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Guarding disjoint triangles and claws in the plane

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Abstract

We consider the problem of guarding triangles in the plane and show that $\lfloor (5n+2)/4 \rfloor$ guards can monitor the boundaries and the free space of n disjoint triangles. This improves the best previously known upper bound $\lfloor 4n/3 \rfloor + 1$ due to Hoffmann, Kaufmann and Kriegel. We also consider the analogous problem for n disjoint claws in the plane and show that $3n/2 + \mathrm{O}(1)$ guards are always sufficient and $3n/2 - \mathrm{O}(1)$ are sometimes necessary.

Keywords: Art gallery; Planar graph; Matching

1. Introduction

What is the minimal number of guards that can jointly monitor the boundaries of any n disjoint closed triangular domains (for short, *triangles*) in the Euclidean plane? This is an interesting special case of a more general problem of guarding the boundary of n disjoint convex compact sets in the plane, which was considered by Fejes Tóth [9]. For n disjoint homothetic triangles, there are almost matching upper and lower bounds $n+1$ and n [5]. It is conjectured to be $n+c$ for triangles [19] with some positive constant c. The best previously known upper bound, $\lfloor 4n/3 \rfloor + 1$, follows from the result of Hoffmann, Kaufmann and Kriegel [2,12]: Every polygon with n vertices and h (polygonal) holes can be monitored by $\lfloor (n+h)/3 \rfloor$ point guards. Their bound is tight for polygons with quadrilateral holes, but it is not known to be tight for polygons with triangular holes.

Given a set B of objects in the plane, a set S of point guards collectively *monitor* a point set P if for every point $p \in P$ there is a point guard $s \in S$ such that the open line segment sp is disjoint from

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all objects in B. They *monitor* the set B of objects if they monitor the boundary of every object in B. Interestingly, whenever the boundaries of our objects are closed Jordan curves, all known bounds equally apply to guarding the *free space* around the objects, which is the complement of the union of the objects. We prove the following theorem.

Theorem 1. *Given a set T of n, $n \geqslant 2$, disjoint triangles in the plane ($\mathbb{E}^2$) in general position, we can place at most $\lfloor (5n+2)/4 \rfloor$ guards at points of $\mathbb{E}^2 \setminus \bigcup T$ such that they collectively monitor the boundary of every triangle in T and the free space $\mathbb{E}^2 \setminus \bigcup T$.*

General position means that the triangles have no three collinear vertices.

Note that our notion of guarding is different from Hadwiger's notion for illuminating a convex body by light sources [10]. There, a light source s illuminates a point p on the boundary of an object b if the open line segment sp is disjoint from all objects and the ray $\overrightarrow{sp}$ intersects the interior of b. Our method do not support this type of illumination at the vertices of the triangles.

If the objects are closed polygonal domains, a variant of the problem asks for the maximum number of necessary *vertex guards*, that is, guards located at vertices of the objects. By our definition of visibility, a guard at a vertex v cannot see the sides of an object incident to v. Therefore we require the vertex guards to monitor the free space only. For a set T of n disjoint triangles in the plane [14], there is no better upper bound on the number of vertex guards than $\lfloor 5n/3 \rfloor$, which is an easy consequence of a theorem of Chvátal [4,8] on guarding simple polygons. The best known lower bound, $\lfloor 4n/3 \rfloor$, follows from a construction where every triangle has a common side with the convex hull $\operatorname{conv} \bigcup T$. If the exterior of the convex hull does not need to be monitored by vertex guards, then we have the following theorem.

Theorem 2. *Given a set T of n, $n \geqslant 2$, disjoint triangles in the plane in general position, we can place at most $\lfloor (5n+2)/4 \rfloor$ guards at points of the set of all vertices of triangles and two points of the free space outside the convex hull of the triangles such that they collectively monitor the the free space $\mathbb{E}^2 \setminus \bigcup T$.*

L. Fejes Tóth [9] showed that $4n-7$ point guards are always sufficient and sometimes necessary to monitor n, $n \geqslant 3$, disjoint convex sets and their free space. The problem is basically settled for n disjoint rectilinear rectangles [11,19], too, where the maximum number of necessary point guards is between $n-1$ and $n+1$ in the worst case. It is conjectured [19] that $n+c$ point guards always suffice to monitor a set Q of n disjoint convex quadrilaterals, c is a constant. García-López [3,8] showed that $2n$ vertex guards are always sufficient and sometimes necessary to monitor the free space of Q. $\lfloor (n+1)/2 \rfloor$ point guards can monitor a n disjoint line segments, and $\lfloor (4n+1)/5 \rfloor$ guards are always sufficient and sometimes necessary to monitor their free space [7,18]. n is both an upper and a worst case lower bounds on the number of vertex guards for n disjoint segments. Variants of monitoring disjoint line segment were considered in [15] and [17] where the definition of visibility is relaxed so that the open line segment between the guard and the target should not properly *cross* the given segments.

Our proof is based upon the so-called *matching technique* [6], used in [15] and [18] for line segments where it yields tight bounds. We partition the *free space* $F = \mathbb{E}^2 \setminus \bigcup T$ into convex regions such that a guard on the common boundary of two or more regions can monitor all incident regions. A small click cover (or in our case, a matching) of regions with common boundary points gives a small set of guard locations. The free space of n disjoint triangles can be partitioned into $2n+1$ convex regions. Sometimes, this is the smallest number of regions a convex partition can have, and at the same time no three convex regions have a common boundary point. In this scenario, matching technique would only yield an upper bound of $n + O(1)$, the best possible bound, if a nearly perfect matching on the regions with common boundaries exists. The lower bound we can establish for the maximum matching on the regions gives only the bound stated in Theorem 1.

Inspired by the methods used, we investigate another problem, that of guarding the free space of n disjoint *claws*.

Definition. A *claw* is the union of three closed line segments with one common endpoint such that the angle between any two consecutive segments is less than π.

The *inner vertex* of a claw is the common endpoint of the segments, the *outer vertices* are the three other segment endpoints.

The analogon of Theorem 1 for claws gives a tight bound apart from an additive constant.

Theorem 3. *Assume that we are given a set L of n, $n \geqslant 2$, disjoint claws in the plane. $\lfloor 3n/2 \rfloor$ guards at points of $\mathbb{E}^2 \setminus \bigcup L$ are always sufficient and $\lfloor 3n/2 \rfloor - 2$ are sometimes necessary to monitor the free space $\mathbb{E}^2 \setminus \bigcup L$.*

The rest of the paper is organized as follows. In Section 2, we describe a simple convex partitioning of the free space of disjoint polygonal objects. Section 3 contains the proofs of Theorems 1 based upon the maximum matching of the dual graph defined on the convex faces. Then Section 4 discusses the claws and the proof of Theorem 2. The proof of a lemma about the convex partition of the free space is postponed to Sections 5 and 6.

2. Convex partitioning of the free space

We introduce some basic notation.

Free space. Let B be the set of disjoint closed objects in the Euclidean plane. The *free space* F_B is the open set $\mathbb{E}^2 \setminus \bigcup B$. The free space is *polygonal* if its boundary is the union of finitely many line segments. The *vertices* of the free space are the segment endpoints in a minimum set of segments along the boundary. A vertex v of F_B is called *reflex* if F_B contains an angular domain with apex at v of size greater than π. In particular, if T is a set of n disjoint triangles, then F_T is polygonal, its vertices are exactly the $3n$ vertices of the triangles, and every vertex is reflex. If L is a set of n disjoint claws, then F_L is polygonal, its vertices are the $4n$ vertices of the claws, and its reflex vertices are the $3n$ outer vertices of the claws.

A *convex partition* for the polygonal free space F_B of a set B of objects is a finite set C of convex open non-overlapping polygonal domains such that $\bigcup\{\mathrm{cl}(f)\colon f \in C\} = \mathrm{cl}(F_B)$ and $f \cap b = \emptyset$ for all pairs $f \in C$, $b \in B$. The elements in C are called *faces* of the partition.

Points in infinity. We want that the boundary of every face of a convex partition is a closed polygonal curve. For this purpose, we add points in infinity to the the Euclidean plane $\mathbb{E}^2$. We associate a point in infinity to every equivalence class of rays in the plane with respect to translation and denote the set of Euclidean and infinite points by $\overline{\mathbb{E}^2}$. A *vertex* of a polygonal face can be a ray on its boundary, that is, a point in infinity. The closure $\mathrm{cl}(f)$ or $\mathrm{cl}(F)$ of a face f or the free space F can be understood in $\overline{\mathbb{E}^2}$. A convex face has at most two vertices in infinity. (This is not the only possible way to assure closed boundaries. Another method would be to consider a triangle τ_0 containing all objects of B in its interior [15]. Then, we would have three additional vertices, namely the vertices of τ_0, and our argument would give a slightly weaker bound than that of Theorem 1.)

One convex partition is produced by the following naive *convex partitioning* algorithm if the free space is polygonal and there are no three collinear vertices.

CONVPART: Input: a polygonal free space F in general position.
For each reflex vertex v of F, let $\alpha(v)$ be a ray emanating from v which partition the reflex angle at v into two convex or straight angles and which does not pass trough any other reflex vertex of F. Draw consecutively a directed segment $\beta(v)$ from every reflex vertex v along $\alpha(v)$ until it hits a previously drawn segment $\beta(v')$ or the boundary of F (possibly in infinity).
Output: convex partition.

The convex partition produced in this manner is not unique: it depends on the directions of the rays $\alpha(v)$ and on the order in which the segments $\beta(v)$ are drawn. For each $\beta(v)$ of the partition, we define a directed polygonal curve $\gamma(v)$ which we call *bar*. A bar $\gamma(v)$ starts from v and ends at the boundary of the free space F (possibly in infinity). Let $v_0 = v$ and $i = 0$. The bar $\gamma(v)$ starts at v_0 and it follows $\beta(v_i)$, $i = 0, 1, \ldots,$ to its endpoint as long as the this endpoint lies in the relative interior of another segment $\beta(v_{i+1})$, for some v_{i+1}.

Proposition 1. *The convex partition of F_B returned by* CONVPART *has always $r + 1 - n$ faces, where r denotes the number of reflex vertices of F_B and n is the number of disjoint objects in B.*

Proof. After every step where the algorithm CONVPART draws the segments $\beta(v_1), \beta(v_2), \ldots, \beta(v_r)$ (one for each reflex vertex of F_B), consider the sets $F(j) := F_B \setminus \bigcup_{i=1}^{j} \beta(v_i)$. By adding one new segment $\beta(.)$, either the number of connected components of $F(j+1)$ increases by one, or the total number of holes in all components of $F(j+1)$ drops by one. As $F(r)$ is a collection of disjoint convex sets without holes, there must be exactly n steps for which the number of holes drops by one. Therefore $r - n$ steps increase the number of connected components by one each. □

Proposition 2. *For every face f of a convex partition returned by* CONVPART*, and for every point a on the boundary of f but not on the boundary of the free space F, there is a reflex vertex v of F such that $a \in \gamma(v)$ and the portion of $\gamma(v)$ between v and a is on the boundary of f.*

Proof. There is no sequence $\beta(v_1), \beta(v_2), \ldots, \beta(v_t)$ where $\beta(v_i)$ hits $\beta(v_{i+1})$, $i = 1, \ldots, t-1$, and $\beta(v_t)$ hits $\beta(v_1)$, because the segments $\beta(v_i)$ are drawn consecutively. Point a belongs to a segment $\beta(v)$. If we follow the segments $\beta(.)$ from a on the boundary ∂f in reverse orientation, the first common point with the boundary of F is a starting point of a segment $\beta(v')$. The reflex vertex v' is on the boundary of f and $a \in \gamma(v')$. □

Corollary 3. *In the convex partition of F returned by* CONVPART*, there is at least one reflex vertex of F on the boundary of every face.*

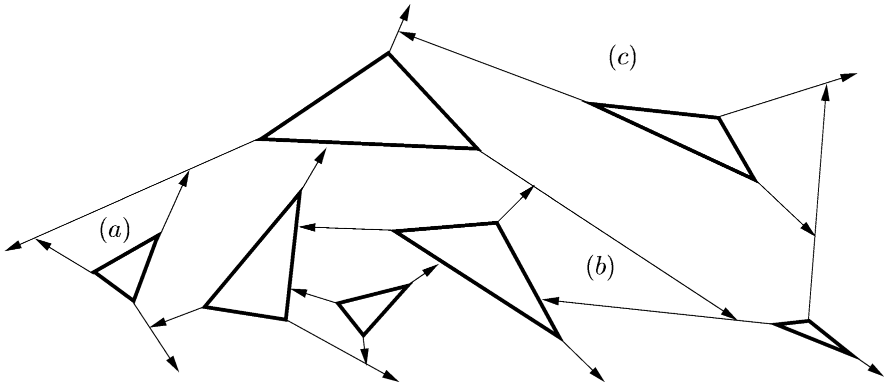

Fig. 1. A possible output of CONPART on the free space of 7 disjoint triangles.

It is possible, however, that a face has only one or two vertices of F on its boundary. See face (a), (b) or (c) in Fig. 1 for examples.

3. Guarding triangles

The main tool to establish Theorems 1 and 2 is the maximum matching of the dual graph G defined below. Let F be a polygonal free space in the plane. Fix a line ℓ_F which is not parallel to any line spanned by a pair of vertices of F. Denote by W the set of vertices of F and the two points in infinity corresponding to the two rays along ℓ_F.

Definition. We define the *dual graph* G for a convex partition C of a polygonal free space F and for ℓ_F. The nodes of G correspond to the faces of C, two nodes are adjacent if and only if the corresponding faces have a common point of W on their boundary.

The next proposition justifies the use of click covers of the dual graph.

Proposition 4. *Assume that F is the free space of disjoint triangles in the plane in general position. For any collection of faces that share a common point of W on their boundaries, the closures of the faces can be monitored by one guard in the free space.*

Proof. Let $a_1, a_2, \ldots, a_k$ denote the faces sharing a common point of W on their boundaries. Since the triangles are in general position and ℓ_F is not parallel to any line through two vertices of triangles, there are overlapping convex sets $A_1, A_2, \ldots, A_k$ in the closure of the free space $\mathrm{cl}(F)$ such that $\mathrm{cl}(a_i) \subset A_i$ for every $i = 1, 2, \ldots, k$. A guard in the interior of the intersection of these convex sets, $\mathrm{int}(\bigcap_{i=1}^{k} A_i)$ monitors all $\mathrm{cl}(a_i)$, $i = 1, 2, \ldots, k$. Note that a guard in the Euclidean plane monitors all these faces even if their common point is in infinity (that is, they contain rays equivalent to translation), because ℓ_F is not parallel to any line through two vertices of F. $\square$

Instead of click covers of the dual graph G, we consider matchings only. Berge's formula [1,16] states that the *deficiency* $d(G)$ of G, the number of nodes not covered by a maximum matching, equals

$\max_{S\subset V(G)}(\operatorname{odd}(G-S)-|S|)$ where $\operatorname{odd}(G-S)$ denotes the number of odd components of the subgraph $G-S$ induced by $V(G)\setminus S$. Moreover, for any $S\subset V(G)$ for which $\operatorname{odd}(G-S)-|S|$ is maximal, there is a matching where every component of $G-S$ with $2k+1$ nodes contains k matched pairs. We will consider a maximum matching of G, and count the number of points in W on the boundaries of faces corresponding to connected components of $G-S$. There is one exceptional configuration we must consider.

Definition. A 5-*group* in a convex partition is a set Q of five faces which jointly have a set U of six points of W on their boundaries with the property that for any $a\in U$ there is a $b\in U$ such that the boundary of any face of Q contains a or b.

For a node $v\in V(G)$, we denote by R_v the closure (in $\overline{\mathbb{E}^2}$) of the face in the convex partition corresponding to v. Similarly, for a set of nodes $D\subset V(G)$, $R_D=\bigcup\{R_v\colon v\in D\}$.

Lemma 5. *For the free space F of n disjoint triangles, there is a convex partition C with $2n+1$ faces such that for any subset $D\subset V(G)$,*

- $|W\cap R_D|\geqslant |D|+2$, *if $|D|$ is odd and D does not form a* 5-*group,*
- $|W\cap R_D|\geqslant |D|+1$, *if $|D|$ is even or D forms a* 5-*group.*

The proof of Lemma 5 is postponed to Sections 5 and 6.

Proof of Theorems 1 and 2. Consider the convex partition C assured by Lemma 5. Choose a set of nodes $S\subset V(G)$ for which $\operatorname{odd}(G-S)-|S|$ is maximal. Denote by $\operatorname{even}(G-S)$ and $\operatorname{odd}(G-S)$ the number of connected components of $G-S$ with even and odd number of nodes, respectively. Let q denote the number of odd components which are 5-groups. If there is no edge between $D\subset V(G-S)$ and $E\subset V(G-S)$, then $W\cap R_D$ and $W\cap R_E$ are disjoint sets. Summing up the number of elements of W in R_D for every connected component D of $G-S$, we have by Lemma 5

$$\operatorname{even}(G-S)+2\operatorname{odd}(G-S)-q+\big(2n+1-|S|\big)\leqslant |W|=3n+2, \tag{1}$$

$$2\operatorname{odd}(G-S)-|S|\leqslant n+q+1-\operatorname{even}(G-S). \tag{2}$$

If $S=\emptyset$, then G is factor critical, and $n+1$ guards are sufficient. If S is non-empty, then we have

$$-|S|\leqslant -1. \tag{3}$$

Combining inequalities (2) and (3), we obtain $d(G)=\operatorname{odd}(G-S)-|S|\leqslant (n+q-\operatorname{even}(G-S))/2\leqslant (n+q)/2$. That is, the deficiency is at most $(n+q)/2$. Moreover, the deficiency is always odd, because the total number of nodes is odd.

We monitor all faces by placing one guard for every pair of faces in a maximum matching by Proposition 4, and one guard at a point of W on the boundary of every face in the deficiency. This set of guards can be rearranged for every 5-group Q containing two matched pair of faces: After a guard is placed on the boundary of the fifth face (either it is in the deficiency or in a matching), the remaining four faces can be monitored by one guard instead of two. The number of necessary guards is therefore at most

$$\frac{2n+1-d(G)}{2}+d(G)-q=n-q+\frac{d(G)+1}{2}\leqslant\left\lfloor\frac{5n+2}{4}\right\rfloor.$$

This completes the proof of Theorem 1. For the proof of Theorem 2, we have shown that the interior of every face is monitored by $\lfloor (5n+2)/4 \rfloor$ guards, and we still need to take care about the boundaries of the open faces. Observe that common boundary of two faces R_a and R_b is always monitored by a vertex guard on the boundary of one of the faces. It might happen that a segment vz on the common boundary is collinear with a side vw of a triangle on the boundary of R_a and the guard assigned to R_a is located at w. In this case, the guard at w does not monitor vz, but the vertex guard assigned to R_b does monitor it, since vz cannot be collinear with two sides of triangles. □

4. Guarding claws

As we will see in Section 5, the proof of Lemma 5 uses the flexibility of bars: the segments $\beta(v)$ can be rotated so that we still have a convex partition. We may ask what happens if the initial segments of the bars are not flexible but rigid and opaque. Replacing triangles by points, we arrive to the definition of claws. The free space has $3n$ reflex vertices, but it has convex vertices at the inner vertex of the claws. In our model, we cannot place a guard at the center of the claw, as all guards should be located in the free space. A guard next to the inner vertex of a claw can possibly monitor only one face.

Consider a set L of n disjoint claws in the plane in general position, and a convex partition C of the free space F_L returned by ConvPart. Let $x(v)$ be a point in the relative interior of $\beta(v)$ for every reflex vertex v of F_L. Let X denote the set of the $3n$ points $x(v)$.

Definition. We define a graph H_X on the faces of a convex partition returned by ConvPart. Two nodes are connected by an edge if and only if the corresponding faces have a common point of X on their boundaries.

Proposition 6. *H is connected.*

Proof. We can create a tree recursively in parallel with the partitioning algorithm ConvPart. We start with one node corresponding to the free space F_L. Whenever one component f of $F(j) := F_B \setminus \bigcup_{i=1}^{j} \beta(v_i)$ is split into two parts by adding segment $\beta(v_{j+1})$, we replace the node corresponding to f by two new nodes, such that they correspond to the two parts (each new node is adjacent to the every node representing a face that has a common point $x(.)$ with the corresponding part), and add an edge between them. We obtain a spanning tree of the graph H_X. □

The graph H_X is not uniquely defined for a convex partitioning, since the points $x(v)$ can be anywhere on the segments $\beta(v)$. Fixing the positions of $x(v)$'s, we obtain unique graphs $H_{X(\varepsilon)}$ described below. Let A be a bounding box which contains all intersections of the lines spanned by the segments of the claws and the segments $\beta(.)$. If $\beta(v)$ ends in infinity, then let $x(v)$ be the intersection point of $\beta(v)$ and the boundary of A. If $\beta(v)$ ends in A, then let $x(v) \in \beta(v)$ be at distance $\varepsilon > 0$ from the endpoint of $\beta(v)$. Let $\varepsilon > 0$ be so small that every $x(v)$ is on the common boundary of exactly two faces which have the endpoint of $\beta(v)$, too, on their common boundary. This implies that $H_{X(\varepsilon)}$ is planar and has exactly $3n$ edges. Now we can formulate an analogon of Lemma 5 for the set $X(\varepsilon)$ and for claws.

Lemma 7. *For the free space F of n disjoint claws,* ConvPart *gives a convex partition C with $2n+1$ faces such that for any subset $D \subset C$, we have $|X(\varepsilon) \cap R_D| \geqslant |D| + 1$.*

Proof. Every convex face has at least three vertices (including the vertices in infinity). Every vertex is either a center of a claw or an endpoint of a segment $\beta(v)$ corresponding to an edge of $H_{X(\varepsilon)}$. Two consecutive vertices on the boundary of a face cannot be centers of claws. Therefore, the minimal degree in graph $H_{X(\varepsilon)}$ is two. $H_{X(\varepsilon)}$ is connected by Proposition 6. The number of edges adjacent to a proper non-empty set of nodes $D \subset V(H_{X(\varepsilon)})$ is at least $|D|+1$. Each edge of $H_{X(\varepsilon)}$ is incident to a node $v \in D$ gives rise to a point of $X(\varepsilon)$ in R_v. $\square$

Proof of Theorem 3. Consider the graph $H = H_{X(\varepsilon)}$. Choose a subset of nodes $S \subset V(H)$ for which $\mathrm{odd}(H-S) - |S|$ is maximal. If $S = \emptyset$, then H is factor critical, and $n+1$ guards are necessary. If S is non-empty then, by Lemma 7, the number of points of $X(\varepsilon)$ on the boundaries of connected components of $H - S$ is

$$\bigl(\mathrm{odd}(H-S) + \mathrm{even}(H-S)\bigr) + \bigl(2n+1-|S|\bigr) \leqslant 3n.$$

We conclude that $d(H) = \mathrm{odd}(H-S) - |S| \leqslant n-1$. Moreover, the deficiency is always odd. Placing one guard at the common point of $X(\varepsilon)$ for every pair in a maximum matching and one guard in the interior of every face in the deficiency, the number of necessary guards is at most

$$\frac{2n+1-d(G)}{2} + d(G) \leqslant \frac{2n+1+d(G)}{2} \leqslant \left\lfloor \frac{3n}{2} \right\rfloor.$$

A lower bound of $\lfloor 3n/2 \rfloor - 2$ follows from the construction depicted in Fig. 2. There are four claws in the center of the construction then ℓ layers, each containing 16 claws, are added. The graph H of these $4+16\ell$ claws has $4+24\ell$ pairwise non-adjacent nodes of degree two. Observe that one guard cannot see

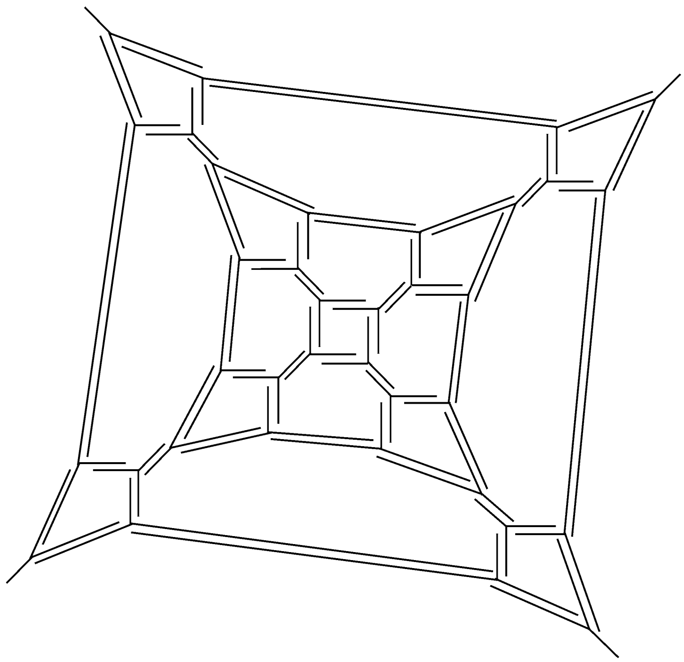

Fig. 2. A pattern of 36 claws where 52 faces have only two reflex vertices of the free space on their boundary.

the centers of two faces corresponding to such nodes at the same time. Hence the construction requires at least $4 + 24\ell$ guards to monitor the free space of $4 + 16\ell$ claws. $\square$

For the sake of completeness we note another variant of the problem, considered by O'Rourke [15] for disjoint line segment. Here, point guards can be located anywhere in the plane, possibly on top of the claws. Furthermore, a guard s sees a point p if the open line segment sp does not properly cross any segment of the claws.

Theorem 4. *If L is a set of n disjoint claws in the plane, then $\lfloor 4n/3 \rfloor$ guards in $\mathbb{E}^2$ are always sufficient and sometimes necessary to monitor the free space $\mathbb{E}^2 \setminus \bigcup L$.*

We prove Theorem 4 by applying Nishizeki's theorem [13]. This theorem states that the maximum matching of a graph on n nodes covers at least $2\lceil (n+4)/3 \rceil$ nodes if the graph is simple, planar, 2-connected, and the minimal degree is at least 3.

Proof. We define the graph J on the faces of a convex partition of n disjoint claws in the plane. Two nodes are connected by an edge if the corresponding faces have at least two common points on their boundary. J is 2-connected and planar. Since unbounded faces may correspond to nodes of degree two, we need to modify the construction: We obtain a new graph J' by adding one node to the graph J which is connected to all nodes corresponding to unbounded faces.

We can apply Nishizeki's theorem to J', which has $2n + 2$ nodes. Place one guard on the common boundary of the $\lceil ((2n+2)+4)/3 \rceil$ pairs of faces of a maximal matching, and one guard in the interior of every face in the deficiency. The total number of guards is therefore $2n + 2 - \lceil (2n+6)/3 \rceil = \lfloor 4n/3 \rfloor$.

For the lower bound $\lfloor 4n/3 \rfloor$, consider the construction depicted in Fig. 3. There are two claws along the convex hull of the construction then k patterns, each containing 3 claws, are added. The total number of claws is $n = 2 + 3k$. The convex partition is unique and contains $2 + 4k$ faces with pairwise disjoint boundaries (marked by asterisks in Fig. 3). One guard cannot see the centers of two such faces at the same time. Hence the construction requires at least $2 + 4k = \lfloor 4(2+3k)/3 \rfloor$ guards. $\square$

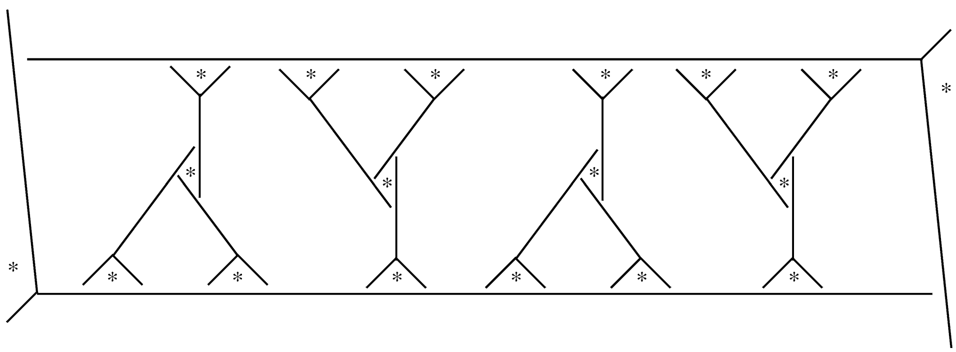

Fig. 3. 14 claws requiring 18 guards in $\mathbb{E}^2$.

5. Three vertices on each face

In this section, we state a decomposition theorem for any polygonal free space. Let F be a free space of h disjoint polygonal objects in general position with a total of r reflex vertices. Let ℓ_F be a line which is not parallel to any line spanned by any two vertices of F. Denote by W the set of vertices of F and the two points in infinity corresponding to the two rays along ℓ_F.

Theorem 5. *F has a convex partition into $r+1-h$ faces such that every face has at least three points of W on its boundary.*

Proof. Consider a convex partition C returned by CONVPART. We make continuous local modifications on the partition until every face has at least three points of W on their boundary. This can be done independently one-by-one for each face.

Let f be a face of C. By Proposition 2, the boundary of f is composed of initial portions of bars $\gamma(v)$, and portions of the boundary of F. Every vertex a of f is either a vertex of the free space F, or belongs to a bar $\gamma(v)$ such that the portion of $\gamma(v)$ between v and a is on the boundary of f. Our goal is to modify f until

($*$) every vertex a of f is either
- a point of W, or
- the endpoint of a side of f containing a vertex v of the free space F in its relative interior such that the directed segment va is part of $\gamma(v)$.

This implies that there are at least three points of W on the boundary of f because every convex face has at least three vertices (including vertices in infinity).

Let v be a reflex vertex of F on the boundary of f. It is possible that v is in the relative interior of a side of f. Let $a_0, a_1, a_2, \ldots, a_k$ be consecutive vertices of f such that $a_0a_1 \not\subset \gamma(v)$, $a_1 \in \gamma(v)$ and $v \in \mathrm{relint}(a_0a_1) \cup \{a_1\}$. (We do not consider bars which are completely in the relative interior of a side of f.)

Observe that ($*$) holds for a_1. Suppose that ($*$) does not hold for a_2. We apply the following step on the boundary of f.

AUGMENT(a_2): Denote by a_2a_3 and a_2b, respectively, the two sides incident to a_2 in the two faces of C along side a_1a_2. We distinguish two cases.

(A) a_2 is in infinity or $\angle a_1a_2a_3 + \angle a_1a_2b \leqslant \pi$: Rotate segment a_1a_2 to a_1a' such that a' moves along a_2b until $a' = b$ or $\angle a_0a_1a' = \pi$. (See Fig. 4.)

(B) $\angle a_1a_2a_3 + \angle a_1a_2b > \pi$: Rotate segments a_1a_2 to a_1a' and a_2a_3 to $a'a_3$ such that a' moves along a_2b until either $a' = b$, or $\angle a_0a_1a' = \pi$, or $\angle a'a_3a_4 = \pi$. (See Fig. 5.)

In the neighboring faces, shorten the sides whose endpoints were in the relative interior of the sides a_1a_2, and a_1a_2 or a_3a_2.

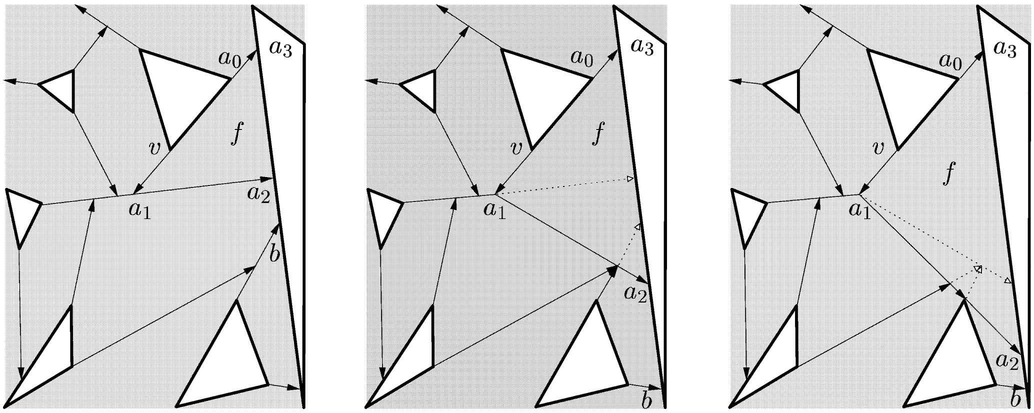

Fig. 4. Three steps of rotation in case (A).

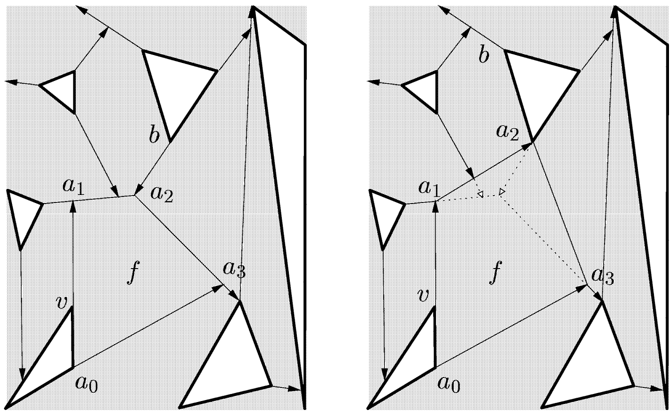

Fig. 5. Rotation step in case (B).

After a step AUGMENT, it can happen that a_1 or a_3 are not vertices of f anymore, if $\angle a_0 a_1 a' = \pi$ or $\angle a' a_3 a_4 = \pi$, respectively. Iterating step AUGMENT on the curve $\gamma(v)$, eventually $(*)$ holds for all vertices of $\gamma(v)$ on the boundary of f. Observe that the continuous modifications in process AUGMENT preserve the orientation of every portion of the bars. We can define the bar $\gamma(v)$ for every reflex vertex of F at all times. If a new reflex vertex w of F appears on a bar $\gamma(v)$ after a rotation step, then necessarily w will be the endpoint of $\gamma(v)$ and the remainder of the previous $\gamma(v)$ will belong to $\gamma(w)$.

We need to worry that a face adjacent to f is contracted to a point while we augment the area of f. This can only happen if a' reaches b and $\gamma(b)$ hits the side va_1. A face might be contracted to a point only if it has only one vertex of P on its boundary. We can avoid this if we first modify the faces until every face has at least two vertices of P on their boundary and then assure a third vertex for every face.

We have a convex partition of F at all times during the process. No point of W is detached from f, nor from any other face. We can assure, independently for every face f, that there are at least three points of W on the boundary of f. $\square$

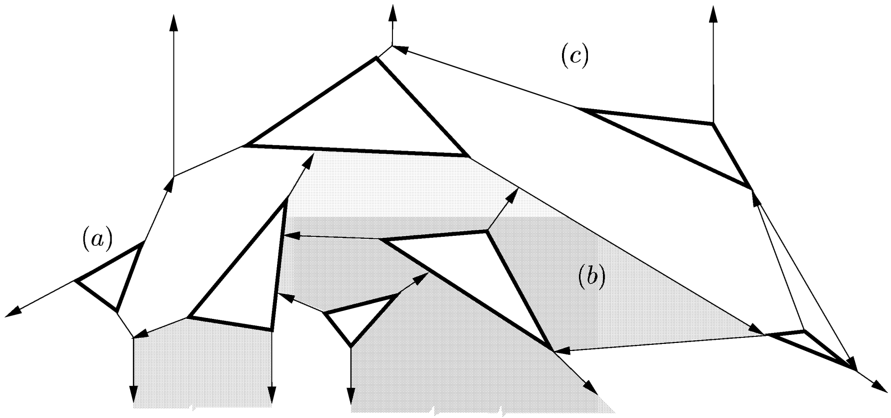

Fig. 6. The result of the modification of bars by AUGMENT on the example of Fig. 1. Line ℓ_F is vertical.

Theorem 5 can be generalized to the following form:

Lemma 8. *F has a convex partition with $r + 1 - h$ faces such that every closed Jordan curve in $\overline{\mathbb{E}^2}$ along the boundaries of faces contains at least three points of W.*

Proof. Consider a convex partition returned by CONVPART. Let κ be a closed Jordan curve along boundaries of faces. Curve κ circumscribes a (possibly unbounded) polygonal region K. We can modify the convex faces containing the convex vertices of K until

(∗∗) every convex vertex a of K is either
- a point of W, or
- the endpoint of a reflex curve along the boundary of K containing a reflex vertex v of the free space F in its relative interior such that the directed curve va is part of $\gamma(v)$.

Let v be a reflex vertex of P on the boundary of f. Let $a_0, a_1, a_2, \ldots, a_k$ be consecutive vertices of f such that $a_0a_1 \not\subset \gamma(v)$, $a_1 \in \gamma(v)$ and $v \in \mathrm{relint}(a_0a_1) \cup \{a_1\}$. Suppose that a_i and a_j, $0 < i < j$, are the first two convex vertices in this sequence. Note that (∗∗) holds for a_i. If (∗∗) does not hold for a_j, we can apply step AUGMENT(a_{i+1}) on the boundary of f until a new vertex of F appears on the reflex chain $a_i, a_{i+1}, \ldots, a_j$ or until the angle at a_i becomes a straight angle and a_j becomes the first convex vertex on the chain between a_0 and a_j.

It might be that as a result of an AUGMENT step, a curve κ is not simple anymore. In this case we can remove this curve for further consideration. No step AUGMENT creates new simple Jordan curves, nor detaches any point of W from closed Jordan curves. Therefore, the statement of our lemma can be assured for every closed Jordan curve one by one. A closed Jordan curve κ visits at least three points of W if the enclosed region K satisfies (∗∗) and has at least three convex vertices. It is possible that K satisfies (∗∗) but has only two convex vertices if K is unbounded and both convex vertices are in infinity. In this case, notice that one point of W in infinity is on the boundary of K but it does not correspond to either convex vertices, so again there are at least three points of W on κ. □

6. Proof of Lemma 5

It is enough to consider $|W \cap R_D|$ for the case where R_D is a connected region in $\overline{\mathbb{E}^2}$. If R_D is not connected, then Lemma 5 follows by adding up the number of elements of W in all connected components of R.

Let A be the bounding box containing all triangles and all Euclidean (non-infinite) vertices of faces of the convex partition. For every $D \subseteq V(G)$, we construct a partition of the bounding box A into at most $2n+1$, not necessarily convex, faces. The faces of the new partition Φ_D correspond to the faces of C: For a node $a \in V(G)$, the closed region R_a corresponds to the closed region $\varphi(R_a) \in \Phi_D$. Using the notation $\varphi(R_D) = \bigcup\{\varphi(R_a)\colon a \in D\}$, we construct Φ_D such that $|W \cap \varphi(R_D)| = |W \cap R_D|$ and for all other $D' \subseteq V(G)$, $|W \cap \varphi(R_{D'})| \geqslant |W \cap R_{D'}|$.

Let the two intersection points of ℓ_F with the boundary of A, o_1 and o_2, represent the two points of W in infinity (Fig. 7). For every reflex vertex v of F, we draw a directed polygonal curve $\lambda(v)$ from v to a point in W. Every curve $\lambda(v)$ will be constructed along the bars $\gamma(.)$ and the boundary of F. Unlike the curves $\gamma(.)$, the relative interiors of the curves $\lambda(.)$ will be disjoint.

Connect o_1 and o_2 by two disjoint directed curves, denoted by $\lambda(o_1)$ and $\lambda(o_2)$, along the boundary of A. For every reflex vertex v of F, we draw the curves $\lambda(v)$ consecutively. $\lambda(v)$ follows $\gamma(v)$ until it reaches either a previously drawn $\lambda(v')$ at point q, or the endpoint $w(v)$ of $\gamma(v)$. If $\lambda(v)$ reaches $w(v)$ at W, then $\lambda(v)$ ends there. If $w(v)$ is at the relative interior of a side ab of a triangle $abc \in T$ then $\lambda(v)$ follows ab to either a or to b. If $w(v)$ is in infinity, then $\lambda(v)$ follows the boundary of A to one intersection with the line ℓ_F. If $\lambda(v)$ reaches a previously drawn $\lambda(v')$ at point q then it follows $\lambda(v')$ parallelly to its starting point or to its endpoint. In the last three cases, there are two possible directions to follow ab, ∂A or $\lambda(v')$ respectively. We choose the direction of $\lambda(v)$ as follows: If region R_D is on one side of $w(v)$ (respectively q) and $F \setminus R_D$ is on the other side then $\lambda(v)$ follows the direction of R_D; otherwise $\lambda(v)$ follows either direction (see Fig. 7).

We have now a planar diagram where the nodes are the points of W, the edges are the curves $\lambda(.)$ and the sides of triangles, and the faces are faces of $\varphi(R_D)$ and triangles. Note that for every closed curve Lemma 8 holds.

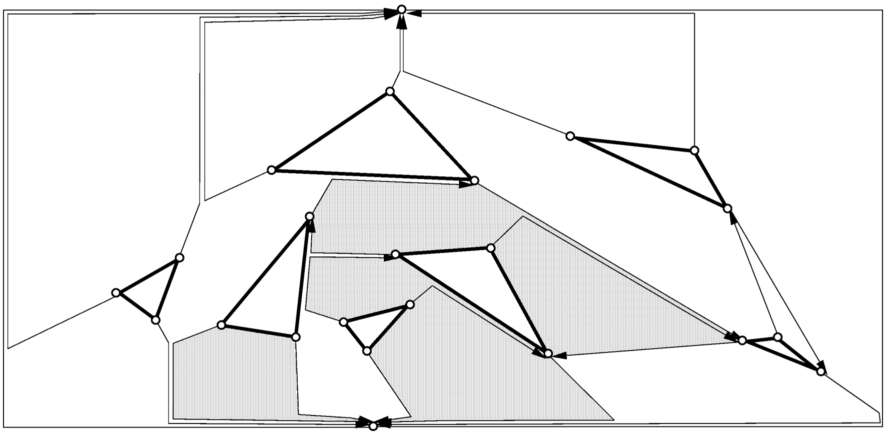

Fig. 7. The new partition where the shaded region $\varphi(R_D)$ corresponds to the shaded region in Fig. 6.

It rests to show that $|W \cap \varphi(R_D)| \geqslant |D| + 2$ if $|D|$ is odd, and $|W \cap \varphi(R_D)| \geqslant |D| + 1$ if $|D|$ is even. It clearly holds if $R_D = \mathrm{cl}(F)$ and $|D| = 2n + 1$. Otherwise consider the boundary of $\varphi(R_D)$. There is a reflex vertex v_0 of F such that $\lambda(v_0)$ is on the boundary of $\varphi(R_D)$. A *circumscribing curve* of $\varphi(R_D)$ is a weakly simple closed curve κ in the boundary of $\varphi(R_D)$ such that $\mathrm{int}(\varphi(R_D))$ is on the positive side of κ and the negative side of κ is connected. (A closed curve is *weakly simple*, if it has no self-crossing but it is not necessarily simple.) Let κ be a circumscribing curve of $\varphi(R_D)$ such that it goes through $\lambda(v_0)$ and has a minimum number of faces in its positive side. Notice that the boundary of $\varphi(F) \setminus \varphi(R_D)$ is not necessarily connected, and κ forms the boundary of one component. We fill up these holes by regarding all components of $\varphi(F) \setminus \varphi(R_D)$ in the positive side κ as new faces, and let D' denote the union of D and the new faces. $|D| \leqslant |D'|$ and we call $\varphi(R_{D'})$ the union of the closures of all faces in D'. If we consider $\varphi(R_{D'})$ and delete one-by-one the points of W and the curves $\lambda(.)$ which are not contained in the curve κ, then we obtain a partition into $|D'|$ faces.

We estimate $|D'|$: Denote by b the number of curves $\lambda(.)$ along κ. Let t_2 and t_3 denote the number of triangles τ such that $\varphi(R_{D'}) \setminus \tau$ has 2 and 3 connected components, respectively. Let T_2 and T_3 be the set of these triangles. For every vertex $v \in \varphi(R_D) \cap W$, let $1 + h_v$ be the number of components of $\varphi(R_D) \setminus v$, and let $h = \sum_{v \in \varphi(R_D) \cap W} h_v$. We have

$$|D'| \leqslant 1 + \bigl(\bigl|W \cap \varphi(R_D)\bigr| - b\bigr) + (h + t_2 + 2t_3). \tag{4}$$

A triangle τ, for which $\varphi(R_{D'}) \setminus \tau$ splits into two (respectively, three) components, has two (respectively, three) vertices on the curve κ. From Lemma 8, we deduce that the boundary of every component of $\varphi(R_{D'}) \setminus (W \cup T_2 \cup T_3)$ contains at least two curves $\lambda(.)$. This implies

$$b \geqslant 2(h + 1) + 2t_2 + 4t_3. \tag{5}$$

Combining inequalities (4) and (5), we obtain

$$|D'| + 1 + t_2 + 2t_3 + h \leqslant \bigl|W \cap \varphi(R_D)\bigr|.$$

This proves Lemma 5 if $t_2 + 2t_3 + h \geqslant 1$, $|D| < |D'|$, or at least one of the inequalities (4) and (5) is not tight.

Suppose that $t_2 = t_3 = h = 0$, $|D| = |D'|$, $b = 2$, and inequality (4) is tight. This latter assumption implies that there are exactly $|W \cap \varphi(R_D)| - 2$ curves $\lambda(.)$ such that each splits the positive side of $\varphi(R_{D'})$ into two parts. Consequently, every triangle on the positive side of κ has at least one common vertex with κ. There are at most two such triangles, since $b = 2$. Denote them by τ_1 and τ_2.

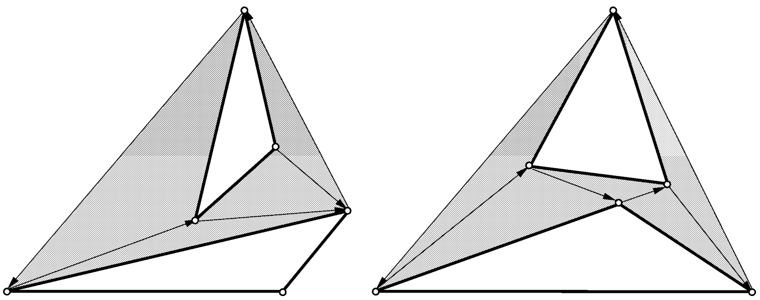

Fig. 8. The two exceptional diagrams, points of W are depicted as small empty circles.

If all elements of $W \cap \varphi(R_D)$ are on κ, then $W \cap \varphi(R_D)$ and the curves $\lambda(.)$ form a planar drawing of an outer-planar graph with $|W \cap \varphi(R_D)|$ nodes and such that every face has at least three nodes on their boundary. In such a graph, the number of nodes exceed the number of faces by at least two.

Assume that not all elements of $W \cap \varphi(R_D)$ are on κ. If τ_1 (or τ_2) has a vertex within the positive side of κ, then it necessarily has two vertices within the same side of κ and one on κ, because κ has a minimum number of faces in its positive side. If both τ_1 and τ_2 have only one-one vertex on κ then κ violates Lemma 8. Therefore, at most one of τ_1 and τ_2 can have one vertex on κ and two vertices within the positive side of κ. There are two possible configurations of the diagram $\varphi(R_D)$, shown in Fig. 8, where $|W \cap \varphi(R_D)| < |D| + 2$ and Lemma 8 is not violated. For every face $\varphi(R_v)$, $v \in D$, we have $|W \cap \varphi(R_v)| = 3$. Therefore $|W \cap R_v| = 3$ and $W \cap \varphi(R_v) = W \cap R_v$ holds, by Lemma 8, for every $v \in D$. One configuration has four faces, the other has five faces and forms a 5-group. This completes the proof of Lemma 5. □

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