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Spherical functions on the Grassmann manifold and generalized Jacobi polynomials – Part 2

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Abstract

Part 1 of this paper presented an explicit formula for generalized Jacobi polynomials of matrix argument. These polynomials constitute a complete system of orthogonal symmetric polynomials with respect to a multivariate beta measure. Zonal spherical functions on the Grassmann manifold may be expressed in terms of generalized Jacobi polynomials, and it is shown in Part 2 that they have an integral representation which generalizes the well-known integrals for Legendre and Gegenbauer polynomials of even order. In particular cases, this integral representation may be used to construct the zonal and associated spherical functions in terms of univariate special functions. © 1999 Elsevier Science Inc. All rights reserved.

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1. Introduction

In Part 1 of this paper, an explicit formula was given for the Jacobi polynomials $P_\kappa(X)$ defined by James and Constantine [11]. Here the argument

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X is a positive definite symmetric $m \times m$ matrix with latent roots lying between zero and unity, and the subscript κ ranges over the set $\mathbf{P}_m$ of ordered partitions

$$\kappa = (k_1, k_2, \ldots, k_m), \quad k_1 \geqslant k_2 \geqslant \cdots \geqslant k_m \geqslant 0, \quad \sum_{i=1}^{m} k_i = k,$$

of all nonnegative integers k into not more than m parts.

The $P_\kappa(X)$ constitute a complete system of orthogonal symmetric polynomials with respect to the beta measure

$$\begin{aligned} \beta(\mathrm{d}X) &= B \cdot |X|^{a-(1/2)(m+1)} |I_m - X|^{c-a-(1/2)(m+1)} (\mathrm{d}X), \\ &\mathrm{Re}(a) > \tfrac{1}{2}(m-1), \ \mathrm{Re}(c-a) > \tfrac{1}{2}(m-1), \ 0 < X < I_m, \end{aligned} \tag{1.1}$$

where I_m denotes the $m \times m$ unit matrix and

$$B = \Gamma_m(c)/\Gamma_m(a)\Gamma_m(c-a),$$

$\Gamma_m(a)$ being the multivariate gamma function. The explicit formula (Part 1, Eq. (3.1)) for $P_\kappa(X)$ was expressed in terms of the author's [4] invariant polynomials in two matrix arguments, but may be expanded in terms of zonal polynomials $C_\lambda(X)$, $\lambda \in \mathbf{P}_{\ell,m}$, $\ell \leqslant k$, where $\mathbf{P}_{\ell,m}$ denotes the set of ordered partitions of the nonnegative integer ℓ into not more than m parts. We refer the reader to Part 1, Section 2 for basic results on zonal and invariant polynomials.

The James Constantine [11] definition was motivated by the fact that when $a = \frac{1}{2}m$, $c = \frac{1}{2}n$ the $P_\kappa(X)$ provide the zonal spherical functions on the Grassmann manifold $G_{m,n-m}$. This manifold consists of the m-dimensional (non-oriented) planes $\mathbf{p}$ passing through the origin in real euclidean n-space $\mathbb{R}^n$. Part 2 of the present paper is concerned with the zonal spherical functions on $G_{m,n-m}$. Section 2 presents an integral formula (Theorem 1) for these functions which generalizes the well-known integrals for Legendre and Gegenbauer polynomials of even order. This result, together with a useful factorization $H \in O(n)$, where $O(n)$ is the orthogonal group of orthogonal $n \times n$ matrices, is applied in Section 4 to construct zonal and associated spherical functions on $G_{m,n-m}$ in terms of univariate special functions, for some small values of m and n. The factorization is based on a construction in Ref. [9]. The associated spherical functions may be regarded as generalizations of the spherical harmonics.

Basic properties of the $P_\kappa(X)$ required in Part 2 are as follows. The orthogonality property was derived in Part 1, Theorem 1:

$$\begin{aligned} &\int_{0<X<I_m} P_\kappa(X) P_\lambda(X) \beta(\mathrm{d}X) = 0 \quad \text{if } \kappa \neq \lambda, \\ &\int_{0<X<I_m} P_\kappa(X) P_\lambda(X) \beta(\mathrm{d}X) = (a)_\kappa (c-a)_\kappa |h_\kappa| \mathscr{Y}_\kappa \quad \text{if } \kappa = \lambda. \end{aligned} \tag{1.2}$$

where b_κ is the coefficient of $C_\kappa(X)$ in $P_\kappa(X)$ (Part 1 Eq. (3.4)), and

$$\mathscr{Y}_\kappa = \sum_{\phi \in \kappa\cdot\kappa} \gamma_{\kappa,\phi} C_\phi(I_m)/(c)_\phi, \tag{1.3}$$

where $\gamma_{\kappa,\phi}$ is the coefficient of $C_\phi(A)$ in the expansion of $C_\kappa(A^2)$. The generalized hypergeometric coefficient $(a)_\kappa$ and notation $\phi \in \kappa\cdot\kappa$ are defined in Part 1, Section 2. Explicit formulas were also derived for b_κ and $\mathscr{Y}_\kappa$ in the case $m = 2$ (Part 1, Eqs. (3.8) and (3.12), respectively).

If $\hat{P}_\kappa(X)$ denotes the generalized Jacobi polynomial normalized to value unity at $X = I_m$,

$$\hat{P}_\kappa(X) = P_\kappa(X)/P_\kappa(I_m), \quad \text{where } P_\kappa(I_m) = \frac{(-1)^k}{k!}(c-a)_\kappa C_\kappa(I_m),$$

then the $\hat{P}_\kappa(X)$ have the generating function (Part 1, Eq. (3.15))

$$\sum_{k=0}^{\infty} \sum_{\kappa\in\mathbf{P}_{k,m}} \frac{C_\kappa(A)\hat{P}_\kappa(X)}{k!(a)_\kappa} = \int_{O(m)} {}_0F_1(a; AHXH') \times {}_0F_1(c-a; -AH(I_m - X)H')(\mathrm{d}H). \tag{1.4}$$

Eq. (1.4) generalizes Bateman's [2] generating function for the univariate Jacobi polynomials. Here $(\mathrm{d}H)$ denotes the invariant measure on $O(m)$, normalized to have unit integral over the group. ${}_0F_1$ is Herz's [8] Bessel function of matrix argument, for which Constantine [3] derived the expansion

$${}_0F_1(a; X) = \sum_{k=0}^{\infty} \sum_{\kappa\in\mathbf{P}_{k,m}} C_\kappa(X)/k!(a)_\kappa. \tag{1.5}$$

Herz showed that

$${}_0F_1\left(\frac{1}{2}n; \frac{1}{4}ZZ'\right) = \int_{\mathscr{V}_{m,n}} \exp(\operatorname{tr} ZK_1)(\mathrm{d}K_1), \tag{1.6}$$

where Z is $m \times n$, K_1 is $n \times m$, $K_1'K_1 = I_m$ and $(\mathrm{d}K_1)$ denotes the invariant measure on the Stiefel manifold $\mathscr{V}_{m,n}$, normalized to have unit integral over the manifold.

As noted in Part 1 (see Eq. (3.16)), Herz [8] raised the problem of constructing a complete system of generalized Jacobi polynomials, and considered the case $P_{(k_1^m)}$. With parameters $a = \frac{1}{2}m$, $c = \frac{1}{2}n$,

$$\hat{P}_{(k_1^m)}(X) = \frac{(-1)^k(\frac{1}{2}m)_{(k_1^m)}}{(\frac{1}{2}(n-m))_{(k_1^m)}} \, {}_2F_1(-k_1, \tfrac{1}{2}(n-m-1)+k_1; \tfrac{1}{2}m; X) \tag{1.7}$$

is Herz's generalized Gegenbauer polynomial $C_{2k_1}^{(1/2)(n-m-1)}$ of even order. However, apart from the integral representation (Theorem 1 of the present paper), Herz's results for this case do not appear to generalize to all $\hat{P}_\kappa$. In this connection we may note that if T is $n \times m$ then $C_\kappa(U'TT'U)$ is a harmonic function of T for all $\kappa \in \mathbf{P}_m$ and all complex $n \times m$ U such that $U'U = O$ (see Appendix A). However, only $C_{(k_1^m)}(U'TT'U) \propto |U'T|^{2k_1}$ satisfies Herz's definition of a H-polynomial, which is basic to his approach.

2. Integral representation of the zonal spherical functions

2.1. As mentioned in Part 1, Section 1, $G_{m,n-m}$ is isomorphic to the coset space $O(n)/(O(m) \times O(n-m))$. The orthogonal group $O(n)$ induces a transitive group of transformations $\mathbf{p} \to \mathbf{p}H$ in $G_{m,n-m}$, where $H \in O(n)$; also, the plane π_0 spanned by the first m coordinate axes in $\mathbb{R}^n$ is invariant under the subgroup $O(m) \times O(n-m)$. If $\mathbf{V}$ denotes the linear space of functions on $G_{m,n-m}$ which are square-integrable with respect to the invariant measure $\mathrm{d}\mathbf{p}$ on $G_{m,n-m}$ then, under the representation $T(H)$ of $O(n)$ in $\mathbf{V}$ defined by

$$[T(H)\zeta](\mathbf{p}) = \zeta(\mathbf{p}H) \quad \text{where } \zeta \in \mathbf{V},\ H \in O(n).$$

$\mathbf{V}$ decomposes when $m \leqslant \frac{1}{2}n$ into a direct sum of invariant subspaces

$$\mathbf{V} = \bigoplus_{\kappa \in \mathbf{P}_m} \mathbf{V}_\kappa,$$

where $\mathbf{V}_\kappa$ carries the irreducible representation (2κ) of $O(n)$. Each $\mathbf{V}_\kappa$ contains a one-dimensional linear subspace of functions which are invariant under $O(m) \times O(n-m)$, generated by the zonal spherical function

$$\zeta_\kappa(\mathbf{p}) = \hat{P}_\kappa(X),$$

where $\hat{P}_\kappa$ is the generalized Jacobi polynomial with parameters

$$a = \tfrac{1}{2}m, \quad c = \tfrac{1}{2}n, \tag{2.1}$$

standardized to the value unity at $X = I_m$. The matrix X is a positive definite symmetric matrix whose latent roots are the squares of the cosines of the critical angles between $\mathbf{p}$ and π_0.

More generally, consider a second Grassmann manifold $G_{q,n-q}$, where $m \leqslant q \leqslant \frac{1}{2}n$, and let $\mathbf{V}^1 = \oplus_{\kappa \in \mathbf{P}_q} \mathbf{V}^1_\kappa$ denote the space of square-integrable functions on $G_{q,n-q}$, decomposed into irreducible invariant subspaces. Then James and Constantine [11] show that the intertwining operator which maps $\mathbf{V}_\kappa$ onto $\mathbf{V}^1_\kappa$, $\kappa \in \mathbf{P}_m$, may be expressed in terms of $P_\kappa(X)$ with parameters $a = \frac{1}{2}q$, $c = \frac{1}{2}n$ and X $m \times m$.

In the case of the zonal spherical functions, the evaluation of the right-hand side of Eq. (1.2) may be circumvented by use of the Peter–Weyl theorem (see Ref. [13], para 4.3 for example). This yields

$$\int_{0<X<I_m} \{\hat{P}_\kappa(X)\}^2 \beta(\mathrm{d}X) = 1/\mathscr{G}_n(\kappa). \tag{2.2}$$

where $\mathscr{G}_n(\kappa)$ is the dimension of the invariant subspace $\mathbf{V}_\kappa$. A convenient formula derived from results in Ref. [12], is

$$\mathscr{G}_n(\kappa) = 2^m \prod_{i<j}^{m} \{2(k_i - k_j) - i + j\}\{2(k_i + k_j) + n - i - j\}$$
$$\times \prod_{r=1}^{m}{}^{*} \{(2k_r + m - r + 1)_{n-2m-1}(2k_r + \tfrac{1}{2}n - r)\} / \prod_{s=1}^{m}(n - 2s)!. \tag{2.3}$$

where $\kappa \in \mathbf{P}_m$, $m \leqslant \frac{1}{2}n$, and for $n = 2m$

$\Pi^* = 1 \quad \text{if } k_m > 0.$

$\Pi^* = 1/2 \quad \text{if } k_m = 0.$

Unfortunately, we have been unable to prove that Eq. (1.2) reduces to Eq. (2.3) for all parameter values (2.1). However, Eq. (1.2) may be checked for $m = 2$ using the results for b_κ and $\mathscr{Y}_\kappa$ in Part 1, Eqs. (3.8) and (3.12). Particular cases will be indicated in Section 4.

2.2. We now derive an integral representation of the zonal spherical functions which generalizes the well-known integrals for the Legendre and Gegenbauer polynomials of even order. Consider the plane $\mathbf{p}$ spanned by the *rows* of the $m \times n$ matrix

$$\mathscr{H} = [\mathscr{H}_1, \mathscr{H}_2], \quad \mathscr{H}\mathscr{H}' = I_m. \tag{2.4}$$

where $\mathscr{H}_1$ is $m \times m$ and $\mathscr{H}_2$ is $m \times (n - m)$. If $\theta_1, \ldots, \theta_m$ are the critical angles between $\mathbf{p}$ and the plane π_0 spanned by the rows of the $m \times n$ matrix $[I_m, O]$, then

$$r_j^2 = \cos^2\theta_j, \quad j = 1, 2, \ldots, m$$

are the latent roots of

$$X = \mathscr{H}_1\mathscr{H}_1'. \tag{2.5}$$

Theorem 1. *Let* $\mathscr{H}$, X *be defined by Eqs.* (2.4) *and* (2.5), *respectively. If* $\hat{P}_\kappa(X)$ *has parameters* $a = \frac{1}{2}m$, $c = \frac{1}{2}n$, *where* $m \leqslant \frac{1}{2}n$, *then*

$$\hat{P}_\kappa(X) = \int_{\mathscr{V}_{m,n-m}} C_\kappa^*[(\mathscr{H}_1 + \mathrm{i}\mathscr{H}_2K_1)'(\mathscr{H}_1 + \mathrm{i}\mathscr{H}_2K_1)](\mathrm{d}K_1). \tag{2.6}$$

Here $\mathscr{V}_{m,n-m}$ *denotes the Stiefel manifold,* $K_1 \in \mathscr{V}_{m,n-m}$ *is* $(n - m) \times m$, $K_1'K_1 = I_m$, *and* $(\mathrm{d}K_1)$ *is the invariant measure on* $\mathscr{V}_{m,n-m}$.

Proof. Using Eq. (1.6) and the invariance of $(\mathrm{d}K_1)$ under $K_1 \to K_1 H'$, $H \in O(m)$, the right-hand side of Eq. (1.4) becomes

$$\int_{V_{m,n-m}} (\mathrm{d}K_1) \int_{O(m)} {}_0F_1(\tfrac{1}{2}m; (\mathscr{H}_1 + \mathrm{i}\mathscr{H}_2 K_1) HAH' (\mathscr{H}_1 + \mathrm{i}\mathscr{H}_2 K_1)')(\mathrm{d}H),$$

and Eq. (2.6) follows. □

Theorem 1 generalizes the even-order case of Herz's [8] Theorem 6.9 which gives the result $\hat{P}_{(k_1^m)}$, recalling that $C^*_{(k_1^m)}(X) = |X|^{k_1}$.

We may note that Eq. (2.6) may be written

$$\hat{P}_\kappa(X) = \int_{O(m)} (\mathrm{d}H) \int_{O(n-m)} C^*_\kappa\left[\mathscr{U}'\begin{bmatrix} H' & O \\ O & K' \end{bmatrix} E \begin{bmatrix} H & O \\ O & K \end{bmatrix} \mathscr{U}\right](\mathrm{d}K). \tag{2.7}$$

where $\mathscr{U}' = [I_m, \mathrm{i}I_m, O]$ is $m \times n$, and $E = \mathscr{H}'\mathscr{H}$ is the $n \times n$ symmetric idempotent matrix of the orthogonal projection of $\mathbb{R}^n$ onto the plane $\mathbf{p}$ spanned by the rows of $\mathscr{H}$. If we define

$$\chi_\kappa(\mathbf{p}) = C^*_\kappa(\mathscr{U}' E \mathscr{U}), \tag{2.8}$$

then Eq. (2.7) may be expressed in the form

$$\hat{P}_\kappa(X) = \int\int_{O(m)\times O(n-m)} \left\{T\left(\begin{bmatrix} H & O \\ O & K \end{bmatrix}\right)\chi_\kappa\right\}(\mathbf{p})(\mathrm{d}H)(\mathrm{d}K).$$

This suggests that

$$\chi_\kappa(\mathbf{p}) \in \mathbf{V}_\kappa. \tag{2.9}$$

To prove Eq. (2.9), we recall Weyl's result ([14], p. 58) that for $m \leqslant \frac{1}{2}n$ the representation $\langle 2\kappa \rangle$ of $\mathrm{GL}(n, \mathbb{R})$, when restricted to $O(n)$, decomposes into the representation (2κ) of $O(n)$, having multiplicity unity, together with lower order representation of $O(n)$. This implies that $\mathscr{V}_\kappa[E]$ decomposes under $O(n)$ into the direct sum of $\mathbf{V}_\kappa$ and certain $\mathbf{V}_\lambda$ such that $\lambda \in \mathbf{P}_{\ell,m}$, $\ell < k$. We have to show that $C_\kappa(\mathscr{U}' E \mathscr{U})$ has no components in the lower order spaces. From Eq. (2.11) of Part 1

$$\int_{O(n)} C_\kappa(\mathscr{U}' H' E H \mathscr{U}) C_\lambda(L' H' E H L)(\mathrm{d}H) = \sum_{\phi \in \kappa\cdot\lambda} C_\phi^{\kappa,\lambda}(A, LL') C_\phi^{\kappa,\lambda}(E, E) / C_\phi(I_m), \tag{2.10}$$

where $A = \mathscr{U}\mathscr{U}'$ and $L \in \mathrm{GL}(n, \mathbb{R})$. Since $\mathscr{U}'\mathscr{U} = O$ we have

$A^2 = O$ and $\operatorname{tr} A = 0$.

Hence for $\ell < k$ the basis elements (Part 1, Eq. (2.12)) of the $C_{\phi}^{\kappa,\ell}(A, LL')$ all vanish, and the integral in Eq. (2.10) is zero. But $C_\lambda(L'EL)$ spans $\mathscr{V}_\lambda[E]$ as L varies over $\mathrm{GL}(n, \mathbb{R})$; in particular it spans $\mathbf{V}_\lambda$, and so $C_\kappa(\mathscr{U}'E\mathscr{U})$ must have zero component in this space. The result (2.9) follows.

We may note that Eq. (2.9) is an extension of the result ([7], p. 17) that $(u'h)^k$, where u and h are n-vectors, $u'u = 0$ and $h'h = 1$, is an eigenfunction of the Laplace–Beltrami operator on the unit sphere $\mathscr{S}^{n-1}$ in $\mathbb{R}^n$.

3. A factorization of orthogonal matrices

To apply Eqs. (2.6) and (2.9) to the construction of zonal and associated spherical functions on $G_{m,n-m}$, we require an approach to the parametrization of $O(n)$ which is suited to the form of these results. Such an approach is provided by

Theorem 2. *If $H \in O(n)$ and $m \leqslant \frac{1}{2}n$ then*

$$H = \begin{bmatrix} U & O \\ O & W \end{bmatrix} \begin{bmatrix} R & S & O \\ -S & R & O \\ O & O & I_{n-2m} \end{bmatrix} \begin{bmatrix} V & O \\ O & Z \end{bmatrix} = \begin{bmatrix} URV & USZ_1 \\ -W_1SV & W_1RZ_1 + W_2Z_2 \end{bmatrix}. \tag{3.1}$$

where $R = \operatorname{diag}(r_1, \ldots, r_m)$, $S = \operatorname{diag}(s_1, \ldots, s_m)$, $r_i = \cos\theta_i$, $s_i = \sin\theta_i$, $0 \leqslant \theta_m \leqslant \cdots \leqslant \theta_1 \leqslant \frac{1}{2}\pi$; $U \in O(m)$; $V \in O(m)$ *has nonnegative elements in column* 1; $W = [W_1, W_2] \in O(n-m)$, *where* W_1 is $(n-m) \times m$;

$$Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \in O(n-m),$$

where $Z_1' \in \mathscr{V}_{m,n-m}$; *if* $m < \frac{1}{2}n$ *then the elements of* Z_2 *are analytic functions of admissible coordinates for* Z_1, *and may be chosen so that* $Z \in \mathrm{SO}(n-m)$.

Here $\mathrm{SO}(n)$ denotes the special orthogonal group of $n \times n$ orthogonal matrices with determinant $+1$.

Professor James has pointed out to the author that Eq. (3.1) is equivalent to the construction in his paper ([9], para 7) for the case $k = q$, in the notation of that paper. This construction elucidates the structure of Eq. (3.1). We write

$$H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix},$$

where H_1 is $m \times n$ and H_2 is $(n-m) \times n$, and consider the planes $\mathbf{p}$ and $\mathbf{p}^\perp$ spanned by the rows of H_1 and H_2, respectively; also let π_0 and $\pi_0^\perp$ denote the planes spanned by the first m and last $n-m$ coordinate axes, respectively. Then

$$U'H_1 = R[V, O] + S[O, Z_1], \tag{3.2}$$
$$W_1'H_2 = -S[V, O] + R[O, Z_1], \tag{3.3}$$
$$W_2'H_2 = [O, Z_2]. \tag{3.4}$$

The orthogonal matrix U thus maps the row vectors of H_1 into the set of m orthonormal vectors in $\mathbf{p}$ which make the critical angles $\theta_1, \ldots, \theta_m$ with π_0. Eq. (3.2) expresses these vectors in terms of their projections $R[V, O]$ and $S[O, Z_1]$ onto π_0 and $\pi_0^\perp$, respectively. The right-hand side of Eq. (3.3) defines m orthonormal vectors in $\mathbf{p}^\perp$ in terms of R, S, V and Z_1; $W_1'H_2$ expresses these vectors in terms of the rows of H_2. If we now define $[O, Z_2]$ to be a further $n-2m$ orthonormal vectors in $\mathbf{p}^\perp$, orthogonal to $[O, Z_1]$ and analytic functions of the latter, then Eq. (3.4) expresses these vectors in terms of the rows of H_2.

An alternative approach to deriving Eq. (3.1) is to partition

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$$

and construct $H_{11} = URV$ by singular-value decomposition. Z_1 and W_1 are then uniquely determined by $Z_1 = S^{-1}U'H_{12}$, $W_1 = -H_{21}V'S^{-1}$, and are found to satisfy $Z_1Z_1' = W_1'W_1 = I_m$. Finally, $W_2Z_2 = H_{22} - W_1RZ_1$, and if Z_2 is defined as above then W_2 is also determined. As required, Eq. (3.1) expresses H in terms of $\frac{1}{2}n(n-1)$ parameters.

The appropriateness of Eq. (3.1) for discussing functions on $G_{m,n-m}$ follows from the fact that it displays the decomposition of $O(n)$ with respect to the subgroup $O(m) \times O(n-m)$. If the plane $\mathbf{p}$ is spanned by the rows of H_1 then

$$E = H_1'H_1 = [RV, SZ_1]'[RV, SZ_1], \tag{3.5}$$

i.e., as indicated in Eq. (3.2) $\mathbf{p}$ is spanned by the rows of

$$\mathscr{H} = [RV, SZ_1] = [R, S, O]\begin{bmatrix} V & O \\ O & Z \end{bmatrix}. \tag{3.6}$$

$G_{m,n-m}$ is thus isomorphic to the set of right cosets of $O(n)$ with respect to $O(m) \times O(n-m)$ specified by R, V and Z_1, and may be parametrized in terms of admissible coordinates for these matrices, involving $m(n-m)$ parameters (see Ref. [11], p. 185).

The zonal spherical functions on $G_{m,n-m}$ are functions which are constants on the two-sided cosets of $O(n)$ with respect to $O(m) \times O(n-m)$, and are thus functions of R. In terms of Eq. (3.1), Eq. (2.6) becomes

$$\hat{P}_\kappa(R^2) = \int_{V_{m,n-m}} C^*_\kappa[(R + iSK_{11})'(R + iSK_{11})](dK_1), \tag{3.7}$$

where

$$K_1 = \begin{bmatrix} K_{11} \\ K_{12} \end{bmatrix}$$

and K_{11} is $m \times m$. From Eqs. (3.5) and (2.8)

$$\chi_\kappa(\mathbf{p}) = C^*_\kappa[(RV + iSZ_{11})'(RV + iSZ_{11})], \tag{3.8}$$

where $Z_1 = [Z_{11}, Z_{12}]$ and Z_{11} is $m \times m$. The integral

$$\hat{P}_\kappa(R^2) = \int_{V_{m,n-m}} \chi_\kappa(\mathbf{p})(dZ_1) \tag{3.9}$$

is equivalent to Eq. (3.7). The $\hat{P}_\kappa(R^2)$ are orthogonal with respect to the measure ([9], (7.13))

$$(d\underset{\sim}{\theta}) = c_{m,n} \prod_{i=1}^{m}(1 - r_i^2)^{(1/2)n-m} \prod_{i<j}^{m}(r_j^2 - r_i^2) \prod_{i=1}^{m} d\theta_i \tag{3.10}$$

where $0 \leqslant \theta_m \leqslant \cdots \leqslant \theta_1 \leqslant \frac{1}{2}\pi$, and

$$c_{m,n} = \frac{2^m \pi^{(1/2)m^2} \Gamma_m(\frac{1}{2}n)}{\{\Gamma_m(\frac{1}{2}m)\}^2 \Gamma_m(\frac{1}{2}(n-m))}.$$

Equation (3.2) may be written

$$\int \{\hat{P}_\kappa(R^2)\}^2 (d\underset{\sim}{\theta}) = 1/\mathscr{D}_n(\kappa). \tag{3.11}$$

The invariant measure $(d\mathbf{p})$ on $G_{m,n-m}$ is given ([9], (7.10)) by

$$(d\mathbf{p}) = (d\underset{\sim}{\theta})(dV)(dZ_1) \tag{3.12}$$

and the invariant measure on $O(n)$ may be written

$$(dH) = (d\mathbf{p})(dU)(dW).$$

This result may also be obtained directly from Eq. (3.1).

Eqs. (3.7)–(3.9) allow us to derive explicit expressions for the zonal spherical functions in terms of univariate special functions for some small m. The associated spherical functions, which together with $\hat{P}_\kappa(R^2)$ constitute a basis in $\mathbf{V}_\kappa$, may also be constructed in particular cases. The dimension $\mathscr{D}_n(\kappa)$ of $\mathbf{V}_\kappa$ has been given in Eq. (2.3). We may also note that, when restricted to SO(n), the representation (2κ) of $O(n)$ in $\mathbf{V}_\kappa$ remains irreducible except when $n = 2m$ and $k_m > 0$, in which case the representation decomposes into the direct sum of two equivalent irreducible representations (2κ) of $SO(n)$ (see Ref. [12]).

4. Examples

4.1. *The case* $m = 1$, $n = 3$. The results in this subsection are known ([6,13]), and are presented only to illustrate the methods of Sections 2 and 3, and introduce notation required later in this section. $G_{1,2}$ consists of all non-oriented straight lines $\mathbf{p}$ through the origin in $\mathbb{R}^3$. Under transformation by $O(3)$ the space of square-integrable functions on $G_{1,2}$ decomposes into the direct sum of irreducible invariant subspaces $\mathbf{V}_{(k)}$ carrying the irreducible representations $((2k))$ of $O(3)$, $k = 0, 1, 2, \ldots$ Since $m < \frac{1}{2}n$, Eq. (3.6) implies that $\mathbf{p}$ is spanned by the unit vector

$$\begin{aligned} \mathscr{H} &= (\cos\theta, \sin\theta, 0)\begin{bmatrix} 1 & 0 \\ 0 & \gamma(\psi) \end{bmatrix} \\ &= (\cos\theta, \sin\theta\cos\psi, \sin\theta\sin\psi), \quad 0 \leqslant \theta \leqslant \tfrac{1}{2}\pi, \end{aligned} \tag{4.1}$$

where

$$\gamma(\psi) = \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix}, \quad -\pi < \psi \leqslant \pi.$$

Hence by Eq. (3.8)

$$\chi_{(k)}(\mathbf{p}) = (\cos\theta + \mathrm{i}\sin\theta\cos\psi)^{2k}, \quad k = 0, 1, 2, \ldots.$$

and by Eq. (3.9) the zonal spherical function in $\mathbf{V}_{(k)}$ is

$$\hat{P}_{(k)}(\cos^2\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (\cos\theta + \mathrm{i}\sin\theta\cos\psi)^{2k}\, \mathrm{d}\psi = P_{2k}(\cos\theta),$$

the Legendre polynomial of order $2k$. In fact, the terminology "zonal spherical function" derives from the fact that the Legendre polynomial is constant on the "zones" θ = constant on the sphere $\mathscr{S}^2$ in $\mathbb{R}^3$. From Eq. (3.10) the $\hat{P}_{(k)}$ are orthogonal with respect to the measure $\sin\theta\, \mathrm{d}\theta, 0 \leqslant \theta \leqslant \frac{1}{2}\pi$, and the integral

$$\int_0^{(1/2)\pi} \{\hat{P}_{(k)}(\cos^2\theta)\}^2 \sin\theta\, \mathrm{d}\theta = \frac{1}{4k+1}.$$

where $\mathscr{D}_3((k)) = 4k + 1$ is the dimension of $\mathbf{V}_{(k)}$, is a well-known property of Legendre polynomials. This result also follows from Eq. (1.2) with $a = \frac{1}{2}$, $c = \frac{3}{2}$ and, from Eqs. (3.7) and (3.11) of Part 1,

$$b_{(k)} = (-1)^k \left(k + \frac{1}{2}\right)_k \Big/ k!, \quad \mathscr{Y}_{(k)} = 1 \Big/ \left(\frac{3}{2}\right)_{2k}.$$

We may note that $(-1)^k b_{(k)}$ is the coefficient of $\cos^{2k}\theta$ in $\hat{P}_{(k)}(\cos^2\theta) = P_{2k}(\cos\theta)$.

Next consider the construction of a basis for the $(4k + 1)$-dimensional space $\mathbf{V}_{(k)}$. If we denote $\hat{P}_{(k)}(\cos^2\theta)$ by $\hat{P}_{(k)}(\mathbf{p})$, then a basis may be constructed by expanding $\hat{P}_{(k)}(\mathbf{p}H)$ for $H \in O(3)$; for as H varies over $O(3)$, $\hat{P}_{(k)}(\mathbf{p}H)$ must span $\mathbf{V}_{(k)}$ or else generate a proper invariant subspace, which would contradict the irreducibility of $\mathbf{V}_{(k)}$. To parametrize $H \in O(3)$, we may note that $H = H_+ \operatorname{diag}(1, 1, \tilde{d})$ where $\tilde{d} = |H|$ and $H_+ \in SO(3)$ and write

$$H_+ = \begin{bmatrix} 1 & 0 \\ 0 & \gamma(\tilde{\phi}) \end{bmatrix} \begin{bmatrix} \gamma(\tilde{\theta}) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \gamma(\tilde{\psi}) \end{bmatrix}, \quad 0 \leqslant \tilde{\theta} \leqslant \pi, \quad -\pi < \tilde{\phi}, \tilde{\psi} \leqslant \pi. \tag{4.2}$$

This factorization expresses the result that a rotation in $\mathbb{R}^3$ may be resolved into successive rotations, for example, about the x, z, and x-axes; it may also be derived from Eq. (3.1). Then $\mathbf{p}^* = \mathbf{p}H$ is spanned by the row vector

$$\mathscr{H}^* = \mathscr{H} H = (\cos\theta^*, \sin\theta^* \cos\psi^*, \sin\theta^* \sin\psi^*),$$

where

$$\cos\theta^* = \cos\theta \cos\tilde{\theta} - \sin\theta \sin\tilde{\theta} \cos\phi^*, \tag{4.3}$$

$$\sin\theta^* \cdot e^{i\psi'} = \cos\theta \sin\tilde{\theta} + \sin\theta \cos\tilde{\theta} \cos\phi^* + i \sin\theta \sin\phi^*, \tag{4.4}$$

and

$$\phi^* = \psi + \tilde{\phi}, \quad \psi^* = \tilde{d}(\psi' + \tilde{\psi}).$$

These equations define θ^* and ψ^* uniquely in the range $0 \leqslant \theta^* \leqslant \pi$, $-\pi < \psi^* \leqslant \pi$. If $\cos\theta^* < 0$ then a parametrization of form (4.1) is obtained by reversing the sign of $\mathscr{H}^*$, but $\hat{P}_{(k)}(\mathbf{p}^*) = P_{2k}(\cos\theta^*)$ holds since the Legendre polynomial is of even order. We may expand $\hat{P}_{(k)}(\mathbf{p}^*)$ using the "generalized spherical functions" $P^\ell_{uv}(\cos\theta)$ of Gelfand and Shapiro (see Ref. [6]). A useful reference for these functions is Ref. [13], Chapter III. It is convenient for our purposes to define the P^ℓ_{uv}, $\ell = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots; u, v = -\ell, -\ell + 1, \ldots, \ell$, by the generating function ([13], III (5.1.3))

$$\left(\cos\frac{1}{2}\theta + \mathrm{i}w\sin\frac{1}{2}\theta\right)^{\ell+v}\left(w\cos\frac{1}{2}\theta + \mathrm{i}\sin\frac{1}{2}\theta\right)^{\ell-v}$$
$$= \sqrt{(\ell-v)!(\ell+v)!}\sum_{u=-\ell}^{\ell}\frac{w^{\ell-u}P^{\ell}_{uv}(\cos\theta)}{\sqrt{(\ell-u)!(\ell+u)!}}, \quad 0\leqslant\theta\leqslant\pi. \tag{4.5}$$

We shall be concerned only with integer values of ℓ, u, and v. Basic properties which may be derived from Eq. (4.5) include

$$P^{\ell}_{uv}(z) = P^{\ell}_{vu}(z) = P^{\ell}_{-u,-v}z; \quad P^{\ell}_{00}(z) = P_{\ell}(z). \tag{4.6}$$

$$P^{\ell}_{uv}(-z) = (-1)^{\ell-u-v}P^{\ell}_{u,-v}(z). \tag{4.7}$$

$$\overline{P^{\ell}_{uv}(z)} = (-1)^{v-u}P^{\ell}_{uv}(z), \quad -1\leqslant z\leqslant 1. \tag{4.8}$$

The P^{ℓ}_{uv} may be expressed in terms of univariate Jacobi polynomials ([13], III (3.9.2)). The composition formula ([13], III (4.1.7)) for the P^{ℓ}_{uv} may be derived from Eq. (4.5) in the case $v = 0$ (and hence in the case $u = 0$ by Eq. (4.6)), assuming Eqs. (4.3) and (4.4),

$$\mathrm{e}^{-\mathrm{i}v\psi'}P^{\ell}_{0v}(\cos\theta') = \sum_{u=-\ell}^{\ell}\mathrm{e}^{-\mathrm{i}u\phi'}P^{\ell}_{0u}(\cos\theta)\,P^{\ell}_{uv}(\cos\tilde{\theta}). \tag{4.9}$$

Taking $v = 0$ in Eq. (4.9)

$$P_{2k}(\cos\theta') = \sum_{u=-2k}^{2k}\mathrm{e}^{-\mathrm{i}u\psi}P^{2k}_{0u}(\cos\theta)\cdot\mathrm{e}^{-\mathrm{i}u\phi}P^{2k}_{u0}(\cos\tilde{\theta}),$$

and so a basis for $\mathbf{V}_{(k)}$ is provided by the $4k+1$ functions

$$Y^{(k)}_{v}(\mathbf{p}) = \mathrm{e}^{-\mathrm{i}v\psi}P^{2k}_{0v}(\cos\theta), \quad v = -2k, -2k+1, \ldots, 2k. \tag{4.10}$$

The $P^{2k}_{0v}(\cos\theta)$ are proportional to the associated Legendre functions ([13], III (3.9.9)), and the $Y^{(k)}_{v}$ are proportional to the well-known harmonic functions. More generally, the spherical harmonics $\mathrm{e}^{-\mathrm{i}v\psi}P^{\ell}_{0v}(\cos\theta)$, $v = -\ell, -\ell+1, \ldots, \ell$ constitute a set of basis functions for a realization of the irreducible representation $((\ell))$ of $O(3)$, $\ell = 0, 1, 2, \ldots$ They play a fundamental role in physics as solutions of the angular wave equation for motion in a spherically symmetric potential field (see Ref. [5], Chapter 19); ℓ and v are then interpreted as the angular momentum and magnetic quantum numbers, respectively, involved in the definition of electron orbitals, and the "shapes" of the orbitals are in part determined by the spherical harmonics.

In particular, $Y^{(k)}_{0}(\mathbf{p}) = P_{2k}(\cos\theta)$, the zonal spherical function in $\mathbf{V}_{(k)}$, and the remaining $Y^{(k)}_{v}(\mathbf{p})$ are the associated spherical functions in $\mathbf{V}_{(k)}$. Applying Eq. (4.9) to the $Y^{(k)}_{v}(\mathbf{p})$,

$$Y_v^{(k)}(\mathbf{p}^*) = \sum_{u=-2k}^{2k} Y_u^{(k)}(\mathbf{p})\tau_{uv}^{(k)}(H), \quad H \in O(3),$$

where

$$\tau_{uv}^{(k)}(H) = t_{u,dv}^{2k}(H_+)$$

and we introduce the notation

$$t_{uv}^{\ell}(H_+) = t_{uv}^{\ell}(\tilde{\phi}, \tilde{\theta}, \tilde{\psi}) = \mathrm{e}^{-\mathrm{i}(u\tilde{\phi}+v\tilde{\psi})} P_{uv}^{\ell}(\cos\tilde{\theta}), \quad u, v = -\ell, -\ell+1, \ldots, \ell. \tag{4.11}$$

Hence the $(4k+1) \times (4k+1)$ matrix

$$\mathbf{T}_{(k)}(H) = (\tau_{uv}^{(k)}(H))$$

is the matrix of the irreducible representation$((2k))$ of $O(3)$ in $\mathbf{V}_{(k)}$ relative to the basis (4.10); the representation is unitary. The basis appears as row $u = 0$ of $\mathbf{T}_{(k)}$. When the representation is restricted to $H_+ \in SO(3)$ it remains irreducible, with

$$\mathbf{T}_{(k)}(H_+) = (t_{uv}^{2k}(H_+)).$$

4.2. *The case $m = 1$, $n > 3$.* We shall construct $\hat{P}_{(k)}$ using Eq. (3.7). For $n > 3$, $\mathscr{V}_{1,n-1}$ is isomorphic to the unit sphere $\mathscr{S}^{n-2}$ in $\mathbb{R}^{n-1}$, and we may take

$$K_1 = (\cos\phi_1, \sin\phi_1 \cos\phi_2, \ldots, \sin\phi_1 \ldots \sin\phi_{n-3}\cos\phi_{n-2}, \sin\phi_1 \ldots \sin\phi_{n-3}\sin\phi_{n-2})',$$

$$(\mathrm{d}K_1) = \frac{\Gamma(\frac{1}{2}(n-1))}{2\pi^{(1/2)(n-1)}} \sin^{n-3}\phi_1 \sin^{n-4}\phi_2 \ldots \sin\phi_{n-3}\, \mathrm{d}\phi_1 \ldots \mathrm{d}\phi_{n-2},$$
$$0 \leqslant \phi_1, \ldots, \phi_{n-3} \leqslant \pi, \quad |\phi_{n-2}| \leqslant \pi$$

(see Ref. [13], IX, para 1.1). Hence by Eq. (3.7) the zonal spherical function in $\mathbf{V}_{(k)}$ is proportional to a Gegenbauer polynomial of even order,

$$\hat{P}_{(k)}(\cos^2\theta) = \frac{\Gamma(\frac{1}{2}(n-1))}{\sqrt{\pi}\Gamma(\frac{1}{2}n-1)} \int_0^{\pi} (\cos\theta + \mathrm{i}\,\sin\theta\,\cos\phi_1)^{2k} \sin^{n-3}\phi_1\, \mathrm{d}\phi_1$$

$$= \binom{n-3+2k}{2k}^{-1} \mathbf{C}_{2k}^{(1/2)n-1}(\cos\theta), \quad k = 0, 1, 2, \ldots$$

([13], IX (4.7.6)). Eq. (1.7) is consistent with this result since

$$\mathbf{C}_{2k}^{(1/2)n-1}(\cos\theta) = (-1)^k \binom{\frac{1}{2}n-2+k}{k} {}_2F_1\left(-k, \frac{1}{2}n-1+k; \frac{1}{2}; \cos^2\theta\right).$$

From Eq. (3.10) the $\hat{P}_{(k)}$ are orthogonal with respect to the measure $\{2\Gamma(\frac{1}{2}n)/\sqrt{\pi}\Gamma(\frac{1}{2}(n-1))\}\sin^{n-2}\theta\,\mathrm{d}\theta, 0 \leqslant \theta \leqslant \frac{1}{2}\pi$, and

$$\frac{2\Gamma(\frac{1}{2}(n)}{\sqrt{\pi}\Gamma(\frac{1}{2}(n-1))}\int_0^{(1/2)\pi}\{\hat{P}_{(k)}(\cos^2\theta)\}^2\sin^{n-2}\theta\,\mathrm{d}\theta = 1/\mathscr{D}_n((k)),$$

where from Eq. (2.3)

$$\mathscr{D}_n((k)) = \frac{n-2+4k}{n-2}\binom{n-3+2k}{2k},$$

is the dimension of $\mathbf{V}_{(k)}$. This results also follows from Eq. (1.2) with $a = \frac{1}{2}$, $c = \frac{1}{2}n$ and

$$b_{(k)} = (-1)^k(\tfrac{1}{2}n - 1 + k)_k/k!, \quad \mathscr{H}_{(k)} = 1(\tfrac{1}{2}n)_{2k}.$$

A basis for $\mathbf{V}_{(k)}$ ([13], IX (3.6.4)) may be constructed using $\chi_{(k)}$, but we omit the details.

4.3. *The case $m = 2$, $n = 4$.* $G_{2,2}$ consists of the 2-dimensional planes $\mathbf{p}$ passing through the origin in $\mathbb{R}^4$. From Eq. (3.6) $\mathbf{p}$ is spanned by the rows of $[RV, SZ]$ where $R = \mathrm{diag}(r_1, r_2)$, $S = \mathrm{diag}(s_1, s_2)$ and, since $m = \frac{1}{2}n$, $V, Z \in O(2)$ where V has nonnegative column 1 elements. For convenience in constructing $\mathbf{p}H$ we shall take $\mathscr{H}$ to be any orthogonal 2-frame in $\mathbf{p}$,

$$\begin{aligned}\mathscr{H} &= U[RV, SZ], \quad U \in O(2),\\ &= \gamma(\phi_1)[\mathscr{R}\gamma(\psi_1), \mathscr{S}\gamma(\psi_2)]\end{aligned}$$

say, where $-\pi < \phi_1, \psi_2 \leqslant \pi$, $-\frac{1}{2}\pi \leqslant |V|\psi_1 \leqslant 0$, and

$$\mathscr{R} = \mathrm{diag}\,(\rho_1, \rho_2) = \mathrm{diag}\,(r_1, g_1 r_2),$$

$$\mathscr{S} = \mathrm{diag}\,(\sigma_1, \sigma_2) = \mathrm{diag}\,(s_1, g_1 g_2 s_2),$$

where $g_1 = |U||V|$, $g_2 = |V||Z|$. Hence for any partition $\kappa = (k_1, k_2)$ of $k = 0, 1, 2, \ldots$ we have from Eq. (2.8)

$$\chi_{k_1 k_2}(\mathbf{p}) = C_{(k_1,k_2)}(Q'Q),$$

where $Q = \mathscr{R}\gamma(\psi_1) + \mathrm{i}\mathscr{S}\gamma(\psi_2)$. But

$$C_{(k_1,k_2)}(A) = |A|^{(1/2)k}P_{k_1-k_2}(\mathrm{tr}\,A/2|A|^{1/2}) \tag{4.12}$$

[10], Eq. (7.9) Hence, if we define, for $\varepsilon = \pm 1$,

$$\xi_\varepsilon = \rho_1\rho_2 - \varepsilon\sigma_1\sigma_2 = g_1\xi_{\varepsilon,0}, \quad \eta_\varepsilon = \sigma_1\rho_2 + \varepsilon\rho_1\sigma_2 = g_1\eta_{\varepsilon,0},$$

where

$$\xi_{\epsilon 0} = \cos(\theta_1 + \epsilon g_2 \theta_2), \quad \eta_{\epsilon 0} = \sin(\theta_1 + \epsilon g_2 \theta_2)$$

we obtain

$$\chi_{(k_1,k_2)}(\mathbf{p}) = \{\xi_1 + \mathrm{i}\eta_1 \cos(\psi_2 - \psi_1)\}^k P_{k_1-k_2}(\xi_{-1}),$$

which is independent of g_1, hence of U. By Eq. (3.9) the zonal spherical function in $\mathbf{V}_{(k_1,k_2)}$ is obtained by integrating over $Z \in O(2)$,

$$\hat{P}_{(k_1,k_2)}(R^2) = \tfrac{1}{2}[P_k(\xi_+)P_{k_1-k_2}(\xi_-) + P_k(\xi_-)P_{k_1-k_2}(\xi_+)], \tag{4.13}$$

where $\xi_\pm = \cos(\theta_1 \pm \theta_2)$. From Eq. (3.10) these functions are orthogonal with respect to the measure $\mathrm{d}\xi_+ \, \mathrm{d}\xi_-$, $0 \leqslant |\xi_+| \leqslant \xi_- \leqslant 1$, and we have

$$\int\int_{|\xi_+| \leqslant \xi_- \leqslant 1} \{\hat{P}_\kappa(R^2)\}^2 \, \mathrm{d}\xi_+ \, \mathrm{d}\xi_- = 1/\mathcal{D}_4(\kappa),$$

where $\mathcal{D}_4(\kappa) = 2(2k+1)[2(k_1 - k_2) + 1]$ when $k_2 > 0$, $(2k+1)^2$ when $k_2 = 0$.

Alternatively, the result may be derived from the orthogonality properties of the Legendre polynomials. It also follows from Eq. (1.2) with $a = 1, c = 2$, noting that

$$b_\kappa = \frac{1}{2}(-1)^k \binom{2k}{k} \left(\frac{1}{2}\right)_{k_2} / (k_1 + 1)_{k_2} \quad \text{when } k_2 > 0$$

and

$$b_{(k,0)} = (-1)^k \binom{2k}{k}.$$

These apparently inconsistent expressions both follow from Eq. (3.8) of Part I, and are consistent with the distinction between $k_m > 0$ and $k_m = 0$ in Eq. (2.3).

We now construct a basis for $\mathbf{V}_{(k_1,k_2)}$ by transforming to $\mathbf{p}^* = \mathbf{p}H$, $H \in O(4)$. Extending the notation for $\mathbf{p}$ to H and $\mathbf{p}^*$,

$$H = \begin{bmatrix} \gamma(\tilde{\phi}_1) & 0 \\ 0 & \gamma(\tilde{\phi}_2) \end{bmatrix} \begin{bmatrix} \tilde{\mathscr{R}} & \tilde{\mathscr{S}} \\ -\tilde{\mathscr{S}}_0 & \tilde{\mathscr{R}}_0 \end{bmatrix} \begin{bmatrix} \gamma(\tilde{\psi}_1) & 0 \\ 0 & \gamma(\tilde{\psi}_2) \end{bmatrix},$$

where

$$\tilde{\mathscr{R}} = \operatorname{diag}(\tilde{\rho}_1, \tilde{\rho}_2) = \operatorname{diag}(\tilde{r}_1, \tilde{g}_1\tilde{r}_2),$$
$$\tilde{\mathscr{S}} = \operatorname{diag}(\tilde{\sigma}_1, \tilde{\sigma}_2) = \operatorname{diag}(\tilde{s}_1, \tilde{g}_1\tilde{g}_2\tilde{s}_2),$$
$$\tilde{\mathscr{R}}_0 = \operatorname{diag}(\tilde{\rho}_1, \tilde{d}\tilde{\rho}_2), \quad \tilde{\mathscr{S}}_0 = \operatorname{diag}(\tilde{\sigma}_1, \tilde{d}\tilde{\sigma}_2), \quad \tilde{d} = |H|,$$

and $\mathbf{p}^*$ is spanned by the rows of

$$\mathscr{H}^* = [\mathscr{H}_1^*, \mathscr{H}_2^*] = \gamma(\phi_1^*)[\mathscr{R}^*\gamma(\psi_1^*), \mathscr{S}^*\gamma(\psi_2^*)],$$

where

$$\mathscr{R}^* = \mathrm{diag}(\rho_1^*, \rho_2^*), \quad \mathscr{S}^* = \mathrm{diag}(\sigma_1^*, \sigma_2^*).$$

Then

$$\mathscr{H}_1^* = \gamma(\phi_1)[\mathscr{R}\gamma(\phi_1^+)\tilde{\mathscr{R}} - \mathscr{S}\gamma(\phi_2^+)\tilde{\mathscr{S}}_0]\gamma(\tilde{\psi}_1),$$
$$\mathscr{H}_2^* = \gamma(\phi_1)[\mathscr{R}\gamma(\phi_1^+)\tilde{\mathscr{S}} + \mathscr{S}\gamma(\phi_2^+)\tilde{\mathscr{R}}_0]\gamma(\tilde{\psi}_2),$$

where $\phi_j^+ = \psi_j + \tilde{\phi}_j$, $j = 1, 2$. Defining

$$\tilde{\xi}_\varepsilon = \tilde{\rho}_1\tilde{\rho}_2 - \varepsilon\tilde{\sigma}_1\tilde{\sigma}_2, \quad \tilde{\eta}_\varepsilon = \tilde{\sigma}_1\tilde{\rho}_2 + \varepsilon\tilde{\rho}_1\tilde{\sigma}_2, \quad \varepsilon = \pm 1,$$

and $\xi_\varepsilon^*, \eta_\varepsilon^*$ similarly, we have for example

$$\rho_1^*\rho_2^* = |\mathscr{H}_1^*|, \quad \sigma_1^*\sigma_2^* = |\mathscr{H}_2^*|,$$
$$\eta_1^* = |\mathscr{H}_1^*| \operatorname{tr}(\mathscr{R}^{*-1}\mathscr{S}^*) = |\mathscr{H}_1^*| \operatorname{tr}\{\gamma(\psi_1^* - \psi_2^*)\mathscr{H}_1^{*-1}\mathscr{H}_2^*\}.$$

The following relations are obtained after some algebra,

$$\xi_\varepsilon^* = \xi_{\varepsilon d}\tilde{\xi}_\varepsilon - \eta_{\varepsilon d}\tilde{\eta}_\varepsilon \cos(\phi_1^+ - \varepsilon\tilde{d}\phi_2^+),$$
$$\eta_\varepsilon^* \cdot \mathrm{e}^{\mathrm{i}(\psi_1' - \varepsilon\psi_2')} = \xi_{\varepsilon d}\tilde{\eta}_\varepsilon + \eta_{\varepsilon d}\tilde{\xi}_\varepsilon \cos(\phi_1^+ - \varepsilon\tilde{d}\phi_2^+)$$
$$+ \mathrm{i}\eta_{\varepsilon d} \sin(\phi_1^+ - \varepsilon\tilde{d}\phi_2^+), \quad \varepsilon = \pm 1,$$

where $\psi_j^* = \psi_j' + \tilde{\psi}_j$, $j = 1, 2$. These equations have the same form as Eqs. (4.3) and (4.4) but, in order to apply Eq. (4.9) to the expansion of $\hat{P}_\kappa(R^{*2})$, the constraint on θ in Eq. (4.5) requires that we substitute $\eta_\varepsilon = g_1\eta_{\varepsilon,0}$ where $\eta_{\varepsilon,0} = \sin(\theta_1 + \varepsilon g_2\theta_2) \geqslant 0$, and similarly for $\tilde{\eta}_\varepsilon$ and η_ε^*. Eq. (4.9). then yields

$$(g_1^*)^v \mathrm{e}^{-\mathrm{i}v(\psi_1' - \varepsilon\psi_2')} P_{0v}^\ell(\xi_\varepsilon^*) = \sum_{u=-\ell}^{\ell} \mathrm{e}^{-\mathrm{i}u(\phi_1^+ - \varepsilon\tilde{d}\phi_2^+)} g_1^u P_{0u}^\ell(\xi_{\varepsilon d}) \cdot \tilde{g}_1^{u+v} P_{uv}^\ell(\tilde{\xi}_\varepsilon). \tag{4.14}$$

Substituting in

$$\hat{P}_{(k_1,k_2)}(R^{*2}) = \frac{1}{2}\sum_{\varepsilon=\pm 1} P_k(\xi_\varepsilon^*) P_{k_1-k_2}(\xi_{-\varepsilon}^*)$$

we obtain $2(2k+1)[2(k_1 - k_2) + 1]$ basis elements in $\mathbf{V}_{(k_1,k_2)}$ when $k_2 > 0$, in accordance with $\mathscr{Y}_4(\kappa)$ above,

$$Y_{uv\varepsilon}^{(k_1,k_2)}(\mathbf{p}) = \mathrm{e}^{-\mathrm{i}u(\psi_1 - \varepsilon\psi_2)} P_{0u}^k(\xi_{\varepsilon,0}) \cdot \mathrm{e}^{-\mathrm{i}v(\psi_1 + \varepsilon\psi_2)} P_{0v}^{k_1-k_2}(\xi_{-\varepsilon,0}), \tag{4.15}$$

where $u = -k, -k+1, \ldots, k$; $v = -k_1 + k_2, -k_1 + k_2 + 1, \ldots, k_1 - k_2$; $\varepsilon = \pm 1$. When $k_2 = 0$, the number of linearly independent basis elements reduces to $(2k+1)^2$ as required.

Applying Eq. (4.14) once again, we obtain for $k_2 > 0$

$$Y^{(k_1,k_2)}_{u_2 v_2 \varepsilon_2}(\mathbf{p}^*) = \sum_{u_1=-k}^{k} \sum_{v_1=-k_1+k_2}^{k_1-k_2} \sum_{\varepsilon_1=\pm 1} Y^{(k_1,k_2)}_{u_1 v_1 \varepsilon_1}(\mathbf{p})\tau^{(k_1,k_2)}_{u_1 v_1 \varepsilon_1 : u_2 v_2 \varepsilon_2}(H),$$

where in the notation (4.11)

$$\tau^{(k_1,k_2)}_{u_1 v_1 \varepsilon_1 : u_2 v_2 \varepsilon_2}(H) = \delta_{\varepsilon_1,\varepsilon_2 \tilde{d}} \cdot \tilde{g}_1^{u_1+u_2+v_1+v_2} t^k_{u_1 u_2}(\tilde{\phi}_1 - \varepsilon_1\tilde{\phi}_2, \hat{\theta}_{\varepsilon_2}, \tilde{\psi}_1 - \varepsilon_2\tilde{\psi}_2)$$
$$\times\, t^{k_1-k_2}_{v_1 v_2}(\tilde{\phi}_1 + \varepsilon_1\tilde{\phi}_2, \hat{\theta}_{-\varepsilon_2}, \tilde{\psi}_1 + \varepsilon_2\tilde{\psi}_2).$$

and $\hat{\xi}_\varepsilon = \cos\hat{\theta}_\varepsilon,\ 0 \leqslant \hat{\theta}_\varepsilon \leqslant \pi$. Hence for $k_2 > 0$

$$\mathbf{T}_{(k_1,k_2)}(H) = (\tau^{(k_1,k_2)}_{u_1 v_1 \varepsilon_1 : u_2 v_2 \varepsilon_2}(H)) \tag{4.16}$$

is the matrix of the irreducible representation $((2k_1, 2k_2))$ of $O(4)$ relative to the basis (4.15). If Eq. (4.16) is partitioned according to $\varepsilon_1 = \pm 1, \varepsilon_2 = \pm 1$, then for $H_+ \in$ SO(4)

$$\mathbf{T}_{(k_1,k_2)}(H_+) = \begin{bmatrix} \mathbf{T}^{11}(H_+) & O \\ O & \mathbf{T}^{22}(H_+) \end{bmatrix}.$$

The basis (4.15) thus displays the reducibility of the representation when restricted to SO(4), T^{11} and T^{22} being the matrices of equivalent representations $((2k_1, 2k_2))$ of SO(4). If $\tilde{g}_1 = -1$ we may write $\tilde{g}_1 = e^{\pm i\pi}$ and incorporate $\tilde{g}_1^{u_1+u_2+v_1+v_2}$ into the exponential factors. Then the elements of $\mathbf{T}^{11}(H_+)$ and $\mathbf{T}^{22}(H_+)$ may be written as $t^k_{u_1 u_2}(\alpha_1, \beta_1, \gamma_1)t^{k_1-k_2}_{v_1 v_2}(\alpha_2, \beta_2, \gamma_2)$ and $t^k_{u_1 u_2}(\alpha_2, \beta_2, \gamma_2)t^{k_1-k_2}_{v_1 v_2}(\alpha_1, \beta_1, \gamma_1)$, respectively. This factorization arises from the underlying result that the Lie algebra of SO(4) has the decomposition so(4) = so(3) ⊕ so(3) ([1], p. 197).

When $k_2 = 0$, the $(2k+1)^2 \times (2k+1)^2$ matrix $\mathbf{T}_{(k,0)}(H)$ has the elements

$$\tau^{(k,0)}_{u_1 v_1 : u_2 v_2}(H) = \tilde{g}^{u_1+u_2+v_1+v_2} t^k_{u_1 u_2}(\tilde{\phi}_1 - \tilde{\phi}_2, \hat{\theta}_{\tilde{d}}, \tilde{\psi}_1 - \tilde{d}\tilde{\psi}_2)$$
$$\times\, t^k_{v_1 v_2}(\tilde{\phi}_1 + \tilde{\phi}_2, \hat{\theta}_{-\tilde{d}}, \tilde{\psi}_1 + \tilde{d}\tilde{\psi}_2),$$
$$u_1, v_1, u_2, v_2 = -k, -k+1, \ldots, k.$$

The representation $(2k)$ of $O(4)$ remains irreducible when restricted to SO(4).

4.4. *The case* $m = 2, n = 5$. The zonal spherical functions have been obtained by the author for $m = 2, n \geqslant 5$ and arbitrary (k_1, k_2). The expressions are complicated and will not be presented here, but are available on request. However, we may consider the case $m = 2, n = 5, k_1 = k_2$ $(k = 2k_1)$, for which $C^*_{(k_1^2)}(A) = |A|^{k_1}$. Since $m < \frac{1}{2}n, \mathbf{p} \in G_{2,3}$ is spanned by the rows of

$\mathscr{H} = [RV, SZ_{11}]$ where $R = \mathrm{diag}(r_1, r_2), S = \mathrm{diag}(s_1, s_2), V \in O(2)$ has nonnegative column 1 elements, and $Z \in \mathrm{SO}(3)$. Then writting $\mathbf{d} = |V|$,

$$\chi_{(k_1^2)}(\mathbf{p}) = |R\gamma(\gamma) + \mathrm{i}\mathscr{S}Z_{11}|^k,$$

where

$$\mathscr{S} = \mathrm{diag}(s_1, \mathbf{d}s_2) = \mathrm{diag}(\sigma_1, \sigma_2),$$

$$Z = \begin{bmatrix} \gamma(\phi) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \gamma(\theta) \end{bmatrix} \begin{bmatrix} \gamma(\psi) & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix},$$

$$Z_{11} = \gamma(\phi) \begin{bmatrix} 1 & 0 \\ 0 & \cos\theta \end{bmatrix} \gamma(\psi),$$

$$(\mathrm{d}Z_1) = (\mathrm{d}Z) = \frac{1}{8\pi^2} \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \, \mathrm{d}\psi,$$

$$-\tfrac{1}{2}\pi \leqslant \mathbf{d} \cdot \gamma \leqslant 0, \quad 0 \leqslant \theta \leqslant \pi, \quad -\pi < \phi, \psi \leqslant \pi. \tag{4.17}$$

We obtain

$$\chi_{(k_1^2)}(\mathbf{p}) = [\cos^2 \tfrac{1}{2}\theta\{\xi_1 + \mathrm{i}\eta_1 \cos(\phi + \psi^-)\} + \sin^2 \tfrac{1}{2}\theta\{\xi_{-1} + \mathrm{i}\eta_{-1}\cos(\phi - \psi^-)\}]^k$$

where $\psi^- = \psi - \gamma$ and $\xi_\varepsilon = r_1 r_2 - \varepsilon\sigma_1\sigma_2$, $\eta_\varepsilon = \sigma_1 r_2 + \varepsilon r_1 \sigma_2$. Hence, from Eq. (3.9),

$$\begin{aligned} \hat{P}_{(k_1^2)}(R^2) &= \frac{1}{8\pi^2} \int_{\theta=0}^{\pi} \int_{\phi=-\pi}^{\pi} \int_{\psi=-\pi}^{\pi} \chi_{(k_1^2)}(\mathbf{p}) \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \, \mathrm{d}\psi \\ &= \frac{1}{k+1} \sum_{r+s=k} P_r(\xi_+) P_s(\xi_-), \end{aligned} \tag{4.18}$$

where $\xi_\pm = \cos(\theta_1 \pm \theta_2)$. According to Eq. (1.7), $\hat{P}_{(k_1^2)} = ((-1)^k/(k+1))\ {}_2F_1(-k_1, k_1+1; 1; R^2)$. These functions are orthogonal with respect to the measure $\frac{3}{2}(\xi_- - \xi_+)\mathrm{d}\xi_+ \, \mathrm{d}\xi_-, 0 \leqslant |\xi_+| \leqslant \xi_- \leqslant 1$, and from Eq. (3.11)

$$\frac{3}{2} \iint_{|\xi_+| \leqslant \xi_- \leqslant 1} \{P_{(k_1^2)}(R^2)\}^2 (\xi_- - \xi_+) \mathrm{d}\xi_+ \, \mathrm{d}\xi_- = 1/\mathscr{D}_5((k_1^2)), \tag{4.19}$$

where from Eq. (5.3) $\mathscr{D}_5((k_1^2)) = \frac{1}{3}(k+1)(2k+1)(2k+3)$ is the dimension of $\mathbf{V}_{(k_1^2)}$. To evaluate the integral directly, we note that $\sum_{r+s=k} P_r(\xi_+)P_s(\xi_-)$ is the coefficient of α^k in the generating function $f(\alpha) = (1 - 2\alpha\xi_+ + \alpha^2)^{-1/2} (1 - 2\alpha\xi_- + \alpha^2)^{-1/2}$. The required integral is given by the coefficient of $(\alpha\beta)^k$ in the integral of $f(\alpha)f(\beta)$ with respect to the invariant measure. The part of the generating function yielding terms $(\alpha\beta)^k$ is

$$\frac{3}{8}(\alpha\beta)^{-3/2}(1+\alpha\beta)\ln\{(1+(\alpha\beta)^{1/2})/(1-(\alpha\beta)^{1/2})\},$$

from which Eq. (4.19) follows. The result also follows from Eq. (1.2) with $a = 1, c = \frac{5}{2}$ and

$$b_{(k_1^2)} = 2^k\left(\frac{1}{2}\right)_k \Big/ k!, \quad \mathscr{Y}_{(k_1^2)} = \mu_{(k_1^2)} \Big/ \left(\frac{5}{2}\right)_{(k^2)}.$$

We may also construct a basis for $\mathbf{V}_{(k_1^2)}$. Taking $v = 0, w = e^{i\beta}$ in Eq. (4.5) we obtain

$$(\cos\alpha + i\ \sin\alpha\ \cos\beta)^\ell = \ell!\sum_{u=-\ell}^{\ell}\frac{e^{-iu\beta}P_{u0}^\ell(\cos\alpha)}{\sqrt{(\ell-u)!(\ell+u)!}},$$

and hence

$$\chi_{(k_1^2)}(\mathbf{p}) = k!\sum_{r+s=k}\sum_{p=-r}^{r}\sum_{q=-s}^{s}\cos^{2r}\frac{1}{2}\theta\ \sin^{2s}\frac{1}{2}\theta\cdot e^{-i[(p+q)\phi+(p-q)\psi]}$$

$$\times\frac{P_{p0}^r(\xi_1)P_{q0}^s(\xi_{-1})}{\sqrt{(r+p)!(r-p)!(s+q)!(s-q)!}}. \tag{4.20}$$

A more useful expression for χ may be obtained using the expansion theorem in Ref. [13], III para 6.3. This result is an application of the Peter–Weyl theorem: the $t_{uv}^\ell(\phi,\theta,\psi)$, being the matrix elements of a complete system of irreducible unitary representations of the compact group SO(3), constitute a complete system of functions on SO(3) with respect to the invariant measure (4.17). Applying the expansion theorem to Eq. (4.20),

$$\chi_{(k_1^2)}(\mathbf{p}) = \sum_{\ell=0}^{k}\sum_{u=-\ell}^{\ell}\sum_{\substack{v=-\ell \\ (u+v\ \text{even})}}^{\ell} e^{-iv\gamma}\ t_{uv}^\ell(\phi,\theta,\psi)\Pi_{\ell;u,v}^{(k_1^2)}(\xi_1,\xi_{-1}), \tag{4.21}$$

where

$$\Pi_{\ell;u,v}^{(k_1^2)}(\xi_1,\xi_{-1}) = k!\sum_{r+s=k}{}^{*}\alpha_{\ell;u,v}^{r,s}\frac{P_{p0}^r(\xi_1)P_{q0}^s(\xi_{-1})}{\sqrt{(r+p)!(r-p)!(s+q)!(s-q)!}}, \tag{4.22}$$

$$p = \tfrac{1}{2}(u+v),\quad q = \tfrac{1}{2}(u-v),$$

$$\alpha_{\ell;u,v}^{r,s} = \frac{1}{2}(2\ell+1)\int_0^\pi \cos^{2r}\frac{1}{2}\theta\ \sin^{2s}\frac{1}{2}\theta\ \overline{P_{uv}^\ell(\cos\theta)}\sin\theta\ d\theta \tag{4.23}$$

$$= (-1)^p(2\ell+1)(r+p)!\sqrt{\frac{(\ell-u)!(\ell-v)!}{(\ell+u)!(\ell+v)!}}$$
$$\times \sum_{j=\max(u,v)}^{\ell} \frac{(-1)^j(\ell+j)!(s+j-p)!}{(\ell-j)!(j-u)!(j-v)!(j+k+1)!}. \tag{4.24}$$

The summation $\sum^*$ in Eq. (4.22) extends over $r \geqslant |p|, s \geqslant |q|$ such that $r+s=k$. The formula ([13], III (3.4.1)) for P^ℓ_{uv} has been used to calculate $\alpha^{r,s}_{\ell:u,v}$. To see that these coefficients are zero for $\ell > k$ we refer to the Rodrigues formula for P^ℓ_{uv} ([13], III (3.4.3)). The result follows by substituting $z = \cos\theta$, $\cos^{2r}\frac{1}{2}\theta\sin^{2s}\frac{1}{2}\theta = 2^{-k}(1+z)^r(1-z)^s$ in Eq. (4.23) and integrating by parts. Note that the properties (4.6) allow us to select the subscripts of P^ℓ_{uv} so that $-u \pm v$ are nonnegative integers. From the properties of the P^ℓ_{uv},

$$\alpha^{r,s}_{\ell:u,v} = \alpha^{r,s}_{\ell:v,u} = \alpha^{r,s}_{\ell:-u,-v} = (-1)^{\ell-u-v}\alpha^{s,r}_{\ell:u,-v},$$
$$\overline{\alpha^{r,s}_{\ell:u,v}} = (-1)^{u-v}\alpha^{r,s}_{\ell:u,v}.$$

We may now use Eq. (4.21) to expand $\chi_{(k_1^2)}(\mathbf{p}H)$ in the case

$$H = \begin{bmatrix} I_2 & O \\ O & \tilde{W} \end{bmatrix},$$

where $\tilde{W} \in SO(3)$ has parameters $(\tilde{\phi}, \tilde{\theta}, \tilde{\psi})$,

$$\chi_{(k_1^2)}(\mathbf{p}H) = \sum_{\ell=0}^{k}\sum_{u=-\ell}^{\ell}\sum_{v=-\ell}^{\ell} t^\ell_{uv}(\tilde{\phi},\tilde{\theta},\tilde{\psi})Y^{(k_1^2)}_{\ell:u,v}(\mathbf{p}),$$

where

$$Y^{(k_1^2)}_{\ell:u,v}(\mathbf{p}) = e^{-iv\gamma}\sum_{\substack{w=-\ell \\ (w+\ell \text{ even})}}^{\ell} t^\ell_{wu}(\phi,\theta,\psi)\Pi^{(k_1^2)}_{\ell:w,v}(\xi_1,\xi_{-1}), \tag{4.25}$$
$$\ell = 0,1,\ldots,k;\quad u,v = -\ell,-\ell+1,\ldots,\ell.$$

Although H was restricted to a subgroup of $O(5)$, we may ask whether the functions defined by Eq. (4.25) in fact constitute a basis for $\mathbf{V}_{(k_1^2)}$. The number of terms is

$$\sum_{\ell=0}^{k}(2\ell+1)^2 = \frac{1}{3}(k+1)(2k+1)(2k+3) = \mathscr{D}_5((k_1^2))$$

as required. If $Y^{(k_1^2)}_{\ell:u,v}$ is a null function then the linear independence of the t^ℓ_{wu} implies that $\Pi^{(k_1^2)}_{\ell:w,v}$ is null, $w = -\ell,-\ell+1,\ldots,\ell$. From Eq. (4.22) the orthogonality of the P^{ℓ}_{uv} for $\ell \geqslant \max(|u|, |v|)$ ([13], III para 6.2) would then require that $\alpha^{r,s}_{\ell:w,v} = 0$ for $r + s = k, w = -\ell, -\ell + 1, \ldots, \ell$; but $\alpha^{r,s}_{\ell:\ell,v} \neq 0$ by Eq. (4.24). The linear independence of the $Y^{(k_1^2)}_{\ell:u,v}$ follows from that of the $e^{-iv\gamma}\, t^{\ell}_{wu}$ for distinct triplets (ℓ, u, v). Hence Eq. (4.25) constitutes a basis for $\mathbf{V}_{(k_1^2)}$. In particular, since $\alpha^{r,s}_{0:0,0} = r!s!/(k+1)!$, the zonal spherical function (4.18) is

$$\hat{P}_{(k_1^2)}(R^2) = Y^{(k_1^2)}_{0:0,0}(\mathbf{p}) = \Pi^{(k_1^2)}_{0:0,0}(\xi_1, \xi_{-1}),$$

and the remaining basis elements are the associated spherical functions. Properties of the $\Pi^{(k_1^2)}_{\ell:u,v}$ include

$$\begin{aligned}
\Pi^{(k_1^2)}_{\ell:u,v}(\xi_1, \xi_2) &= \Pi^{(k_1^2)}_{\ell:v,u}(\xi_1, \xi_2) = \Pi^{(k_1^2)}_{\ell:-v,-u}(\xi_1, \xi_2), \\
\Pi^{(k_1^2)}_{\ell:u,v}(\xi_2, \xi_1) &= (-1)^{\ell-u-v}\,\Pi^{(k_1^2)}_{\ell:u,-v}(\xi_1, \xi_2), \\
\Pi^{(k_1^2)}_{\ell:u,v}(-\xi_1, -\xi_2) &= (-1)^{u}\,\Pi^{(k_1^2)}_{\ell:u,v}(\xi_1, \xi_2), \\
\overline{\Pi^{(k_1^2)}_{\ell:u,v}(\xi_1, \xi_2)} &= (-1)^{u}\,\Pi^{(k_1^2)}_{\ell:u,v}(\xi_1, \xi_2).
\end{aligned} \tag{4.26}$$

The properties (4.26) imply that it is sufficient to tabulate the $\Pi^{(k_1^2)}_{\ell:u,v}$ for $\ell = 0, 1, \ldots, k; u = 0, 1, \ldots, \ell; v = -u, -u + 2, \ldots, u$ (see Table 1).

$Y^{(k_1^2)}_{\ell:u,v}(\mathbf{p})$ and $Y^{(k_1'^2)}_{\ell':u',v'}(\mathbf{p})$ are orthogonal with respect to the invariant measure $(d\mathbf{p})$ (Eq. (3.12)) for $(k_1, \ell, u, v) \neq (k_1', \ell', u', v')$. For $k_1' > k_1$ this implies the following orthogonality property of the $\Pi^{(k_1^2)}_{\ell:u,v}$,

Table 1
The polynomials $\Pi^{(1^2)}_{\ell,u,v}$

ℓ	u	v	$\Pi^{1^2}_{\ell,u,v}(\xi_1, \xi_2)$
0	0	0	$\hat{P}_{(1^2)}(R^2) = \frac{1}{2}(\xi_1^2 + \xi_2^2) + \frac{1}{3}(\xi_1\xi_2 - 1)$
1	0	0	$\frac{3}{4}(\xi_1^2 - \xi_2^2)$
1	1	-1	$-\frac{1}{4}i\eta_2(\xi_1 + 3\xi_2)$
1	1	1	$\frac{1}{4}i\eta_1(3\xi_1 + \xi_2)$
2	0	0	$\frac{1}{4}(\xi_1^2 + \xi_2^2) - \frac{1}{3}\xi_1\xi_2 - \frac{1}{6}$
2	1	-1	$\frac{1}{4}\, i\eta_2(\xi_2 - \xi_1)$
2	1	1	$\frac{1}{4}i\eta_1(\xi_1 - \xi_2)$
2	2	-2	$-\frac{1}{4}\eta_2^2$
2	2	0	$\frac{1}{2\sqrt{6}}\eta_1\eta_2$
2	2	2	$-\frac{1}{4}\eta_1^2$

$$\int\int_{|\xi_1| \leqslant \xi_2 \leqslant 1} \left\{ \sum_{\substack{w=-\ell \\ (u+w \text{ even})}}^{\ell} \Pi^{(k_1^{\prime 2})}_{\ell;w,v}(\xi_1,\xi_2)\overline{\Pi^{(k_1^2)}_{\ell,w,v}(\xi_1,\xi_2)} \right\}(\xi_2-\xi_1)\mathrm{d}\xi_1 k_1^{\prime 2}\mathrm{d}\xi_2 = 0$$

for $\ell = 0, 1, \ldots, k; v = -\ell, -\ell+1, \ldots, \ell$.

4.5. *The case* $m = 3, n = 6$. For $m > 2$ no general formula exists for $C_\kappa(A)$. However, to show we are not entirely restricted to $m = 2$ we consider $m = 3$, $n = 6$, $\kappa = (k_1^3)$, for which $C^*_{(k_1^3)}(A) = |A|^{k_1}$. Any $\mathbf{p} \in G_{3,3}$ is spanned by the rows of $[RV, SZ]$, where $R = \mathrm{diag}(r_1, r_2, r_3)$, V and $Z \in O(3)$, and V has nonnegative column 1 elements. Equivalently, $\mathbf{p}$ is spanned by the rows of

$$\mathscr{H} = [RV_+, \mathbf{d}SZ_+],$$

where $V_+, Z_+ \in \mathrm{SO}(3)$ and $\mathbf{d} = |V||Z|$. Writing $Q = Z_+V'_+ \in \mathrm{SO}(3)$.

$$\chi_{(k_1^3)}(\mathbf{p}) = |R + \mathrm{i}\mathbf{d}SQ|^{2k_1},$$

and if we take Q in the form (4.2) with parameters (ϕ, θ, ψ), then

$$\begin{aligned}\chi_{(k_1^3)}(\mathbf{p}) = {}& [\cos^2\tfrac{1}{2}\theta \cdot \mathrm{e}^{-\mathrm{i}\mathbf{d}\theta_1}\{\xi_+ + \mathrm{i}\mathbf{d}\eta_+\cos(\phi+\psi)\} \\ & + \sin^2\tfrac{1}{2}\theta \cdot \mathrm{e}^{-\mathrm{i}\mathbf{d}\theta_1}\{\xi_- - \mathrm{i}\mathbf{d}\eta_-\cos(\phi-\psi)\}]^{2k_1},\end{aligned}$$

where $\xi_\pm = \cos(\theta_2 \pm \theta_3)$ and $\eta_\pm = \sin(\theta_2 \pm \theta_3)$. Applying the expansion theorem as in Section 4.4,

$$\chi_{(k_1^3)}(\mathbf{p}) = \sum_{\ell=0}^{2k_1} \sum_{\substack{u=-\ell \\ (u+\ell \text{ even})}}^{\ell} \sum_{v=-\ell}^{\ell} t^\ell_{uv}(Q)\Gamma^{(k_1^3)}_{\ell;u,v}(\underset{\sim}{\theta}, \mathbf{d}), \tag{4.27}$$

where

$$\Gamma^{(k_1^3)}_{\ell;u,v}(\underset{\sim}{\theta}, \mathbf{d}) = (-1)^q\mathbf{d}^u(2k_1)! \sum_{r+s=2k_1}^{*} \alpha^{r,s}_{\ell;u,v} \frac{\mathrm{e}^{\mathrm{i}(r-s)\mathbf{d}\theta_1} P^r_{p0}(\xi_+)P^s_{q0}(\xi_-)}{\sqrt{(r+p)!(r-p)!(s+q)!(s-q)!}},$$
$$p = \tfrac{1}{2}(u+v), \quad q = \tfrac{1}{2}(u-v),$$

and the summation $\sum^*$ extends over $r \geqslant |p|, s \geqslant |q|$ such that $r + s = 2k_1$. The coefficients $\alpha^{r,s}_{\ell;u,v}$ were defined in Eq. (4.24). Basic properties of the $\Gamma^{(k_1^3)}_{\ell;u,v}$ are

$$\Gamma^{(k_1^3)}_{\ell;u,v}(\underset{\sim}{\theta}, \mathbf{d}) = \Gamma^{(k_1^3)}_{\ell;v,u}(\underset{\sim}{\theta}, \mathbf{d}) = \Gamma^{(k_1^3)}_{\ell;-u,-v}(\underset{\sim}{\theta}, \mathbf{d}),$$
$$\overline{\Gamma^{(k_1^3)}_{\ell;u,v}(\underset{\sim}{\theta}, \mathbf{d})} = \Gamma^{(k_1^3)}_{\ell;u,v}(\underset{\sim}{\theta}, -\mathbf{d}),$$

(see Table 2). A basis in $\mathbf{V}_{(k_1^3)}$ may be constructed by first transforming to $\mathbf{p}H$ where

Table 2
The polynomials $\Gamma^{(1^3)}_{\ell;u,v}$

ℓ	u	v	$\Gamma^{(1^3)}_{\ell;u,v}(\theta,\underset{\sim}{\mathbf{d}})$
0	0	0	$\hat{P}_{(1^3)} - \frac{1}{2}\mathrm{i}\mathbf{d}\,\Pi^3_{i=1}\sin 2\theta_i$
			where $\hat{P}_{(1^3)} = \frac{1}{2}\Pi^3_{i=1}\cos 2\theta_i + \frac{1}{6}\sum^3_{i=1}\cos 2\theta_i$
1	0	0	$-\frac{3}{4}\cos 2\theta_1\sin 2\theta_2\sin 2\theta_3 + \frac{1}{4}\mathrm{i}\mathbf{d}\sin 2\theta_1(3\cos 2\theta_2\cos 2\theta_3 + 1)$
1	1	−1	$\frac{1}{8}\mathrm{i}\mathbf{d}\{3\mathrm{e}^{-2\mathrm{i}\mathbf{d}\theta_1}\sin(2\theta_2 - 2\theta_3) + \sin 2\theta_2 - \sin 2\theta_3\}$
1	1	1	$\frac{1}{8}\mathrm{i}\mathbf{d}\{3\mathrm{e}^{2\mathrm{i}\mathbf{d}\theta_1}\sin(2\theta_2 + 2\theta_3) + \sin 2\theta_2 + \sin 2\theta_3\}$
2	0	0	$\frac{1}{2}\hat{P}_{(1^3)} - \frac{1}{4}(\cos 2\theta_2 + \cos 2\theta_3)$
2	1	−1	$\frac{1}{8}\mathrm{i}\mathbf{d}\{-\mathrm{e}^{-2\mathrm{i}\mathbf{d}\theta_1}\sin(2\theta_2 - 2\theta_3) + \sin 2\theta_2 - \sin 2\theta_3\}$
2	1	1	$\frac{1}{8}\mathrm{i}\mathbf{d}\{\mathrm{e}^{2\mathrm{i}\mathbf{d}\theta_1}\sin(2\theta_2 + 2\theta_3) - \sin 2\theta_2 - \sin 2\theta_3\}$
2	2	−2	$-\frac{1}{4}\mathrm{e}^{-2\mathrm{i}\mathbf{d}\theta_1}\sin^2(\theta_2 - \theta_3)$
2	2	0	$\frac{1}{4\sqrt{6}}(\cos 2\theta_2 - \cos 2\theta_3)$
2	2	2	$-\frac{1}{4}\mathrm{e}^{2\mathrm{i}\mathbf{d}\theta_1}\sin^2(\theta_2 + \theta_3)$

$$H = \begin{bmatrix} I_3 & O \\ O & \tilde{W} \end{bmatrix}.$$

$\tilde{W} \in \mathrm{SO}(3)$. If we set $Q = Z_-\tilde{W}V'_-$ in Eq. (4.27) we obtain

$$Z_{(k_1^3)}(\mathbf{p}H) = \sum_{\ell=0}^{2k_1}\sum_{u=-\ell}^{\ell}\sum_{v=-\ell}^{\ell} t^\ell_{vu}(\tilde{W})Y^{(k_1^3)}_{\ell;u,v}(\mathbf{p}),$$

where

$$Y^{(k_1^3)}_{\ell;u,v}(\mathbf{p}) = \sum_{\substack{h=-\ell \\ (h+\ell\ \text{even})}}^{\ell}\sum_{j=-\ell}^{\ell} t^\ell_{hu}(Z_+)t^\ell_{vj}(V'_+)\Gamma^{(k_1^3)}_{\ell;h,j}(\theta,\underset{\sim}{\mathbf{d}}), \tag{4.28}$$

$$\ell = 0, 1, \ldots, 2k_1;\quad u, v = -\ell, -\ell+1, \ldots, \ell.$$

The number of functions (4.28) is

$$\sum_{\ell=0}^{2k_1}(2\ell+1)^2 = \frac{1}{3}(2k_1+1)(4k_1+1)(4k_1+3) = \frac{1}{2}\mathscr{D}_6((k_1^3)),$$

where $\mathscr{D}_6((k_1^3))$ is the dimension of $\mathbf{V}_{(k_1^3)}$. Replacing $\underset{\sim}{W}$ by $-\underset{\sim}{W}$, in H, we derive a second set of basis elements

$$\left\{Y^{(k_1^3)-}_{\ell;u,v}(\mathbf{p})\right\}$$

say, obtained from Eq. (4.28) by replacing $\Gamma^{(k_1^3)}_{l;h,j}(\theta, \underset{\sim}{\mathbf{d}})$ by its conjugate $\Gamma^{(k_1^3)}_{l;h,j}(\theta, -\mathbf{d})$. The resulting basis may be expected to display the reducibility of the irreducible representation $((2k_1)^3)$ of $O(6)$ when restricted to SO(6).

In terms of the above basis the zonal spherical functions are given by

$$\hat{P}_{(k_1^3)}(R^2) = \frac{1}{2}\left\{Y^{(k_1^3)}_{0;0,0}(\mathbf{p}) + Y^{(k_1^3)-}_{0;0,0}(\mathbf{p})\right\} = \mathrm{Re}\left\{\Gamma^{(k_1^3)}_{0;0,0}(\theta, \underset{\sim}{1})\right\}$$

$$= \frac{1}{2k_1+1}\sum_{r+s=2k_1} \cos((r-s)\theta_1)P_r(\xi_+)P_s(\xi_-).$$

The latter expression is the coefficient of β^{2k_1} in the expansion of

$$(2k_1+1)^{-1}\mathrm{Re}[(1-\beta\,\mathrm{e}^{\mathrm{i}(\theta_1+\theta_2+\theta_3)})(1-\beta\,\mathrm{e}^{\mathrm{i}(\theta_1-\theta_2-\theta_3)})(1-\beta\,\mathrm{e}^{\mathrm{i}(-\theta_1+\theta_2-\theta_3)})$$
$$\times (1-\beta\,\mathrm{e}^{\mathrm{i}(-\theta_1-\theta_2+\theta_3)})]^{1/2},$$

which is a symmetric function of θ_1, θ_2 and θ_3. Eq. (1.7) gives $\hat{P}_{(k_1^3)} = (-1)^k\,{}_2F_1(-k_1, k_1+1; \frac{3}{2}; R^2)$.

From Eq. (3.10),

$$\frac{96}{\pi}\int\int\int_{0<\theta_3<\theta_2<\theta_1<\frac{1}{2}\pi} \{\hat{P}_{(k_1^3)}(R^2)\}^2 \prod_{i<j}^{3}(r_j^2 - r_i^2)\mathrm{d}\theta_1\,\mathrm{d}\theta_2\,\mathrm{d}\theta_3 = 1/\mathscr{D}_6((k_1^3)).$$

This result also follows from Eq. (1.2) with $a = \frac{3}{2}$, $c = 3$, and from Eq. (3.9) of Part 1

$$b_{(k_1^3)} = (-1)^k(k_1+1)_{(k_1^3)}/k!, \quad \mathscr{Y}_{(k_1^3)} = \mu_{(k_1^3)}/(3)_{((2k_1)^3)},$$

where $\mu_{(k_1^3)} = C_{(k_1^3)}(I_3) = 2^{2k_1+1}k!/k_1!(2k_1+2)!$.

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Appendix A

Let $T = t_{ij}$ be $n \times m$ and define the Laplacian $\nabla_T^2 = \sum_{i=1}^n \sum_{j=1}^m (\partial/\partial t_{ij})^2$. Then if U is $n \times m$

$$\nabla_T^2 \operatorname{etr}(U'TT'U) = 2 \operatorname{etr}(U'TT'U)\{\operatorname{tr}(U'U \cdot U'TT'U) + m \operatorname{tr}(U'U)\}$$

vanishes identically if $U'U = O$. Hence, for any positive definite symmetric $m \times m$ A,

$$\begin{aligned} 0 &= \nabla_T^2 \int_{O(m)} \operatorname{etr}(AH'U'TT'UH)(\mathrm{d}H) \\ &= \sum_{k \in \mathbf{P}_m} (k!)^{-1}(-1)^k C_\kappa^*(A)\{\nabla_T^2 C_\kappa(U'TT'U)\}. \end{aligned}$$

Hence

$$\nabla_T^2 C_\kappa(U'TT'U) = 0, \quad \kappa \in \mathbf{P}_m,$$

for any $n \times m$ matrix U such that $U'U = O$.

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