non-empty open subset of \( \mathbb{R}^n \).

**Theorem 1.1.** Suppose the following conditions are satisfied.

For any \( \xi(0) \) from \( \Sigma_0 \) and for any positive number \( R \) there are solutions \( \zeta(j) \) to the equations \( P_j(\zeta) = 0 \) such that

\[ \zeta(1) + \zeta(2) = \xi(0) \text{ and } |\zeta(j)| > R. \quad (1.2) \]

There is positive number \( C \) such that for \( \zeta(j) \) satisfying the condition (1.2) we have

\[ \left(1/C |\zeta(j)|\right) \leq \bar{P}_j(\xi + \zeta(j)) \quad \text{for all } \xi \text{ from } \mathbb{R}^n. \quad (1.3) \]

Then the set

\[ \text{span}\{u_1 u_2 : u_j \text{ is a } L_2\text{-solution to Eq. (1.1) near } \bar{\Omega}\} \]

is dense in \( L_1(\Omega) \).

In the proof we use the following results.
Theorem 1.2. Let $P$ be a linear differential operator of order $m$ in $\mathbb{R}^n$ and let $\Omega$ be a bounded open set in $\mathbb{R}^n$. Then there is a bounded linear operator $E$ in $L_2(\Omega)$ such that

$$PEf = f \quad \text{for all } f \text{ from } L_2(\Omega) \quad (1.4)$$

and for any differential operator $Q$ with constant coefficients we have

$$\|Q(D) Ef\|_2(\Omega) \leq C(\sup |\tilde{Q}(\xi)/\tilde{P}(\xi)|) \|f\|_2(\Omega) \quad (1.5)$$

where $C$ depends only on $m, n$ and on $\Omega$ and $\sup$ is over $\xi \in \mathbb{R}^n$.

This result is similar to Theorem 10.3.7 from Hörmander's book [3]; however, it does not follow directly from Theorem 10.3.7, so we deduce it from other results of [3]. Such $P$-independent estimates have been used by L. Nirenberg [10] while studying uniqueness in the Cauchy problem for general differential equations.

Note that in [3] the following norm is used

$$\|u\|_{p,h} = ((2\pi)^{-n} \int |k(\xi) \hat{u}(\xi)|^p d\xi)^{1/p},$$

where $k$ is a certain weight function and $\hat{u}(\xi)$ is the Fourier transform of $u$. For $p = \infty$ this norm is defined in a standard way.

Proof of Theorem 1.2. From Theorem 10.3.7 of [3] and from its proof it follows that for $P$ there is a linear operator $E$ with the property (1.4) such that the left side in (1.5) is bounded by

$$\|F_0\|_{\infty, p} \sup |\tilde{Q}(\xi)/\tilde{P}(\xi)| \|f\|_2(\Omega)$$

with $F_0 = \psi E_0$ where $\psi$ belongs to $C_c^\infty(\mathbb{R}^n)$ and depends only on the diameter of $\Omega$ and $E_0$ is a regular fundamental solution of $P$ described in [3, formula (7.3.22)]. From the proof of Theorem 10.2.1 in [3] we obtain that $\|E_0/g\|_{\infty, p} \leq C_1$ where $g(x) = \exp(|x|) + \exp(-|x|)$ and $C_1$ depends only on $m$ and $n$. We have

$$\|\psi E_0\|_{\infty, p} = \|(\psi g)(E_0/g)\|_{\infty, p} \leq C \|E_0/g\|_{\infty, p}$$

by Theorem 10.1.15 from [3] where $M(\xi)$ is $\sup |\tilde{P}(\xi + \eta)/\tilde{P}(\eta)|$ over $\eta \in \mathbb{R}^n$. Due to the inequality (10.1.8) of [3] we have $M(\xi) \leq (1 + C|\xi|)^m$ with $C$ depending only on $m$ and $n$. Summing up we get the estimate (1.5).

The proof is complete.
THEOREM 1.3. Suppose the conditions (1.2), (1.3) are satisfied; then for any \( \xi(0) \in \Sigma_0 \) and for \( \xi(j) \) from these conditions there are solutions \( u_j \) to Eq. (1.1) near \( \Omega \) such that

\[
u_j(x) = e^{i\xi(j)x}(1 + w_j(x; \xi(j))), \tag{1.6}\]

where \( \|w_j\|_2(\Omega) \) tends to zero as \( |\xi(j)| \) goes to infinity.

Proof. From Leibniz' formula we have

\[
P(D)(uv) = \Sigma_{a} P^{(a)}(D) uD^a v/\alpha!. \]

Letting \( u = \exp(i\zeta \cdot x), v = 1 + w_j \) and using the relations \( P^{(a)}(D) u = P^{(a)}(\zeta) \exp(i\zeta \cdot x) \) we conclude that \( u_j \) of the form (1.6) is a solution to (1.1) if

\[
P_j(D + \zeta(j)) w_j = \Sigma_{a} P^{(a)}(\zeta(j)) D^a w_j/\alpha! = u_j(1 + w_j), \tag{1.7}\]

where we also used the relation \( P_j(\zeta(j)) = 0 \).

Let \( E \) be the operator from Theorem 1.2 for \( P = P_j(\cdot; \zeta) \). From the property (1.4) of \( E \) it follows that any solution \( w_j \) to the equation

\[
w_j = -E(a_j(1 + w_j)) \quad \text{in} \quad L_2(\Omega) \tag{1.8}\]

is a solution to Eq. (1.7). From the condition (1.3) and from the estimate (1.5) with \( Q(D) = |\xi(j)| \) we have

\[
\|Ef\|_2(\Omega) \leq C |\xi(j)|^{-1} \|f\|_2(\Omega). \tag{1.9}\]

In view of the condition (1.2) we can choose \( \xi(j) \) so that

\[
2C \|a_j\|_{\infty}(\Omega) < |\xi(j)|; \tag{1.10}\]

then the operator mapping \( w \) into \( -E(a_j(1 + w)) \) is a contraction in \( L_2(\Omega) \). From (1.9) and (1.10) it also follows that it maps the ball

\[
\{w : \|w\|_2(\Omega) \leq 2C \|a_j\|_{\infty}(\Omega)(\text{meas}_\Omega)^{1/2}/|\xi(j)|\}\}

into itself. So by Banach's Contraction Theorem there is a solution \( w_j \) to Eq. (1.8) in this ball which is a solution to Eq. (1.7). So the function \( u_j \) from (1.6) satisfies all the requirements of Theorem 1.3.

The proof is complete.

Proof of Theorem 1.1. Assume that linear combinations of products of solutions are not dense in \( L_1(\Omega) \); then there is a non-zero continuous linear
functional \( v \) on \( L_1(\Omega) \) such that \( v(u_1u_2) = 0 \) for all \( L_2(\Omega) \)-solutions \( u_j \) to Eq. (1.1) near the closure of \( \Omega \).

Let \( \xi(0) \) be from \( \Sigma_0 \) and let \( u_j \) be solutions of the form Theorem 1.3; then \( \zeta(1) + \zeta(2) = \xi(0) \) and by Theorem 1.2 the functions

\[
\begin{align*}
\Phi(x) &= e^{i\Phi(0) \cdot x}(1 + w_1(x; \zeta(1)) + w_2(x; \zeta(2)) \\
&\quad + w_1(x; \zeta(1)) w_2(x; \zeta(2)))
\end{align*}
\]

converge to \( \exp(i\xi(0) \cdot x) \) in \( L_1(\Omega) \) as \( |\xi(j)| \) goes to infinity. So \( v_x(\exp(i\xi(0) \cdot x)) = 0 \) for \( \xi(0) \) from \( \Sigma_0 \). The distribution \( v \) is compactly supported; therefore its Fourier transform is an analytic function. As shown, this function is zero on \( \Sigma_0 \), and so it is zero on \( \mathbb{R}^n \). By the Fourier inversion formula we conclude that \( v \) is zero, which is a contradiction.

The proof is complete.

2. Applications to Elliptic Equations

Consider the equations

\[
-\Delta u_j + a_j u_j = 0, \quad j = 1, 2
\]

which is Eq. (1.1) with \( P_j(\zeta) = P(\zeta) = \zeta \cdot \zeta \).

Check the conditions (1.2), (1.3) with \( \Sigma_0 = \mathbb{R}^n \) for \( 3 \leq n \).

Let \( \xi(0) \) belong to \( \mathbb{R}^n \). Due to the rotational invariancy we may assume \( \xi(0) = (\xi_1(0), 0, \ldots, 0) \). The vectors

\[
\begin{align*}
\zeta(1) &= (\xi_1(0)/2, i\xi_1(0)^2/4 + R^2)^{1/2}, R, 0, \ldots, 0) \\
\zeta(2) &= (\xi_1(0)/2, -i\xi_1(0)^2/4 + R^2)^{1/2}, -R, 0, \ldots, 0)
\end{align*}
\]

are solutions to the equation \( \xi \cdot \zeta = 0 \) with the absolute values greater than \( R \), so the condition (1.2) is satisfied. To check the condition (1.3) we observe that

\[
\begin{align*}
\overline{\Phi}^2(\xi + \zeta) &= |2\zeta_1 + 2\xi_1|^2 + \cdots + |2\zeta_n + 2\xi_n|^2 + 12 \\
&\geq 4(|\text{Im} \, \zeta_1|^2 + \cdots + |\text{Im} \, \zeta_n|^2) \geq |\zeta|^2
\end{align*}
\]

provided \( \zeta \cdot \xi = 0 \), for then \( |\text{Re} \, \zeta| = |\text{Im} \, \zeta| \).

So from Theorem 1.1 we obtain

**Corollary 2.1.** If \( 3 \leq n \) then

\[
\text{span}\{u_1u_2 : u_j \text{ is a solution to Eq. (2.1) near } \text{cl } \Omega\}
\]

is dense in \( L_1(\Omega) \).
This result was obtained in fact by Sylvester and Uhlmann in [12] for smooth \(a_j\), later Nachman et al. [9] observed that the proofs in [12] are valid for measurable and bounded coefficients \(a_j\).

Consider the fourth order equation

\[
\Delta^2 u_j + a_j u_j = 0 \tag{2.2}
\]

which is Eq. (1.1) with \(P_j(\zeta) = P(\zeta) = (\zeta \cdot \zeta)^2\).

Again check the conditions (1.2), (1.3).

The condition (1.2) is satisfied as above.

Check the condition (1.3). We have

\[
P(\xi + \zeta) = ((\xi + \zeta) \cdot (\xi + \zeta))^2 = (\xi \cdot \xi + 2\xi \cdot \zeta)^2
\]

\[
= (\xi \cdot \xi)^2 + 4(\xi \cdot \zeta)(\xi \cdot \xi) + 4(\xi \cdot \zeta)^2
\]

provided \(P(\zeta) = 0\). We have

\[
\frac{\partial^3 P(\xi + \zeta)}{\partial \xi^3} = 24(\xi_j + \zeta_j);
\]

therefore, as above,

\[
\Phi(\xi + \zeta)^3 \geq \sum |\frac{\partial^3 P(\xi + \zeta)}{\partial \xi^3}|^2
\]

\[
\geq 4\Sigma |\text{Im} \zeta_j|^2 \geq |\zeta|^2.
\]

**Corollary 2.2.** If \(3 \leq n\) then the set

\[
\text{span}\{ u_1, u_2 : u_j \text{ is a solution to Eq. (2.2) near } \partial \Omega \}
\]

is dense in \(L^1(\Omega)\).

We give an application to an inverse problem for Eq. (2.2).

Suppose the boundary of \(\Omega\) is of class \(C^4\). It is well known (see, e.g., Morrey [7, p. 255, Theorem 6.5.4]) that for any functions \(g_k\) from the Sobolev space \(H^{(3/2) - 4}(\partial \Omega)\) there is a unique solution \(u_j \in H^{(4)}(\Omega)\) to Eq. (2.2) satisfying the boundary data

\[
u_j = g_0, \quad \frac{\partial u_j}{\partial N} = g_1, \quad \text{on } \partial \Omega \tag{2.3}
\]

provided zero is not an eigenvalue of this problem. That is the case if \(0 < a_j\).

**Theorem 2.3.** Let \(3 \leq n\). If for any \(g_0, g_1\) mentioned solutions \(u_j\) to the boundary value problem (2.2), (2.3) satisfy the conditions

\[
\frac{\partial^3 u_1}{\partial N^3} = \frac{\partial^2 u_2}{\partial N^2}, \quad \frac{\partial^3 u_1}{\partial N^3} = \frac{\partial^3 u_2}{\partial N^3} \quad \text{on } \partial \Omega \tag{2.4}
\]

then \(a_1 = a_2\).
Proof. Let \( u_1 \) be any solution to Eq. (2.2) near \( \partial \Omega \); then \( u_1 \) belongs to \( H^{(4)}(\Omega) \). Let \( g_0, g_1 \) be \( u_1 \) and \( \partial u_1/\partial N \) on \( \partial \Omega \) and let \( u_2 \) be a solution to the boundary value problem (2.2), (2.3) with \( j = 2 \). Subtracting Eqs. (2.2) with \( j = 1 \) from Eqs. (2.2) with \( j = 2 \), using the boundary conditions (2.3), (2.4), and letting \( u = u_2 - u_1, a = a_1 - a_2 \) we get

\[
\begin{align*}
A^2 u + a_2 u &= a u_1 \quad \text{in } \Omega \\
\partial^k u/\partial N^k &= 0 \quad \text{on } \partial \Omega, k = 0, \ldots, 3.
\end{align*}
\]

Let \( v_2 \) be any solution to Eq. (2.2) with \( j = 2 \) near \( \partial \Omega \). Multiplying Eq. (2.5) by \( v_2 \) and applying Green's formula twice we conclude that

\[
\int_{\Omega} u_1 v_2 = 0
\]

for any solutions \( u_1, v_2 \) to Eqs. (2.2) near \( \partial \Omega \). From Corollary 2.2 it follows that \( a = 0 \), so \( a_1 = a_2 \).

This proof is complete.

3. PARABOLIC EQUATIONS

Consider the parabolic partial differential equation

\[
\frac{\partial u_j}{\partial t} - A u_j + a_j u_j = 0, \quad j = 1, 2
\]

with the coefficient \( a_j \) from \( L_\infty(\Omega) \) where \( \Omega \) is a bounded open set in \( \mathbb{R}^{n+1} \), \( t = x_{n+1} \). For \( \xi = (\xi_1, \ldots, \xi_{n+1}) \), we denote \( \xi' = (\xi_1, \ldots, \xi_n, 0) \). In this case we have \( P_1(\xi') = P_2(\xi') = P(\xi) = \xi \cdot \xi' + i \xi_{n+1} \).

Let \( 2 \leq n \). Check the condition (1.3).

Let \( \xi(0) \in \mathbb{R}^{n+1} \). Due to the \( \xi' \)-rotational invariancy of \( P \) we may assume \( \xi(0) = (\xi_1(0), 0, \ldots, 0, \xi_{n+1}(0)) \). For large \( R \) the equations

\[
2\eta^2_1 - 2\xi_1(0) \eta_1 + \xi_1^2(0) = 2R^2, \quad 16\eta^4_2 - 16R^2\eta^2_2 - \xi_{n+1}^2(0) = 0
\]

have positive solutions \( \eta_1, \eta_2 \); moreover the absolute values of these solutions are between \( R/2 \) and \( R \). The vectors

\[
\begin{align*}
\xi(1) &= (\eta_1, -\xi_{n+1}(0)/(4)\eta_2 + i\eta_2, 0, \ldots, 0, \xi_{n+1}(0)/2 \\
&\quad + i(-\xi_1^2(0)/2 + \xi_1(0) \eta_1)) \\
\xi(2) &= (\xi_1(0) - \eta_1, \xi_{n+1}(0)/(4)\eta_2 - i\eta_2, 0, \ldots, 0, \xi_{n+1}(0)/2 \\
&\quad + i(\xi_1^2(0)/2 - \xi_1(0) \eta_1))
\end{align*}
\]

satisfy the characteristic equation \( P(\xi) = 0 \) and their absolute values are between \( R/2 \) and \( CR \) for large \( R \) where \( C \) depends only on \( \xi(0) \).
Check the condition (1.3). We have
\[ P(\xi + \zeta) = 2\zeta_1 \xi_1 + \cdots + 2\zeta_n \xi_n + i\xi_{n+1} + \xi_1^2 + \cdots + \xi_n^2 \]
provided \( P(\zeta) = 0 \). So
\[ P^2(\xi + \zeta) \geq |\partial P(\xi + \zeta)/\partial \xi_2|^2 \geq 4|\text{Im} \ \zeta_2|^2 > R^2 > |\zeta|^2/C \]
for \( \zeta \) which are \( \zeta(j) \) found above as roots of the polynomial \( P(\zeta) \).
So from Theorem 1.1 follows

**Corollary 3.1.** If \( 2 \leq n \) then the set
\[ \text{span}\{u_1 u_2 : u_j \text{ is a solution to Eq. (3.1) near } \partial \Omega \} \]
is dense in \( L_1(\Omega) \).

While studying coefficients identification problems for differential equation we however need a different family of solutions; that is why we consider the equations
\[ au_1/\partial t - \Delta u_1 + a_1 u_1 = 0, \quad -\partial v_2/\partial t - \Delta v_2 + a_2 v_2 = 0 \tag{3.2} \]
with the coefficients \( a_j \). We have
\[ P_1(\zeta) = \zeta \cdot \zeta' + i\xi_{n+1}, \quad P_2(\zeta) = \zeta \cdot \zeta' - i\xi_{n+1}. \]
We let \( \Sigma_0 = \{ \zeta(0) : |\zeta'(0)| \neq 0 \} \).

Check the condition (1.2). As above, we may assume \( \zeta(0) = (\zeta_1(0), 0, \ldots, 0, \xi_{n+1}(0)) \) with \( \xi_1(0) \neq 0 \) due to the choice of \( \Sigma_0 \). The equations \( P_j(\zeta(j)) = 0 \) for vectors
\[ \zeta(1) = (\zeta_1, \zeta_2, 0, \ldots, 0, \zeta_{n+1}), \]
\[ \zeta(2) = (\zeta_1(0) - \zeta_1, -\zeta_2, 0, \ldots, 0, \xi_{n+1}(0) - \zeta_{n+1}) \]
are equivalent to the equations
\[ \zeta_1^2 + \zeta_2^2 + i\xi_{n+1} = 0, \quad \xi_1(0) - 2\zeta_1(0) \zeta_2 - i\xi_{n+1}(0) = 0 \]
which have the solutions
\[ \zeta_1 = \xi_{n+1}(0)/2 - i\xi_{n+1}(0)/(2\xi_1(0)), \quad \zeta_2 = iR, \]
\[ \zeta_{n+1} = \xi_{n+1}(0)/2 + i(\xi_1^2(0)/4 - \zeta_{n+1}(0)/(2\xi_1(0))^2 - R^2) \]
with \( |\text{Im} \ \zeta| \geq R \).

The condition (1.3) is checked as above.
So again from Theorem 1.1 we have
COROLLARY 3.2. Let \( 2 \leq n \); then the set
\[
\text{span}\{u_1, v_2 : u_1, v_2 \text{ are solutions to Eqs. (3.2) near } \partial \Omega \}
\]
is dense in \( L_1(\Omega) \).

We give an application to inverse problems. Suppose \( \Omega = \Omega' \times (0, T) \) where \( \Omega' \) is a bounded domain in \( \mathbb{R}^n \) with the \( C^2 \)-boundary. It is well known (see Friedman [2, in particular, p. 319]) that for any functions \( g \) from \( C(\partial \Omega' \times [0, T]) \) and \( h \) from \( C(\partial \Omega' \times \{0\}) \) there is a unique solution \( u_j \) to Eq. (3.1) in \( \Omega \) with \( u_j \) from \( C(\partial \Omega), \partial u_j / \partial x_k \)
from \( C(\partial \Omega' \times (0, T)), k = 1, ..., n \), satisfying the following initial and boundary value data
\[
\begin{align*}
    u_j &= h & \text{on } \Omega' \times \{0\} & \quad (3.3) \\
    \partial u_j / \partial N &= g & \text{on } \partial \Omega' \times (0, T). & \quad (3.4)
\end{align*}
\]

THEOREM 3.3. Let \( 3 \leq n \). If for any \( g, h \) satisfying the condition above solutions \( u_1, u_2 \) to the initial boundary value problem (3.1), (3.3), (3.4) satisfy the condition
\[
\text{then } u_1 = u_2 \text{ on } \Omega.
\]

**Proof.** Let \( u_1 \) be a solution to Eq. (3.1), \( j = 1 \), near \( \partial \Omega \); then \( g = \partial u_1 / \partial N \) belongs to \( C(\partial \Omega' \times [0, T]) \) and \( h = u_1 \) belongs to \( C(\partial \Omega' \times \{0\}) \). Let \( u_2 \) be a solution to the problem (3.1), \( j = 2 \), (3.3), (3.4). Substracting Eqs. (3.1), \( j = 1 \), from Eqs. (3.1), \( j = 2 \), using the boundary conditions (3.3), (3.4), and (3.5), and letting \( u = u_2 - u_1, a = a_1 - a_2 \) we get
\[
\begin{align*}
    \partial u / \partial t - \Delta u + a_2 u &= au_1 & \text{in } \Omega & \quad (3.6) \\
    u &= 0 & \text{on } \partial \Omega, & \quad \partial u / \partial N = 0 & \text{on } \partial \Omega' \times (0, T).
\end{align*}
\]

Let \( v_2 \) be a solution to Eq. (3.2) near \( \partial \Omega \). Multiplying the both sides of Eq. (3.6) by \( v_2 \), applying Green's formula to the domains \( \omega \times (\tau, T) \) with \( \text{cl } \omega \) in \( \Omega' \) and positive \( \tau \), and then approximating \( \Omega' \) by \( \omega \) and letting \( \tau \) tend to zero and using the boundary data (3.6) for \( u \) on \( \partial \Omega \) we obtain
\[
\int_{\Omega} au_1 v_2 = 0
\]
for any \( u_1, v_2 \) mentioned. From Corollary 2.2 it follows that \( a = 0 \), so \( a_1 = a_2 \).

The proof is complete.
4. Hyperbolic Equations

Consider the second order ultrahyperbolic equation

$$A_i u_j - A_j u_i + a_j u_j = 0$$  \hspace{1cm} (4.1)

with the coefficient $a_j$ from $L_\infty(\Omega)$ where $\Omega$ is a bounded open set in $\mathbb{R}^{n+m}$, $2 \leq n, 1 \leq m$. Here $i \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$ and $A_i, A_j$ are the related Laplace operators. Let $\zeta = (\zeta_1, \ldots, \zeta_n, \zeta_n+1, \ldots, \zeta_{n+m})$, $\zeta' = (\zeta_1, \ldots, \zeta_n, 0, \ldots, 0)$, and $\zeta'' = \zeta - \zeta'$. We have $P_1(\zeta) = P_2(\zeta) = P(\zeta) = \zeta \cdot \zeta' - \zeta \cdot \zeta''$.

Check the condition (1.2). Let $\Sigma_0$ be the open set $\{\zeta(0) \in \mathbb{R}^{n+m} : 0 < |\zeta'(0)| < |\zeta(0)|\}$. Let $\zeta(0)$ be from this set. Due to the $\zeta'$- and $\zeta''$-rotational invariancy of $P$ and $\Sigma_0$ we may assume $\zeta(0) = (\xi_1(0), 0, \ldots, 0, \xi_{n+1}(0), 0, \ldots, 0)$. Let $R$ be large positive and let

$$r = \frac{\xi_1(0)}{\xi_{n+1}(0)} R - \frac{\xi_{n+1}(0) - \xi_1(0)}{\xi_{n+1}(0)} R = R/C$$

The vectors

$$\zeta(1) = (R, i(R^2 - r^2)^{1/2}, 0, \ldots, 0, r, 0, \ldots, 0)$$
$$\zeta(2) = (\xi_1(0) - R, -i(R^2 - r^2)^{1/2}, 0, \ldots, 0, \xi_{n+1}(0) - r, 0, \ldots, 0)$$

are solutions to the characteristic equation $P(\zeta) = 0$ and satisfy the condition $|\zeta(j)| > R/C$ where $C$ depends only on $\zeta(0)$.

We now check the condition (1.3). If $P(\zeta) = 0$ we have

$$P(\zeta + \zeta') = 2(\xi_1 \xi_1 + \cdots + \xi_n \xi_n) - 2(\xi_{n+1} \xi_{n+1} + \cdots + \xi_{n+m} \xi_{n+m})$$
$$+ \xi_1^2 + \cdots + \xi_n^2 - \xi_{n+1}^2 - \cdots - \xi_{n+m}^2$$

So

$$\bar{P}^2(\zeta + \zeta(1)) \geq \Sigma |\partial P(\zeta + \zeta(1))/\partial \xi_j|^2$$
$$= 4\Sigma |\zeta_1(1) + \zeta_1|^2$$
$$\geq 4 |\text{Im} \zeta(1)|^2 \geq R^2/C_1 \geq |\zeta(1)|^2/C_2,$$

where $C_2, C_2$ depend only on $\zeta(0)$. The same is valid for $\zeta(2)$.

As above Theorem 1.1 implies the following

**Corollary 4.1.** If $2 \leq n, 1 \leq m$ then the set

$$\text{span}\{u_1, u_2 : u_j \text{ is a } L_2(\Omega) \text{ solution to Eq. (4.1)}\}$$

is dense in $L_1(\Omega)$.

We give an application to an inverse problem for hyperbolic equations.
Let $m = 1$. Let $\Omega$ be the cylindrical domain from Section 3. Assume that the coefficients $a_j, j = 1, 2$, to Eq. (4.1) satisfy the conditions

$$a_j, \quad \partial a_j / \partial t \quad \text{belong to } L_\infty(\Omega).$$

It is well known (see, e.g., Ladyzenskaja [6]) that for any initial data $h_0 \in H^{(2)}(\Omega), h_1 \in H^{(1)}(\Omega)$ and for Neumann data $g_1 \in H^{(1)}(\partial \Omega' \times (0, T))$ there is an unique solution $u_j \in H^{(2)}(\Omega)$ to the initial boundary value problem

$$\begin{align*}
d^2 u_j / \partial t^2 - \Delta u_j + a_j u_j &= 0 \quad \text{on } \Omega \\
u_j &= h_0, \quad \partial u_j / \partial t = h_1 \quad \text{on } \Omega' \times \{0\} \\
\partial u_j / \partial N &= g_1 \quad \text{on } \partial \Omega' \times (0, T)
\end{align*}$$

provided the natural compatibility condition $\partial h_0 / \partial N = g_1$ on $\partial \Omega' \times \{0\}$ is satisfied.

**Theorem 4.2.** Let $2 \leq n$. If for any mentioned $h_0, h_1, g_1$ solutions $u_j$ to the initial boundary value problem (4.2), (4.3), (4.4) satisfy the conditions

$$u_1 = u_2 \quad \text{on } \partial \Omega, \quad \partial u_1 / \partial t = \partial u_2 / \partial t \quad \text{on } \Omega \times \{0\}$$

then $a_1 = a_2$ on $\Omega$.

The proof is similar to the proof of Theorem 3.3 if we use Corollary 4.1 instead of Corollary 3.2.

**References**


