Pergamon

Computers Math. Applic. Vol. 34, No. 12, pp. 37–47, 1997

0898-1221/97 $17.00 + 0.00

PII: S0898-1221(97)00232-0

An Extremal Problem of Erdős in Interpolation Theory

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(Received February 1997; accepted June 1997)

Abstract—One of the intriguing problems of interpolation theory posed by Erdős in 1961 is the problem of finding a set of interpolation nodes in $[-1,1]$ minimizing the integral I_n of the sum of squares of the Lagrange fundamental polynomials. The guess of Erdős that the optimal set corresponds to the set F of the Fekete nodes (coinciding with the extrema of the Legendre polynomials) was disproved by Szabados in 1966.

Another aspect of this problem is to find a sharp estimate for the minimal value I_n^* of the integral. It was conjectured by Erdős, Szabados, Varma and Vertesi in 1994 that asymptotically $I_n^* - I_n(F) = o(1/n)$.

In the present paper, we use a numerical approach in order to find the solution of this problem. By applying an appropriate optimization technique, we found the minimal values of the integral with high precision for n from 3 up to 100. On the basis of these results and by using Richardson's extrapolation method, we found the first two terms in the asymptotic expansion of I_n^*, and thus, disproved the above-mentioned conjecture. Moreover, by using some heuristic arguments, we give an analytic description of nodes which are, for all practical purposes, as useful as the optimal nodes.

Keywords—Lagrange interpolation, Fundamental polynomials, Extremal problem.

1. INTRODUCTION

One of the most important problems in interpolation theory is the problem of determining an optimal set of interpolation nodes. According to the numerical analysis approach, the error of approximation is represented in the Lagrangian form

$$f(x) - L_{n-1}(x) = \frac{f^{(n)}(\xi)}{n!} \prod_{k=1}^{n} (x - x_k), \qquad \xi \in [-1,1],$$

and the set of interpolation nodes $\{x_k\}_{k=1}^n$ in $[-1,1]$ is called an optimal set if the sup-norm of the polynomial term in the above representation is minimal. As is well known, the optimal set of nodes in the above sense is the set of the roots of the Chebyshev polynomial of the first kind $T_n(x)$.

According to the functional analysis approach, the criteria for optimality are expressed in terms of the Lagrange fundamental polynomials

$$l_k(x) = \prod_{\substack{j=1 \\ j\neq k}}^{n} \frac{x - x_j}{x_k - x_j}, \qquad k = 1,2,\dots,n.$$

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In 1932, Fejer [1] considered the following extremal problem. Find a set of nodes $X = \{x_1, x_2, \dots, x_n\}$ in $[-1,1]$ minimizing

$$M_n(X) := \max_{-1\le x\le 1} \sigma_n(X;x) := \max_{-1\le x\le 1} \sum_{k=1}^{n} l_k^2(X;x). \tag{1}$$

Fejer proved that $M_n(X)$ is minimal if and only if $X = F$, the set of the Fekete nodes (coinciding with the roots of the integral of the Legendre polynomial $\int_{-1}^{x} P_{n-1}(t)\,dt$). He also showed that $M_n^\star := \min_X M_n(X) = M_n(F) = 1$.

A revival of interest in the above-mentioned extremal problem was motivated by the development in the sixties of the statistical theory of design of experiments. Thus, in the monograph of Karlin and Studden [2], a whole chapter is devoted to the extremal problem (1) and its generalizations to weighted interpolation, interpolation in infinite intervals, and trigonometric interpolation. It should be mentioned that the proofs of Karlin and Studden are based on deep techniques of convex analysis and game theory. In 1979, Balázs [3] found an elementary proof (based on standard arguments of approximation theory) of the result of Karlin and Studden concerning optimal weighted interpolation.

In 1961, Erdős [4] posed a problem similar to (1), where the sup-norm in the criterion of optimality was replaced by the integral norm

$$\min_X I_n(X) := \min_X \int_{-1}^{1} \left[\sum_{k=1}^{n} l_k^2(X;x)\right] dx. \tag{2}$$

By analogy with the solution of (1), Erdős conjectured that the F-set of nodes provides an optimal solution of (2). This conjecture was disproved by Szabados [5] in 1966, and since then the problem of determining an optimal set of points has been considered to be one of the difficult open problems in approximation theory.

Another aspect of this problem is to find a sharp estimate for the minimal value $I_n^\star$ of the integral $I_n(X)$. In his paper of 1968, Erdős [6] claimed (without proof) that the following estimate holds:

$$I_n^\star > 2 - O\left(\frac{\log n}{n}\right). \tag{3}$$

However, the efforts of Hungarian mathematicians to "reproduce" the "proof" of Erdős were unsuccessful, and in a recent paper, Erdős, Szabados, Varma and Vertesi [7] proved the following weaker result:

$$I_n^\star > 2 - O\left(\frac{\log^2 n}{n}\right). \tag{4}$$

On the other hand, they conjectured that asymptotically

$$I_n^\star = 2 - \frac{1}{n} - o\left(\frac{1}{n}\right), \qquad n \to \infty, \tag{5}$$

which, in view of the well-known formula (see, e.g., [8])

$$I_n(F) = 2 - \frac{2}{2n-1} = 2 - \frac{1}{n} - O\left(\frac{1}{n^2}\right), \tag{6}$$

means that $I_n^\star - I_n(F) = o(1/n)$. This conjecture appears also in a more recent paper of Erdős [9].

In the present paper, we treat this problem numerically. By applying an appropriate optimization technique, we found the minimal values of the integral with high precision for n from 3 up to 100. On the basis of these results and by using Richardson's extrapolation method, we found the first two terms in the asymptotic expansion of $I_n^\star$, and thus, disproved the above-mentioned conjecture. Moreover, by using some heuristic arguments, we give an analytic description of the A-nodes which are, for all practical purposes, as useful as the optimal nodes.

The paper is organized as follows. Section 2 deals with the evaluation of M_n and I_n for some important sets of interpolation nodes. Special attention is given to the so-called extended Chebyshev nodes which play an important role in the optimal norm interpolation (see, e.g., [10]).

In Section 3, we present the results of numerical computations and apply them to the analysis of the asymptotic behavior of I_n^* and $I_n(A)$.

2. THE EVALUATION OF M_n AND I_n FOR SPECIFIC SETS OF INTERPOLATION NODES

We start by noting that, for some sets of interpolation nodes, explicit expressions for $M_n(X)$ and $I_n(X)$ may be found in the book of Turetskii [8]. However, in proving these results for different sets of nodes, Turetskii used different arguments. In the following, we derive a general formula for $\sigma_n(X;x)$, valid for an arbitrary set of interpolation points which is the main key in evaluating $M_n(X)$ and $I_n(X)$.

THEOREM 2.1. *Let $X = \{x_1, x_2, \dots, x_n\}$ where $-1 \le x_n < \cdots < x_2 < x_1 \le 1$, and let $\omega_n(x) = \omega_n(X;x) = (x-x_1)(x-x_2)\cdots(x-x_n)$. Then the function $\sigma_n(X;x)$ defined by (1) has the following representation:*

$$\sigma_n(x) = \sigma_n(X;x) = 1 + \omega_n^2(x) \sum_{k=1}^{n} \frac{\omega_n''(x_k)}{[\omega_n'(x_k)]^3} \frac{1}{x - x_k}. \tag{7}$$

PROOF. Since $\sigma_n(x_k) = 1$ $(k = 1, 2, \dots, n)$, $\sigma_n(x)$ may be represented in the form

$$\sigma_n(x) = 1 + \omega_n(x) q_{n-2}(x), \tag{8}$$

where $q_{n-2}(x)$ is a polynomial of degree at most $n-2$. Differentiation of (8) yields

$$2 \sum_{j=1}^{n} l_j(x) l_j'(x) = \omega_n'(x) q_{n-2}(x) + \omega_n(x) q_{n-2}'(x),$$

from which it follows for $x = x_k$,

$$2l_k'(x_k) = \omega_n'(x_k)\, q_{n-2}(x_k).$$

Taking into account the well-known formula (see, e.g., [11, p. 24])

$$l_k'(x_k) = \frac{\omega_n''(x_k)}{2\omega_n'(x_k)},$$

we find

$$q_{n-2}(x_k) = \frac{\omega_n''(x_k)}{[\omega_n'(x_k)]^2}, \qquad k = 1, 2, \dots, n. \tag{9}$$

Let $y_k = \omega_n''(x_k)/[\omega_n'(x_k)]^2$. Then, $q_{n-2}(x)$ is the interpolating polynomial for the table $\{x_k, y_k\}_{k=1}^n$, and since $q_{n-2}(x)$ is of degree at most $n-2$, its $(n-1)^{\text{th}}$ divided difference vanishes:

$$q_{n-2}[x_1, x_2, \dots, x_n] = \sum_{k=1}^{n} \frac{y_k}{\omega_n'(x_k)} = 0.$$

Thus, we obtained, as a by-product, the following identity valid for an arbitrary set of nodes:

$$\sum_{k=1}^{n} \frac{\omega_n''(x_k)}{[\omega_n'(x_k)]^3} = 0. \tag{10}$$

Moreover, $q_{n-2}(x)$ may be represented in the Lagrangian form

$$q_{n-2}(x) = \sum_{k=1}^{n} \frac{\omega_n''(x_k)}{[\omega_n'(x_k)]^2} \frac{\omega_n(x)}{\omega_n'(x_k)(x-x_k)} = \omega_n(x) \sum_{k=1}^{n} \frac{\omega_n''(x_k)}{[\omega_n'(x_k)]^3} \frac{1}{(x-x_k)}. \tag{11}$$

Substitution of (11) in (8) yields the desired result. ∎

Next, we apply this general formula to the the set T of the Chebyshev roots in order to find a new representation for $\sigma_n(T;x)$.

THEOREM 2.2. *Let* $X = T := \{\xi_k^{(n)} = \cos[(2k-1)\pi/(2n)]\}_{k=1}^{n}$. *Then for* $n \geq 2$, *the following representations hold:*

$$\sigma_n(T;x) = 1 + \frac{1}{n} U_{n-2}(x) T_n(x) = \frac{2n-1}{2n} + \frac{1}{2n} U_{2n-2}(x), \tag{12}$$

where $U_n(x)$ *is the Chebyshev polynomial of the second kind of degree* n.

PROOF. Note first that for the Chebyshev roots $\omega_n(T;x) = 2^{-(n-1)} T_n(x)$, and therefore, (7) yields

$$\sigma_n(T;x) = 1 + T_n^2(x) \sum_{k=1}^{n} \frac{T_n''\left(\xi_k^{(n)}\right)}{\left[T_n'\left(\xi_k^{(n)}\right)\right]^3} \frac{1}{\left(x - \xi_k^{(n)}\right)}.$$

Further, by using the differential equation of the Chebyshev polynomials

$$\left(1 - x^2\right) T_n''(x) - x T_n'(x) + n^2 T_n(x) = 0,$$

we find

$$\frac{T_n''\left(\xi_k^{(n)}\right)}{T_n'\left(\xi_k^{(n)}\right)} = \frac{\xi_k^{(n)}}{1 - \left(\xi_k^{(n)}\right)^2},$$

and since $T_n'(\xi_k^{(n)}) = (-1)^{k-1} n / \sqrt{1 - (\xi_k^{(n)})^2}$, we have

$$\sigma_n(T;x) = 1 + \frac{T_n^2(x)}{n^2} \sum_{k=1}^{n} \frac{\xi_k^{(n)}}{x - \xi_k^{(n)}}.$$

Thus, in order to prove the first representation in (12), we have to verify that the following identity holds:

$$\frac{U_{n-2}(x)}{T_n(x)} = \frac{1}{n} \sum_{k=1}^{n} \frac{\xi_k^{(n)}}{x - \xi_k^{(n)}}. \tag{13}$$

But $U_{n-2}(x)/T_n(x)$ has the unique partial fraction decomposition of the form

$$\frac{U_{n-2}(x)}{T_n(x)} = \sum_{k=1}^{n} \frac{c_k}{x - \xi_k^{(n)}},$$

where c_k is defined by the formula for the residue at the simple pole as follows:

$$c_k = \operatorname{Res} \left[\frac{U_{n-2}(x)}{T_n(x)}\right]_{x=\xi_k^{(n)}} = \frac{U_{n-2}\left(\xi_k^{(n)}\right)}{T_n'\left(\xi_k^{(n)}\right)} = \frac{T_{n-1}'\left(\xi_k^{(n)}\right)}{(n-1) T_n'\left(\xi_k^{(n)}\right)} = \frac{\xi_k^{(n)}}{n}.$$

This proves the first representation in (12). Finally, the second formula in (12) follows from the first by using $2U_{n-2}(x)T_n(x) = U_{2n-2}(x) - 1$. ∎

Note, that by applying (12), one can easily derive the following well-known formulas (see, e.g., [8]):

$$M_n(T) = 2 - \frac{1}{n}, \tag{14}$$

$$I_n(T) = 2 - \frac{2}{2n-1} + \frac{2}{n(2n-1)}. \tag{15}$$

Now let us consider the set $\hat{T}$ of the extended Chebyshev nodes obtained from the T-set by a linear transformation which maps the first and the last nodes to ± 1, namely $\hat{T} = \cos[(2k-1)\pi)/(2n)]/\cos[\pi/(2n)]$. Note that in the minimum norm interpolation problem, this set of nodes is a very good approximation to the optimal set (see, e.g., [3]). It will be shown shortly that the extended Chebyshev nodes are of special importance in our study as well.

THEOREM 2.3. *For $n \geq 2$, the following estimates hold:*

$$M_n\left(\hat{T}\right) \leq \frac{5}{4}\left(1 - \frac{1}{2n}\right), \tag{16}$$

$$I_n\left(\hat{T}\right) = 2 - \frac{2}{2n-1}. \tag{17}$$

PROOF. We start by proving (16). It may be easily verified that

$$\sigma_n\left(\hat{T}; x\right) = \sigma_n\left(T; \xi_1^{(n)} x\right), \qquad \text{where } \xi_1^{(n)} = \cos\frac{\pi}{2n}, \tag{18}$$

and hence,

$$M_n\left(\hat{T}\right) = \max_{-1 \leq x \leq 1} \sigma_n\left(T; \xi_1^{(n)} x\right) = \max_{-\xi_1^{(n)} \leq x \leq \xi_1^{(n)}} \sigma_n(T; x).$$

Thus, by applying the representation (12), we find

$$M_n\left(\hat{T}\right) = \frac{2n-1}{2n} + \frac{1}{2n} \max_{-\xi_1^{(n)} \leq x \leq \xi_1^{(n)}} U_{2n-2}(x). \tag{19}$$

Next,

$$|U_n(x)| = \frac{|\sin(n+1)\arccos x|}{\sqrt{1-x^2}} \leq \frac{1}{\sqrt{1-x^2}}, \qquad x \in (-1, 1),$$

with equality only at the points $x = \cos[(2k-1)\pi/(2n+2)]$, $k = 1, 2, \ldots n+1$. Therefore, denoting the roots of $U_n(x)$ by $\{\eta_k^{(n)} = \cos[k\pi/(n+1)]\}_{k=1}^{n}$, we have

$$\begin{aligned} \max_{-\xi_1^{(n)} \leq x \leq \xi_1^{(n)}} U_{2n-2}(x) &= \max_{\eta_3^{(2n-2)} \leq x \leq \eta_2^{(2n-2)}} U_{2n-2}(x) \leq \frac{1}{\sqrt{1 - \left[\eta_2^{(2n-2)}\right]^2}} \\ &= \frac{1}{\sin(2\pi/2n-1)} \leq \frac{1}{4}(2n-1). \end{aligned} \tag{20}$$

Substitution of (20) in (19) completes the proof of the estimate (16).

Let us turn to the proof of (17). By using representation (12), once again we get

$$I_n\left(\hat{T}\right) = \frac{2n-1}{n} + \frac{1}{2n}\int_{-1}^{1} U_{2n-2}\left(x\xi_1^{(n)}\right) dx = \frac{2n-1}{n} + \frac{1}{2n\xi_1^{(n)}}\int_{-\xi_1^{(n)}}^{\xi_1^{(n)}} U_{2n-2}(z)\, dz.$$

But as is well known, $\int U_n(x)\,dx = T_{n+1}(x)/(n+1)$, and therefore,

$$I_n\left(\hat{T}\right) = \frac{2n-1}{n} + \frac{T_{2n-1}\left(\xi_1^{(n)}\right)}{\xi_1^{(n)} n(2n-1)}.$$

It remains to note that $T_{2n-1}(\xi_1^{(n)}) = -\xi_1^{(n)}$ and the result follows. ∎

REMARK. Comparison of (6) and (17) leads to the following surprising identity:

$$I_n\left(\hat{T}\right) = I_n(F), \qquad n = 2, 3, \dots. \tag{21}$$

Figure 1 illustrates the equality of integrals of two different functions for the special case $n = 6$. It should also be mentioned that the estimate (16) is a bit conservative and $M_n(\hat{T})$ is very close to the minimal value 1. Finally, it is worthwhile to emphasize that, from a practical point of view, it is much more convenient to work with the extended Chebyshev nodes rather than with the Fekete ones, since for these nodes, we have an explicit and simple representation.

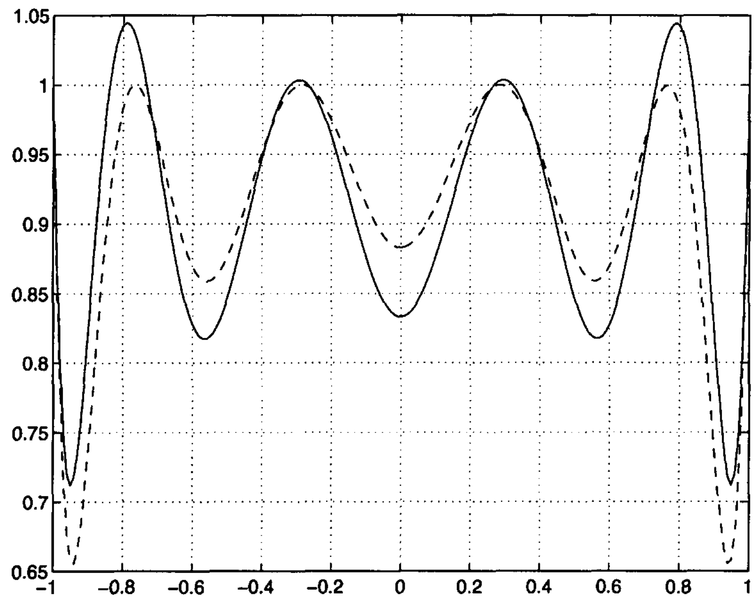

Figure 1. The functions $\sigma_6(\hat{T}; x)$ (solid curve) and $\sigma_6(F; x)$ (dashed curve).

3. NUMERICAL RESULTS AND ASYMPTOTIC BEHAVIOR OF $I_n^\star$

It is clear that the extremal problem (2) may be treated as the following constrained optimization problem. Minimize

$$I_n(x_1, x_2, \dots, x_n) = \int_{-1}^{1} \sigma_n(x_1, x_2, \dots, x_n, x)\,dx, \tag{22}$$

where

$$\sigma_n(x_1, x_2, \dots, x_n, x) = \sum_{k=1}^{n} \prod_{\substack{j=1\\ j\neq k}}^{n} \left(\frac{x - x_j}{x_k - x_j}\right)^2 \tag{23}$$

under the constraints

$$-1 \le x_1 < x_2 < \cdots < x_n \le 1. \tag{24}$$

Our approach consists in applying an appropriate optimization technique in order to find the optimal values of the integral $I_n^\star$, as well as the optimal sets of nodes $X_n^\star$ numerically. Unfortunately, the optimization problem we are dealing with is a nonstandard, difficult problem due to the fact the integral $I_n(x_1, x_2, \ldots, x_n)$ is a complicated rational function of the nodes. To overcome this difficulty, at the first stage of our study, the simple direct search method of Hooke and Jeeves [12] was applied in order to find the optimal solution numerically for small values of n (up to 16). Since the integrand $\sigma_n(x_1, x_2, \ldots, x_n, x)$ in (22) is a polynomial of degree $2n-2$ with respect to x, for numerical evaluation of the integrals, the Gaussian quadrature formula was used with the number of nodes sufficiently large to guarantee that the evaluation of the integral is exact (see, e.g., [13]) To make the results more reliable, the initial vector $(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)})$ was chosen randomly. The numerical results obtained strongly suggest that *the solution* of the extremal problem (22)–(24) is *unique*, the *optimal nodes* are *symmetric* on $[-1, 1]$, and *include the end-points* of the interval.

At the second stage, our purpose was to find the optimal solution for large values of n and with great precision. To this end, taking into account that $I_n(x_1, x_2, \ldots, x_n)$ is a smooth function of the nodes, the second-order Newton method was used. In numerical computations, explicit analytic expressions for the gradient and the Hessian of $\sigma_n(x_1, x_2, \ldots, x_n, x)$ have been used along with the Gaussian quadrature formula. The set of the extended Chebyshev nodes was used as an initial guess. It should be mentioned that theoretical results concerning the convergence of the Newton method (see, e.g., [14]) are based on the assumption that the minimizing function is convex. To verify the convexity of $I_n(x_1, x_2, \ldots, x_n)$ numerically, the Cholesky decomposition was used in solving the corresponding linear systems. The numerical experiment strongly suggests that, at least in the neighborhood of the optimal solution, the integral $I_n(x_1, x_2, \ldots, x_n)$ is a convex function of the nodes.

The computations have been performed on a supercomputer Cray J92 using code written in C++ with 30 decimal digit accuracy. In all the cases, the tolerance of 10^{-28} was achieved after a few iterations. We ran the program for degrees from 3 to 50 and found the optimal values of the integral $I_n^\star$ with at least 25 correct digits.

These numerical values have been used in order to verify an asymptotic behavior of $I_n^\star$. To this end, we assume that

$$I_n^\star = 2 - \frac{c_1}{n} - \frac{c_2}{n^2} - \cdots, \tag{25}$$

and define the sequence $d_n^\star (n = 1, 2, \ldots)$ by

$$d_n^\star := (2 - I_n^\star)\, n = c_1 + \frac{c_2}{n} + \cdots. \tag{26}$$

Then

$$\lim_{n\to\infty} d_n^\star = c_1. \tag{27}$$

But this sequence tends to its limit very slowly (see Table 1), and therefore, to accelerate the convergence, Richardson's extrapolation technique was used (see, e.g., [15]).

To describe Richardson's extrapolation method, let $\{S_n\}_{n=1}^N$, with $N > 2$, be a given sequence of real numbers. On setting $\mathbb{R}_0^{(n)} := S_n (n = 1, 2, \ldots, N)$, regard $\{\mathbb{R}_0^{(n)}\}_{n=1}^N$ as the zeroth column of the Richardson extrapolation table for $\{S_n\}_{n=1}^N$. The first column of the Richardson extrapolation table, consisting of $N-1$ numbers, is defined by

$$\mathbb{R}_1^{(n)} := \frac{x_n \mathbb{R}_0^{(n+1)} - x_{n+1}\mathbb{R}_0^{(n)}}{x_n - x_{n+1}}, \qquad (n = 1, 2, \ldots, N-1),$$

and inductively, the $(k+1)^{\text{st}}$ column of the Richardson extrapolation table, consisting of $N-k-1$ numbers, is defined by

$$\mathbb{R}_{k+1}^{(n)} := \frac{x_n \mathbb{R}_k^{(n+1)} - x_{n+k+1}\mathbb{R}_k^{(n)}}{x_n - x_{n+k+1}}, \qquad (n = 1, 2, \ldots, N-k-1),$$

for each $k = 0, 1, \ldots, N-2$, where the $\{x_n\}_{n=1}^N$ are given constants.

Table 1. The values of the sequence $\{d_n^*\}_{n=20}^{50}$ along with the 11^{th} and 12^{th} columns of the Richardson extrapolation table.

n	d_n^*	Rich11	Rich12
20	1.09953053690152049130	1.094219687170	1.094219716687
21	1.09919590402847313501	1.094219698239	1.094219724898
22	1.09889868217055995621	1.094219707933	1.094219732104
23	1.09863312670444988771	1.094219716464	1.094219738463
24	1.09839458803132105327	1.094219724006	1.094219744093
25	1.09817926618389322777	1.094219730702	1.094219749101
26	1.09798402792476931384	1.094219736669	1.094219753577
27	1.09780626866681586523	1.094219742008	1.094219757578
28	1.09764380700083271896	1.094219746799	1.094219761188
29	1.09749480325512798259	1.094219751116	1.094219764436
30	1.09735769598056354276	1.094219755014	1.094219767368
31	1.09723115195540070277	1.094219758544	1.094219770059
32	1.09711402649254932819	1.094219761757	1.094219772436
33	1.09700533167298618196	1.094219764670	1.094219774730
34	1.09690421073185187532	1.094219767352	1.094219776619
35	1.09680991726056868002	1.094219769770	1.094219778677
36	1.09672179820828702671	1.094219772044	1.094219780085
37	1.09663927990267148243	1.094219774054	1.094219782028
38	1.09656185648678350698	1.094219776007	1.094219782964
39	1.09648908030195654915	1.094219777677	
40	1.09642055384767953874		
41	1.09635592302690784070		
42	1.09629487144490689485		
43	1.09623711557608017652		
44	1.09618240064945976978		
45	1.09613049713203259919		
46	1.09608119771162252011		
47	1.09603431469899101332		
48	1.09598967778317498522		
49	1.09594713208562559890		
50	1.09590653646804320520		

In our case, in accordance with (26), $x_n = 1/n$. To conserve space, in Table 1, we give the values of the subsequence $\{d_n^*\}_{n=20}^{50}$, along with the 11^{th} and 12^{th} columns of the Richardson extrapolation method applied to it. The values of d_n^* have been truncated to 20 decimal digits, while the results of extrapolation are given with 12 decimal digits. In addition to this, in Table 2, we present the diagonal elements of the first 12 columns of the Richardson table.

It is evident from these tables that the constant c_1 in the asymptotic expansion of I_n^* is different from 1. More precisely, with at least seven significant digits, we have

$$c_1 = 1.094219\ldots. \tag{28}$$

This result disproves the above-mentioned conjecture of Erdős *et al.* It also raises the question of finding another set of interpolation nodes (instead of the extended Chebyshev nodes), which may be described analytically and may be considered an "almost optimal" set.

In the process of searching for the solution of this problem, several heuristic arguments have been examined. In the following, we describe and analyze two configurations of nodes based on heuristic arguments.

(1) The first heuristic (which was suggested by Szabados), consists of using the so-called double extended Chebyshev nodes $\tilde{T}$, obtained from the $\hat{T}$-nodes by repeated extension of its inner

Table 2. Diagonal elements of the Richardson extrapolation table for the sequence $\{d_n^*\}_{n=20}^{50}$.

k	$\mathbb{R}_k^{(50-k)}$
1	1.0939 1735
2	1.0942 1525
3	1.0942 1858
4	1.0942 1936
5	1.0942 1957
6	1.0942 1967
7	1.0942 1971
8	1.0942 1974
9	1.0942 1976
10	1.0942 1977
11	1.0942 1978
12	1.0942 1978

points, namely

$$\tilde{T} = \left\{ x_k = \frac{\cos[(2k-1)\pi)/(2n)]}{\cos^2[\pi/(2n)]},\ k = 2,3,\ldots,n-1,\ x_1 = -x_n = 1 \right\}. \tag{29}$$

In order to check the efficiency of this heuristic, we have calculated numerically the values of $I_n(\tilde{T})$ for n in the range 3–100. These results are presented in Table 3, along with the values of the integral for some other important sets of nodes, as well as with the optimal values of the integral. As we see from this table, the repeated extension leads to a decrease in the value of the integral (i.e., $I_n(\tilde{T}) < I_n(\hat{T})$, $n \geq 5$). However, this improvement is insignificant, since the values $I_n(\tilde{T})$ are much closer to $I_n(\hat{T})$ than to the optimal values I_n^*. Indeed, we performed the analysis of the asymptotic behavior of $I_n(\tilde{T})$ (in a way similar to that which was used in the analysis of I_n^*) and found that asymptotically

$$I_n\left(\tilde{T}\right) = 2 - \frac{1}{n} + o\left(\frac{1}{n}\right). \tag{30}$$

Thus, the asymptotic behaviors of $I_n(\tilde{T})$ and $I_n(\hat{T})$ are similar.

(2) Another heuristic was motivated by the results of a numerical experiment. Note first that, as was shown in the previous section, for two different sets of nodes, the Fekete nodes and the extended Chebyshev nodes, the values of the integral are the same. Therefore, one can assume that the optimal set corresponds to the location of nodes somewhere between these two specific sets. Indeed, the numerical results indicate that (for fixed $n \geq 4$), the optimal set of nodes X^* is located approximately in the middle between the $\hat{T}$-set and the F-set. Unfortunately, there is no explicit expression for the Fekete nodes, and to overcome this difficulty, we start with the following observation.

It is known (see, e.g., [16]) that if the Fekete nodes of order n are written in the form $x_k = \cos\theta_k$ $(k = 1,2,\ldots,n)$, where $0 = \theta_1 < \theta_2 < \cdots < \theta_n = \pi$, then

$$\frac{2k-2}{2n-1}\pi \leq \theta_k \leq \frac{2k-1}{2n-1}\pi, \qquad k = 1,2,\ldots,n. \tag{31}$$

In view of this inequality, it seems reasonable to consider the set of nodes $\tilde{F}$ which in addition to the end-points of the interval $[-1,1]$ will contain the points of the form $\cos\tilde{\theta}_k$ $(k = 2,3,\ldots,n-1)$, where $\tilde{\theta}_k$ is the arithmetic mean of the left-hand and right-hand sides of (31), namely

$$\tilde{F} := \left\{ x_k = \cos\frac{(4k-3)\pi}{4n-2},\ k = 2,3,\ldots,n-1;\ x_1 = -x_n = 1 \right\}. \tag{32}$$

Table 3. The values of the integral for several specific sets of nodes in comparison with the optimal values.

n	ExtCheb	DoubleExt	NFekete	Average	Optimal
3	1.600000	1.600000	1.600000	1.600000	1.600000
4	1.714286	1.714782	1.710892	1.711326	1.710758
5	1.777778	1.775415	1.773250	1.773147	1.772609
6	1.818182	1.814176	1.813625	1.812806	1.812367
7	1.846154	1.841486	1.841942	1.840516	1.840165
8	1.866667	1.861869	1.862889	1.861011	1.860728
9	1.882353	1.877687	1.879001	1.876803	1.876571
10	1.894737	1.890321	1.891767	1.889352	1.889159
15	1.931034	1.928026	1.929331	1.926618	1.926527
20	1.948718	1.946653	1.947634	1.945076	1.945023
25	1.959184	1.957697	1.958436	1.956107	1.956073
30	1.966102	1.964986	1.965556	1.963445	1.963421
35	1.971014	1.970147	1.970599	1.968680	1.968663
40	1.974684	1.973990	1.974356	1.972603	1.972589
45	1.977528	1.976962	1.977264	1.975652	1.975642
50	1.979798	1.979327	1.979580	1.978091	1.978082
55	1.981651	1.981253	1.981469	1.980085	1.980078
60	1.983193	1.982852	1.983038	1.981746	1.981740
65	1.984496	1.984201	1.984362	1.983152	1.983147
70	1.985611	1.985353	1.985495	1.984356	1.984352
75	1.986577	1.986350	1.986475	1.985400	1.985396
80	1.987421	1.987219	1.987331	1.986313	1.986310
85	1.988166	1.987985	1.988085	1.987119	1.987116
90	1.988827	1.988664	1.988754	1.987835	1.987832
95	1.989418	1.989271	1.989353	1.988476	1.988473
100	1.989950	1.989816	1.989890	1.989052	1.989050

This set of nodes will be called the Near-Fekete nodes. Note that this set consists of $n-2$ roots of the Chebyshev polynomial $T_{2n-1}(x)$ (the roots with odd indices excluding the first and the last).

The values of the integral for the Near-Fekete nodes have been computed numerically, and they are presented in the fourth column of Table 3. It is worthwhile to indicate that there is a slight decrease in the values of the integral $I_n(\tilde{F})$ in comparison with $I_n(F)$.

Finally, we define the A-set of nodes as the arithmetic mean of the $\tilde{F}$- and the $\hat{T}$-sets, namely

$$A := \begin{cases} x_k = \dfrac{1}{2}\left[\dfrac{\cos((2k-1)\pi)/(2n))}{\cos^2(\pi/(2n))} + \cos\dfrac{(4k-3)\pi}{4n-2}\right], & k = 2,3,\dots,n-1, \\ x_1 = -x_n = 1. \end{cases} \tag{33}$$

This set of nodes will be called the Average set. The values of $I_n(A)$ computed numerically are presented in Table 3. Comparison of these values with the optimal values of the integral $I_n^\star$ shows that $I_n(A)$ are very close to $I_n^\star$. Thus, we conclude that the readily available Average set of nodes is, *for all practical purposes, as useful as the optimal set.*

Moreover, we have performed the analysis of the asymptotic behavior of $I_n(A)$ following the same method which was used for the analysis of $I_n^\star$. Namely, under the assumption that

$$I_n(A) = 2 - \frac{b_1}{n} - \frac{b_2}{n^2} - \cdots, \tag{34}$$

we have calculated numerically (with the precision of 26 decimal digits), the sequence $e_n(A) = [2-I_n(A)]*n$. In order to be consistent with our previous analysis of the sequence $d_n^\star$, Richardson's

Table 4. Diagonal elements of the Richardson extrapolation table for the sequence $\{e_n(A)\}_{n=20}^{50}$.

k	$\mathbb{R}_k^{(50-k)}$
1	1.0939 1146
2	1.0942 1955
3	1.0942 2035
4	1.0942 2022
5	1.0942 2005
6	1.0942 1996
7	1.0942 1991
8	1.0942 1988
9	1.0942 1986
10	1.0942 1985
11	1.0942 1984
12	1.0942 1984

extrapolation method was applied to the subsequence $\{e_n(A)\}_{n=20}^{50}$. The diagonal elements of the first 12 columns of the Richardson table are presented in Table 4.

On the basis of these results, we conclude that at least with seven significant digits, we have

$$b_1 = 1.094219\ldots. \tag{35}$$

Comparison of (28) and (35) leads to the following conjecture.

CONJECTURE. *The following asymptotic relationship holds:*

$$I_n^\star - I_n(A) = o\left(\frac{1}{n}\right), \qquad n \to \infty. \tag{36}$$

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