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On a Question of Pietsch about Hilbert–Schmidt Multilinear Mappings

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In 1983, Pietsch asked if, for $n \geq 3$ and all Hilbert spaces $E_1, \ldots, E_n$, the vector space of the scalar valued absolutely $(r; r_1, \ldots, r_n)$-summing multilinear mappings on $E_1 \times \cdots \times E_n$ coincides with the vector space of the n-linear Hilbert–Schmidt functionals on $E_1 \times \cdots \times E_n$, for some choice of $r, r_1, \ldots, r_n \in]0, +\infty]$, satisfying $1/r \leq 1/r_1 + \cdots + 1/r_n$. We show that the answer to this question is no. Moreover, we show that the same question, for $n \geq 2$ and mappings with values in infinite dimensional Hilbert spaces, has the answer no.

1. INTRODUCTION

Recently many authors studied certain ideals of multilinear mappings and polynomials in Banach spaces motivated by important counterparts within the theory of linear operators in Banach spaces. In the linear theory these operator ideals usually are natural extensions of natural classes in Hilbert spaces. Hence it is important to know what the interesting classes of multilinear mappings and polynomials mean in Hilbert spaces. In particular, it is important to answer the question posed by Pietsch in 1983.

In this paper $F, F_1, \ldots, F_n, G$ are Hilbert spaces over $\mathbb{K}(= \mathbb{R}$ or $\mathbb{C})$ and n is a natural number.

For $s \in]0, +\infty[$, we denote by $l_s(F)$ the vector space of all sequences $(x_j)_{j=1}^{\infty}$ of elements of F, such that

$$\left\| (x_j)_{j=1}^{\infty} \right\|_s = \left(\sum_{j=1}^{\infty} \|x_j\|^s \right)^{\frac{1}{s}} < +\infty.$$

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The norm $\|.\|_s$ (s-norm if $s < 1$) defines a complete metric on $l_s(F)$. We indicate by $l_s^w(F)$ the vector space of all sequences $(x_j)_{j=1}^\infty$ of elements of F, such that $(\phi(x_j))_{j=1}^\infty \in l_s := l_s(\mathbb{K})$, for every ϕ in the topological dual F' of F. The following norm (s-norm, if $s < 1$) defines a complete metrizable topology on this space:

$$\left\|(x_j)_{j=1}^\infty\right\|_{w,\,s} := \sup_{\phi\in B_{F'}} \left\|(\phi(x_j))_{j=1}^\infty\right\|_s.$$

Here $B_{F'}$ denotes the closed unit ball of F' centered at 0. The vector space $l_\infty(F) = l_\infty^w(F)$, formed by all bounded sequences $(x_j)_{j=1}^\infty$ of elements of F, is a Banach space for the norm

$$\left\|(x_j)_{j=1}^\infty\right\|_\infty = \left\|(x_j)_{j=1}^\infty\right\|_{w,\,\infty} := \sup_{j\in\mathbb{N}}\left\|x_j\right\|.$$

We denote by $\mathscr{L}(F_1, \ldots, F_n; G)$ the vector space of all continuous n-linear mappings from $F_1 \times \cdots \times F_n$ into G. If $F_1 = \cdots = F_n = F$, this space is also denoted by $\mathscr{L}({}^nF; G)$.

1.1. DEFINITION. For $r, r_1, \ldots, r_n \in\,]0, +\infty]$, with $\frac{1}{r} \leq \frac{1}{r_1} + \cdots + \frac{1}{r_n}$, $T \in \mathscr{L}(F_1, \ldots, F_n; G)$ is *absolutely* $(r; r_1, \ldots, r_n)$*-summing* if $(T(x_j^1, \ldots, x_j^n))_{j=1}^\infty \in l_r(G)$, for every $(x_j^k)_{j=1}^\infty$ in $l_{r_k}^w(F_k)$, $k = 1, \ldots, n$.

The vector space of all such mappings is denoted by

$$\mathscr{L}_{as}^{(r;\, r_1,\ldots,\, r_n)}(F_1, \ldots, F_n; G).$$

For each $T \in \mathscr{L}_{as}^{(r;\, r_1,\ldots,\, r_n)}(F_1, \ldots, F_n; G)$, we write

$$\|T\|_{as,(r;\, r_1,\ldots,\, r_n)} = \sup_{\|(x_j^k)_{j=1}^\infty\|_{w,\, r_k} \leq 1} \|(T(x_j^1, \ldots, x_j^n))_{j=1}^\infty\|_r.$$

This defines a norm (r-norm, if $r < 1$) on $\mathscr{L}_{as}^{(r;\, r_1,\ldots,\, r_n)}(F_1, \ldots, F_n; G)$ and makes this space metrizable and complete.

1.2. DEFINITION. $T \in \mathscr{L}(F_1, \ldots, F_n; G)$ is *Hilbert–Schmidt* if, for each complete orthonormal system $\{e_j^k; j \in I_k\}$ in F_k, $k = 1, \ldots, n$,

$$\|T\|_{HS} = \left(\sum_{j_k\in I_k} \|T(e_{j_1}^1, \ldots, e_{j_n}^n)\|^2\right)^{\frac{1}{2}} < +\infty.$$

The vector space of all such mappings is denoted by $\mathscr{L}_{HS}(F_1, \ldots, F_n; G)$.

In this paper we shall write $l_2(\mathbb{K}) = E$.

We recall the following result from the Linear Operator Theory (see [2, Theorem 22.1.8, p. 302]).

1.3. THEOREM. *$\mathcal{L}_{HS}(F;G) = \mathcal{L}_{as}^{(r;r)}(F;G)$, for all Hilbert spaces F, G and every strictly positive real number r.*

If $q \in]0, +\infty[$, we denote by $\mathcal{S}_q(F;G)$ the vector space of all $S \in \mathcal{L}(F;G)$ of the form

$$S(x) = \sum_{j \in I} \sigma_j \phi_j(x) e_j, \qquad \forall\, x \in F,$$

with $(\phi_j)_{j\in I}$, $(e_j)_{j\in I}$ being orthonormal families of elements of F', G, respectively, and $(\sigma_j)_{j\in I}$ an absolutely q-summable family of scalars. We have $\mathcal{S}_2(F;G) = \mathcal{L}_{HS}(F;G)$ and, when $q < r$, $\mathcal{S}_q(F;G)$ is a proper vector subspace of $\mathcal{S}_r(F;G)$.

In Pietsch [2, p. 303] we find the following result:

1.4. THEOREM. *For $0 < p < s < +\infty$,*

(1) *If $p \le 2$ and $\frac{1}{p} - \frac{1}{s} < \frac{1}{2}$, then $\mathcal{L}_{as}^{(s;p)}(F;G) = \mathcal{S}_q(F;G)$, with $\frac{1}{q} = \frac{1}{s} - \frac{1}{p} + \frac{1}{2}$.*

(2) *If $p \le 2$ and $\frac{1}{p} - \frac{1}{s} \ge \frac{1}{2}$, then $\mathcal{L}_{as}^{(s;p)}(F;G) = \mathcal{L}(F;G)$.*

(3) *If $p \ge 2$, then $\mathcal{S}_q(F;G) \subset \mathcal{L}_{as}^{(s;p)}(F;G)$, with $\frac{1}{q} = \frac{p}{2s}$.*

1.5. COROLLARY. *If $0 < p < s < +\infty$, then there is $S \in \mathcal{L}_{as}^{(s;p)}(E;E)$, that is not Hilbert–Schmidt.*

Proof. If $p \le 2$ and $\frac{1}{p} - \frac{1}{s} < \frac{1}{2}$, we consider $\frac{1}{t} = \frac{1}{s} - \frac{1}{p} + \frac{1}{2} < \frac{1}{2}$. We have $\mathcal{L}_{as}^{(s;p)}(E;E) = \mathcal{S}_t(E;E)$, by Theorem 1.4(1). Since $t > 2$, we consider $(\alpha_j)_{j=1}^{\infty} \in l_t(\mathbb{K}) \backslash l_2(\mathbb{K})$ and define

$$S(x) = \sum_{j=1}^{\infty} \alpha_j x_j e_j \qquad \forall\, x = (x_j)_{j=1}^{\infty} \in l_2(\mathbb{K}) = E.$$

Hence $S \in \mathcal{L}_{as}^{(s;p)}(E;E) = \mathcal{S}_t(E;E)$ and $S \notin \mathcal{L}_{HS}(E;E) = \mathcal{S}_2(E;E)$.

If $p \le 2$ and $\frac{1}{p} - \frac{1}{s} \ge \frac{1}{2}$, then $\mathcal{L}_{as}^{(s;p)}(E;E) = \mathcal{L}(E;E)$, by Theorem 1.4(2). We certainly have $\mathcal{L}(E;E) \neq \mathcal{L}_{HS}(E;E)$.

If $p \ge 2$, we use Theorem 1.4(3) in order to write $\mathcal{S}_q(E;E) \subset \mathcal{L}_{as}^{(s;p)}(E;E)$, with $q = \frac{2s}{p}$. Since $s > p$ is equivalent to $q > 2$, we can find $S \in \mathcal{S}_q(E;E) \subset \mathcal{L}_{as}^{(s;p)}(E;E)$, that is not in $\mathcal{S}_2(E;E) = \mathcal{L}_{HS}(E;E)$. ∎

Finally we note that $\mathcal{L}_{HS}(E;E) \neq \mathcal{L}(E;E) = \mathcal{L}_{as}^{(+\infty;r)}(E;E) = \mathcal{L}_{as}^{(+\infty,+\infty)}(E;E)$.

Since such results are known for linear operators, it is natural to conjecture about what can happen when multilinear mappings are considered.

2. THE MOST NATURAL PROBLEM

The following result gives an interesting characterization of the linear absolutely (s, s)-summing mappings. See [3] for scalar-valued mappings. We denote by $W(B_{F'})$ the set of all regular probability measures on the σ-algebra of the Borel subsets of $B_{F'}$, for the weak * topology on F' restricted to $B_{F'}$.

2.1. Grothendieck–Pietsch Domination Theorem. *If F and G are Banach spaces, then $T \in \mathcal{L}(F; G)$ is absolutely $(s; s)$-summing if and only if there are $\mu \in W(B_{F'})$ and $C \geq 0$, such that*

$$\|T(x)\| \leq C\left(\int_{B_{F'}} |\phi(x)|^s d\mu(\phi)\right)^{\frac{1}{s}},$$

for every $x \in F$. The infimum over all these possible C is equal to $\|T\|_{as,(s;s)}$.

The corresponding result for multilinear applications is stated as follows:

2.2. Theorem. *If $T \in \mathcal{L}(F_1, \ldots, F_n; G)$ and $r, r_1, \ldots, r_n \in]0, +\infty[$ are such that $1/r = 1/r_1 + \cdots + 1/r_n$, then T is absolutely $(r; r_1, \ldots, r_n)$-summing if and only if there are $C \geq 0$ and $\mu_k \in W(B_{F'_k})$, $k = 1, \ldots, n$, such that*

$$\|T(x^1, \ldots, x^n)\| \leq C\left(\int_{B_{F'_1}} |\phi(x^1)|^{r_1} d\mu_1(\phi)\right)^{\frac{1}{r_1}} \cdots \left(\int_{B_{F'_n}} |\phi(x^n)|^{r_n} d\mu_n(\phi)\right)^{\frac{1}{r_n}},$$

for every $x^k \in F_k$, $k = 1, \ldots, n$. The infimum of all these possible C is equal to $\|T\|_{as,(r;r_1,\ldots,r_n)}$.

These results motivate the following question for n-linear mappings, $n \geq 2$.

2.3. *Natural Problem.* If $n \geq 2$, is it possible to find $r, r_1, \ldots, r_n \in]0, +\infty[$, with $1/r = 1/r_1 + \cdots + 1/r_n$, such that $\mathcal{L}_{HS}(F_1, \ldots, F_n; G) = \mathcal{L}_{as}^{(r;r_1,\ldots,r_n)}(F_1, \ldots, F_n; G)$, for all Hilbert spaces $F_1, \ldots, F_n, G$?

We start with the case $n = 2$.

2.4. Example. If $G \neq \{0\}$, there is $T_b \in \mathcal{L}_{HS}(E, E; G)$, such that $T_b \notin \mathcal{L}_{as}^{r;r_1,r_2)}(E, E : G)$, for all $r, r_1, r_2 \in]0, +\infty[$ satisfying $1/r = 1/r_1 + 1/r_2$.

As usual, if $k \in \mathbb{N}$, we write e_k to denote the element $(0, \ldots, 0, 1, 0, \ldots) \in l_2$, with 1 in the position k. If $x \in l_2$, we write $x = (x_j)_{j=1}^{\infty} = \sum_{j=1}^{\infty} x_j e_j$. We shall use the notation $\overline{x} = (\overline{x_j})_{j=1}^{\infty}$, when $x = (x_j)_{j=1}^{\infty} \in l_2 = E$. It is easy to show that $(\overline{x^k})_{k=1}^{\infty} \in l_p^w(E)$, for $(x^k)_{k=1}^{\infty} \in l_p^w(E)$. We define $T \in \mathcal{L}(l_2, l_2; \mathbb{K})$ by

$$T(x, y) = \sum_{j=1}^{\infty} \frac{1}{j} x_j y_j \qquad \forall\, x, y \in l_2.$$

We note that T is Hilbert–Schmidt since

$$\sum_{j,k=1}^{\infty} |T(e_j, e_k)|^2 = \sum_{j=1}^{\infty} |T(e_j, e_j)|^2 = \sum_{j=1}^{\infty} \frac{1}{j^2} < +\infty.$$

Now we consider $S \in \mathcal{L}(l_2; l_2)$, given by

$$S(x) = \sum_{j=1}^{\infty} \frac{1}{j^{1/2}} x_j e_j \qquad \forall\, x \in l_2.$$

We have

$$\|S(x)\|^2 = |T(x, \overline{x})| \qquad \forall\, x \in l_2.$$

If T were in $\mathcal{L}_{as}^{(r;2r,2r)}(l_2, l_2; \mathbb{K})$, we would have

$$\sum_{j=1}^{\infty} \|S(x^j)\|^{2r} = \sum_{j=1}^{\infty} |T(x^j, \overline{x^j})|^r < +\infty,$$

for every $(x^j)_{j=1}^{\infty} \in l_{2r}^{w}(l_2)$. Thus S would be absolutely $(2r; 2r)$-summing, hence Hilbert–Schmidt. But

$$\sum_{j=1}^{\infty} \|S(e_j)\|^2 = \sum_{j=1}^{\infty} \frac{1}{j} = +\infty.$$

Therefore T cannot be in $\mathcal{L}_{as}^{(r;2r,2r)}(l_2, l_2; \mathbb{K})$. If $b \in G$, $b \neq 0$, we define $T_b(x, y) = T(x, y)b$, for all $x, y \in l_2$. It is easy to see that T_b is Hilbert–Schmidt. If $\phi \in G'$ is such that $\phi(b) = 1$, we have $T = \phi \circ T_b$. Hence, T_b cannot be in $\mathcal{L}_{as}^{(r;2r,2r)}(l_2, l_2; G)$. If this were true, T would be in $\mathcal{L}_{as}^{(r;2r,2r)}(l_2, l_2; \mathbb{K})$, a contradiction.

For $r, r_1, r_2 \in]0, +\infty[$, such that $1/r = 1/r_1 + 1/r_2$, by Theorem 2.2, we have

$$\mathcal{L}_{as}^{(r:r_1,r_2)}(l_2, l_2; G) \subset \mathcal{L}_{as}^{(s;2s,2s)}(l_2, l_2; G) \qquad \text{when } s = \max\{r_1, r_2\}.$$

Hence $T_b \notin \mathcal{L}_{as}^{(r:r_1,r_2)}(l_2, l_2; G)$.

It is possible to prove that for $0 < r, p, q < +\infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, $\mathcal{L}_{as}^{r;p,q}(l_2, l_2; \mathbb{K})$ is the space of nuclear forms. It is properly contained in the space of the Hilbert–Schmidt forms, since the diagonal form

$$T(x, y) = \sum_{j=1}^{+\infty} \frac{1}{j} x_j y_j$$

is Hilbert–Schmidt but not nuclear.

2.5. EXAMPLE. If $n \geq 2$, $G \neq \{0\}$, there is $R \in \mathcal{L}_{HS}({}^nE; G)$, such that R is not absolutely $(r; r_1, \ldots, r_n)$-summing, for all $r, r_1, \ldots, r_n \in]0, +\infty[$, satisfying $1/r = 1/r_1 + \cdots + 1/r_n$.

We consider $\psi \in E'$ and $c \in E$ such that $\psi(c) = 1$. We define $R \in \mathcal{L}({}^nE; G)$ by

$$R(x^1, \ldots, x^n) = T_b(x^1,\ x^2)\psi(x^3)\ldots\psi(x^n),$$

where T_b is as in the previous example. We note that R is Hilbert–Schmidt, since

$$\sum_{j_1=1,\ldots,j_n=1}^{\infty} \|R(e_{j_1}, \ldots, e_{j_n})\|^2 = \|\psi\|^{2(n-2)} \sum_{j_1=1,\ j_2=1}^{\infty} \|T_b(e_{j_1},\ e_{j_2})\|^2 < +\infty.$$

We consider $r, r_1, \ldots, r_n \in]0, +\infty[$, such that $1/r = 1/r_1 + \cdots + 1/r_n$. If R were absolutely $(r; r_1, \ldots, r_n)$-summing, there would be $C \geq 0$ and $\mu_k \in W(B_{E'})$, $k = 1, \ldots, n$, such that

$$\begin{aligned} &\|R(x^1, \ldots, x^n)\| \\ &\quad \leq C\left(\int_{B_{E'}} |\phi(x^1)|^{r_1} d\mu_1(\phi)\right)^{\frac{1}{r_1}} \cdots \left(\int_{B_{E'}} |\phi(x^n)|^{r_n} d\mu_n(\phi)\right)^{\frac{1}{r_n}}, \end{aligned}$$

for every $x^k \in E$, $k = 1, \ldots, n$. Hence, in a particular case, we could write

$$\|R(x^1, x^2, c, \ldots, c)\| \leq C \prod_{k=1}^{2} \left(\int_{B_{E'}} |\phi(x^k)|^{r_k} d\mu_k(\phi)\right)^{\frac{1}{r_k}} \|c\|^{n-2},$$

for every $x^k \in E$, $k = 1, 2$. For $c \in E$, $\psi(c) = 1$, we have $\|T_b(x^1, x^2)\| = \|R(x^1, x^2, c, \ldots, c)\|$. This would imply that T_b is absolutely $(s; r_1, r_2)$-summing, with $1/s = 1/r_1 + 1/r_2$, a contradiction to Example 2.4.

This example can be proved directly, without the Pietsch domination theorem.

3. MORE GENERAL PROBLEM

In 1983, at a Conference in Leipzig (see [3]), Pietsch proposed the following question:

3.1. PIETSCH'S PROBLEM. *If $n \geq 3$, is it possible to find $r, r_1, \ldots, r_n \in]0, +\infty]$, with $1/r \leq 1/r_1 + \cdots + 1/r_n$, such that $\mathcal{L}_{HS}(F_1, \ldots, F_n; \mathbb{K}) = \mathcal{L}_{as}^{(r; r_1, \ldots, r_n)}(F_1, \ldots, F_n; \mathbb{K})$, for all Hilbert spaces $F_1, \ldots, F_n$?*

3.1.a. PROPOSITION. *If F and G are Hilbert spaces over $\mathbb{K}$, then* $\mathscr{L}_{as}^{(2;2,2)}(F, G; \mathbb{K}) = \mathscr{L}_{HS}(F, G; \mathbb{K})$.

This result is a consequence of a linear result, by using the isomorphism Ψ from $\mathscr{L}(F; G')$ onto $\mathscr{L}(F, G; \mathbb{K})$. See Pietsch [2, p. 239, 17.5.2]. Hence the answer to the question of Pietsch is yes when $n = 2$.

It is easy to show that $\mathscr{L}_{HS}(F_1, \dots, F_{n-1}; F'_n)$ is isometric to $\mathscr{L}_{HS}(F_1, \dots, F_n; \mathbb{K})$ by the isomorphism Ψ, defined by

$$\Psi(T)(x^1, \dots, x^{n-1}, x^n) = T(x^1, \dots, x^{n-1})(x^n),$$

when $T \in \mathscr{L}_{HS}(F_1, \dots, F_{n-1}; F'_n)$ and $x^k \in F_k$, $k = 1, \dots, n-1, n$. This motivates the following question:

3.2. *The Infinite Dimensional Vector Valued Problem.* If $n \geq 2$, is it possible to find $r, r_1, \dots, r_n \in]0, +\infty]$, with $1/r \leq 1/r_1 + \cdots + 1/r_n$, such that $\mathscr{L}_{HS}(F_1, \dots, F_n; G) = \mathscr{L}_{as}^{(r; r_1, \dots, r_n)}(F_1, \dots, F_n; G)$, for Hilbert spaces $F_1, \dots, F_n$ and every infinite dimensional Hilbert space G?

We shall prove that the answer to each of these problems is negative. Our examples for Problem 3.2 will depend partially on the examples we give to answer Problem 3.1. Of course, the examples of Section 2 give part of the solution to these two problems.

4. THE INFINITE DIMENSIONAL VECTOR VALUED PROBLEM

In this section we always consider $n \geq 2$.

4.1. EXAMPLE. If $r, r_1, \dots, r_n \in]0, +\infty]$ are such that $1/r \leq 1/r_1 + \cdots + 1/r_n$, with $r_k = +\infty$ for at least one $k \in \{1, \dots, n\}$, then there is $T \in \mathscr{L}_{as}^{(r;r_1,\dots,r_n)}(^nE; E)$ that is not Hilbert–Schmidt.

With no loss of generality we consider $r_1 = +\infty$. For $\phi \in E'$, $\phi \neq 0$, we define

$$T(x^1, \dots, x^n) = x^1 \phi(x^2) \dots \phi(x^n) \qquad \forall\, x^k \in E, k = 1, \dots, n.$$

T is not Hilbert–Schmidt, since

$$\sum_{j_k=1,\, k=1,\dots,n}^{\infty} \|T(e_{j_1}, \dots, e_{j_n})\|^2 = \|\phi\|^{2(n-1)} \sum_{j_1=1}^{\infty} \|e_{j_1}\|^2 = +\infty.$$

On the other hand, we have

$$\left\|(T(x_j^1,\ldots,x_j^n))_{j=1}^\infty\right\|_r \leq \left\|(x_j^1)_{j=1}^\infty\right\|_\infty \prod_{k=2}^n \left\|(\phi(x_j^k))_{j=1}^\infty\right\|_{r_k}$$
$$\leq \|\phi\|^{n-1}\left\|(x_j^1)_{j=1}^\infty\right\|_\infty \prod_{k=2}^n \left\|(x_j^k)_{j=1}^\infty\right\|_{w,r_k}.$$

This shows that $T \in \mathscr{L}_{as}^{(r;r_1,\ldots,r_n)}(^nE;E)$, since $r_1 = +\infty$.

4.2. EXAMPLE. If $r, r_1, \ldots, r_n \in]0,+\infty]$ are such that $1/r < 1/r_1 + \cdots + 1/r_n$, with $r_k \leq r$ for at least one $k \in \{1,\ldots,n\}$, then there is $T \in \mathscr{L}_{as}^{(r;r_1,\ldots,r_n)}(^nE;E)$ that is not Hilbert–Schmidt.

We suppose that $r_n \leq r$, with no loss of generality. Since $1/r \leq 1/r_n$, we can write $1/r \leq (n-1)/s + 1/r_n$, where $s = +\infty$. By Example 4.1, there is $T \in \mathscr{L}_{as}^{(r;+\infty,\ldots,+\infty,r_n)}(^nE;E)$ that is not Hilbert–Schmidt. But $\mathscr{L}_{as}^{(r;+\infty,\ldots,+\infty,r_n)}(^nE;E) \subset \mathscr{L}_{as}^{(r;r_1,\ldots,r_{n-1},r_n)}(^nE;E)$.

4.3. EXAMPLE. If $r, p, q \in]0,+\infty]$, with $\frac{1}{r} < \frac{1}{p} + \frac{1}{q}$, $p > r$, and $q > r$, then there is $T \in \mathscr{L}_{as}^{(r;p,q)}(^2E;E)$ that is not Hilbert–Schmidt.

Since $r < p$, we have $r < +\infty$. By our hypothesis, we cannot have $p = +\infty$ or $q = +\infty$. Thus $p, q \in]0,+\infty[$. We have

$$0 < \frac{1}{s} = \frac{1}{r} - \frac{1}{q} < \frac{1}{p} \qquad \text{and} \qquad 0 < p < s < +\infty.$$

By Corollary 1.5, there is $S \in \mathscr{L}_{as}^{(s;p)}(E;E)$ that is not Hilbert–Schmidt. For $\phi \in E'$, $\phi \neq 0$, we define

$$T(x,y) = S(x)\phi(y) \qquad \forall\, x, y \in E.$$

T is not Hilbert–Schmidt:

$$\sum_{j,k=1}^\infty \|T(e_k,e_j)\|^2 = \sum_{j,k=1}^\infty \|S(e_k)\|^2|\phi(e_j)|^2 = \|\phi\|^2 \sum_{k=1}^\infty \|S(e_k)\|^2 = +\infty.$$

Now, since $\frac{1}{r} = \frac{1}{s} + \frac{1}{q}$, we have

$$\|(T(x^j,y^j))_{j=1}^\infty\|_r \leq \|(S(x^j))_{j=1}^\infty\|_s \|(\phi(y^j))_{j=1}^\infty\|_q$$
$$\leq \|S\|_{as,(s;p)}\|(x^j)_{j=1}^\infty\|_{w,p}\|\phi\|\|(y^j)_{j=1}^\infty\|_{w,q}.$$

Thus $T \in \mathscr{L}_{as}^{(r;p,q)}(^2E;\, E)$.

4.4. EXAMPLE. If $1/r < 1/r_1 + \cdots + 1/r_n$, with $r, r_1, \ldots, r_n \in]0, +\infty]$ and $r_k > r$, for every $k = 1, \ldots, n$, then there is $T \in \mathscr{L}_{as}^{(r;r_1,\ldots,r_n)}({}^nE; E)$ that is not Hilbert–Schmidt.

For $n = 2$, Example 4.3 gives the result.

For $n > 2$, we have $r < +\infty$ for at least two r_k, $k = 1, \ldots, n$, finite. With no loss of generality we may suppose $r_1, r_2 \in]0, +\infty[$. Now we write $1/t_1 = 1/r_1 + 1/r_2$, and find $t > t_1$, $t < +\infty$, such that $1/r < 1/t + 1/r_3 + \cdots + 1/r_n$. Since $1/t < 1/r_1 + 1/r_2$, we can use either Example 4.2, or Example 4.3, in order to find $R \in \mathscr{L}_{as}^{(t;r_1,r_2)}({}^2E; E)$ that is not Hilbert–Schmidt. Now we consider $\phi \in E'$, $\phi \neq 0$, and define

$$T(x^1, \ldots, x^n) = R(x^1, x^2)\phi(x^3)\ldots\phi(x^n) \qquad \forall\, x^k \in E,\ k = 1, \ldots, n.$$

T is not Hilbert–Schmidt:

$$\sum_{j_k=1,\, k=1,\ldots,n}^{\infty} \|T(e_{j_1}, \ldots, e_{j_n})\|^2 = \|\phi\|^{n-2} \sum_{j,\, k=1}^{\infty} \|R(e_j, e_k)\|^2 = +\infty.$$

On the other hand

$$\begin{aligned}\big\|(T(x_j^1, \ldots, x_j^n))_{j=1}^{\infty}\big\|_r &\le \big\|(R(x_j^1, x_j^2))_{j=1}^{\infty}\big\|_t \prod_{k=3}^{n} \big\|(\phi(x_j^k))_{j=1}^{\infty}\big\|_{r_k} \\ &\le \|R\|_{as,\,(t;r_1,r_2)} \|\phi\|^{n-2} \prod_{k=1}^{n} \|(x_j^k)_{j=1}^{\infty}\|_{w,\,r_k},\end{aligned}$$

and T is absolutely $(r; r_1, \ldots, r_n)$-summing.

With this example we have completed the answer to Problem 3.2.

5. THE PROBLEM OF PIETSCH

Propositions 5.1 and 3.1.a show the reason that made Pietsch state his problem with $n \geq 3$.

5.1. PROPOSITION. *$\mathscr{L}_{as}^{(r;r,+\infty)}(F, G; \mathbb{K}) = \mathscr{L}_{HS}(F, G; \mathbb{K})$, if $r \in]0, +\infty[$ and F, G are Hilbert spaces over $\mathbb{K}$.*

Proof. If $T \in \mathscr{L}_{as}^{(r;r,+\infty)}(F, G : \mathbb{K})$, we consider $\Psi^{-1}(T)(x)(y) = T(x, y)$, for $x \in F$ and $y \in G$. We have

$$\|(\Psi^{-1}(T)(x^j))_{j=1}^{\infty}\|_r = \|(T(x^j, y^j))_{j=1}^{\infty}\|_r = (*),$$

for a convenient choice of $y^j \in B_G$. Thus

$$(*) \le \|T\|_{as,\,(r;r,+\infty)} \|(x^j)_{j=1}^{\infty}\|_{w,\,r} \|(y^j)_{j=1}^{\infty}\|_{\infty} = \|T\|_{as,\,(r;r,+\infty)} \|(x^j)_{j=1}^{\infty}\|_{w,\,r}$$

and we have $\Psi^{-1}(T) \in \mathscr{L}_{as}^{(r;r)}(F;G') = \mathscr{L}_{HS}(F;G')$. Thus $T = \Psi(\Psi^{-1}(T))$ is Hilbert–Schmidt.

On the other hand, if $T \in \mathscr{L}_{HS}(F, G;\mathbb{K})$, then $\Psi^{-1}(T) \in \mathscr{L}_{HS}(F;G') = \mathscr{L}_{as}^{(r;r)}(F;G')$. Thus

$$\begin{aligned}\|(T(x^j, y^j))_{j=1}^{\infty}\|_r &\leq \|(\Psi^{-1}(T)(x^j))_{j=1}^{\infty}\|_r \|(y^j)_{j=1}^{\infty}\|_{\infty} \\ &\leq \|\Psi^{-1}(T)\|_{as,\,(r;r)} \|(x^j)_{j=1}^{\infty}\|_{w,\,r} \|(y^j)_{j=1}^{\infty}\|_{\infty}\end{aligned}$$

and T is absolutely $(r;r,+\infty)$-summing. ∎

5.2. EXAMPLE. If $n \geq 3$, $r, r_1, \ldots, r_n \in]0,+\infty]$, with $1/r = 1/r_1 + \cdots 1/r_n$, and, for at least one $k \in \{1,\ldots,n\}$, $r_k = +\infty$ (say, $r_n = +\infty$), then

(i) there is T Hilbert–Schmidt, $T \notin \mathscr{L}_{as}^{(r;r_1,\ldots,r_n)}({}^nE;\mathbb{K})$, when $r_1,\ldots,r_{n-1} \in]0,+\infty[$;

(ii) there is $T \in \mathscr{L}_{as}^{(r;r_1,\ldots,r_n)}({}^nE;\mathbb{K})$, that is not Hilbert–Schmidt, when $r_j = +\infty$, for some j in $\{1,\ldots,n-1\}$.

Of course, we have to consider the cases:

(i) $r_1,\ldots,r_{n-1} \in]0,+\infty[$,

(ii) For at least one $k \in \{1,\ldots,n-1\}$, $r_k = +\infty$.

In the case (i), we have $1/r = 1/r_1 + \cdots + 1/r_{n-1}$ and $n-1 \geq 2$. By Example 2.5, there is S Hilbert–Schmidt defined on E^{n-1} with values in E', that is not absolutely $(r;r_1,\ldots,r_{n-1})$-summing. It follows that $T = \Psi(S) \in \mathscr{L}_{HS}({}^nE;\mathbb{K})$. If $\Psi(S)$ were absolutely $(r;r_1,\ldots,r_n)$-summing, we would have

$$\|(S(x_j^1,\ldots,x_j^{n-1})_{j=1}^{\infty}\|_r = \|(|\Psi(S)(x_j^1,\ldots,x_j^{n-1},x_j^n)|)_{j=1}^{\infty}\|_r = (*),$$

for convenient choices of $x_j^n \in B_E$, $j = 1,\ldots$. Thus

$$(*) \leq \|\Psi(S)\|_{as,\,(r;r_1,\ldots,r_n)} \prod_{k=1}^{n-1} \|(x_j^k)_{j=1}^{\infty}\|_{r_k} \|(x_j^n)_{j=1}^{\infty}\|_{\infty}.$$

Therefore we would have

$$\|(S(x_j^1,\ldots,x_j^{n-1})_{j=1}^{\infty}\|_r \leq \|\Psi(S)\|_{as,\,(r;r_1,\ldots,r_n)} \prod_{k=1}^{n-1} \|(x_j^k)_{j=1}^{\infty}\|_{r_k}$$

and S would be absolutely $(r;r_1,\ldots,r_{n-1})$-summing, a contradiction.

In the case (ii), we have $1/r = 1/r_1 + \cdots + 1/r_{n-1}$ and $n-1 \geq 2$. By Example 4.1, we can find $S \in \mathscr{L}_{as}^{(r;r_1,\ldots,r_{n-1})}({}^{n-1}E;E)$, that is not Hilbert–Schmidt. We define $T \in \mathscr{L}({}^nE;\mathbb{K})$ by

$$T(x^1,\ldots x^n) = \sum_{k=1}^{\infty} (S(x^1,\ldots,x^{n-1}))_k (x^n)_k = \Psi(S)(x^1,\ldots,x^{n-1},\overline{x^n}).$$

Since S is not Hilbert–Schmidt, $\Psi(S)$ and T are not Hilbert–Schmidt. On the other hand,

$$\begin{aligned}\|(T(x_j^1,\ldots,x_j^n))_{j=1}^{\infty}\|_r &\le \|(S(x_j^1,\ldots,x_j^{n-1}))_{j=1}^{\infty}\|_r \|(x_j^n)_{j=1}^{\infty}\|_{\infty}\\ &\le \|S\|_{as,\,(r;r_1,\ldots,r_{n-1})} \prod_{k=1}^{n} \|(x_j^k)_{j=1}^{\infty}\|_{w,\,r_k}\end{aligned}$$

and T is absolutely $(r;r_1,\ldots,r_n)$-summing.

5.3. Example. If $n\ge 3$, $r,r_1,\ldots,r_n\in\,]0,+\infty]$, $1/r<1/r_1+\cdots+1/r_n$, with $r_k\le r$, for at least one $k\in\{1,\ldots,n\}$, then there is $T\in\mathscr{L}_{as}^{(r;r_1,\ldots,r_n)}({}^nE;\mathbb{K})$, that is not Hilbert–Schmidt.

With no loss of generality we suppose $r_1\le r$. If $r>r_1$, we have $\frac{1}{r}<\frac{1}{r_1}$ and

$$\frac{1}{r}<\frac{1}{r_1}+\cdots+\frac{1}{r_{n-1}},\qquad \text{with } n-1\ge 2.$$

If $r=r_1$, we must have $r_j<+\infty$, for some $j\in\{2,\ldots,n\}$.With no loss of generality we may take this $j=2$ and write

$$\frac{1}{r}<\frac{1}{r_1}+\cdots+\frac{1}{r_{n-1}},\qquad \text{with } n-1\ge 2.$$

By Example 4.2, there is $S\in\mathscr{L}_{as}^{(r;r_1,\ldots,r_{n-1})}({}^{n-1}E;E)$ that is not Hilbert–Schmidt. Now, we can define $T\in\mathscr{L}({}^nE;\mathbb{K})$ in terms of S, as in case (ii) of Example 5.2. As it was proved there, T is absolutely $(r;r_1,\ldots,r_{n-1},+\infty)$-summing (hence absolutely $(r;r_1,\ldots,r_{n-1},r_n)$-summing), but it is not Hilbert–Schmidt.

5.4. Example. If $n\ge 2$, $r,r_1,\ldots,r_n\in\,]0,+\infty]$, $1/r<1/r_1+\cdots+1/r_n$, with $r_k>r$, for $k=1,\ldots,n$, at least one of them $+\infty$, then there is $T\in\mathscr{L}_{as}^{(r;r_1,\ldots,r_n)}({}^nE;\mathbb{K})$, that is not Hilbert–Schmidt.

With no loss of generality we may consider $r_n=+\infty$. Hence we have

$$\frac{1}{r}<\frac{1}{r_1}+\cdots+\frac{1}{r_{n-1}},\qquad \text{with } r_k>r,\ \text{for } k=1,\ldots,n-1,\, n-1\ge 1.$$

By Example 4.4, if $n>2$, and by Corollary 1.5, if $n=2$, we can find $S\in\mathscr{L}_{as}^{(r;r_1,\ldots,r_{n-1})}({}^{n-1}E;E)$, that is not Hilbert–Schmidt. Now, we may define $T\in\mathscr{L}({}^nE;\mathbb{K})$ in terms of S, as in case (ii) of Example 5.2. Hence, as it was proved there, T is absolutely $(r;r_1,\ldots,r_n)$-summing, but it is not Hilbert–Schmidt.

5.5. EXAMPLE. If $n \geq 2$, $r, r_1, \ldots, r_n \in]0, +\infty]$, $1/r < 1/r_1 + \cdots + 1/r_n$, with $+\infty > r_k > r$, for $k = 1, \ldots, n$, then there is $T \in \mathscr{L}_{as}^{(r; r_1, \ldots, r_n)}({}^nE; \mathbb{K})$, that is not Hilbert–Schmidt.

We consider $s > r_n$ such that $1/r < 1/r_1 + \cdots + 1/r_{n-1} + 1/s$. Now we take $t > 0$, such that $1/t = 1/r - 1/s$. We have two possibilities:

(1) $t < r_k$, for $k = 1, \ldots, r_{n-1}$.

(2) $t \geq r_k$ for some $k \in \{1, \ldots, n-1\}$.

In case (1), by Example 4.4, if $n \geq 3$, or by Corollary 1.5, if $n = 2$, there is $R \in \mathscr{L}_{as}^{(t; r_1, \ldots, r_{n-1})}({}^{n-1}E; E)$, that is not Hilbert–Schmidt. By Corollary 1.5, there is $S \in \mathscr{L}_{as}^{(s; r_n)}(E; E)$, that is not Hilbert–Schmidt. We define $T \in \mathscr{L}({}^nE; \mathbb{K})$ by

$$\begin{aligned} T(x^1, \ldots, x^n) &= \sum_{j=1}^{\infty} (R(x^1, \ldots, x^{n-1}))_j (S(x^n))_j \\ &= \Psi(R)(x^1, \ldots, x^{n-1}, \overline{S(x^n)}), \end{aligned}$$

for $x^k \in E$, $k = 1, \ldots, n-1$. We have

$$\begin{aligned} \left(\sum_{j=1}^{\infty} |T(x_j^1, \ldots, x_j^n)|^r \right)^{\frac{1}{r}} &\leq \left(\sum_{j=1}^{\infty} \|R(x_j^1, \ldots, x_j^{n-1})\|^t \right)^{\frac{1}{t}} \left(\sum_{j=1}^{\infty} \|S(x_j^n)\|^s \right)^{\frac{1}{s}} \\ &\leq \|R\|_{as,(t;r_1,\ldots,r_{n-1})} \|S\|_{as,(s;r_n)} \prod_{k=1}^{n} \|(x_j^k)_{j=1}^{\infty}\|_{w, r_k}. \end{aligned}$$

Therefore T is absolutely $(r; r_1, \ldots, r_n)$-summing. Since R and S are not Hilbert–Schmidt, is easy to see that T is not Hilbert–Schmidt.

In case (2), if $n = 2$, we have that $1/t + 1/s = 1/r < 1/r_1 + 1/s$. These inequalities imply $t > r_1$. Hence, by Example 4.2, if $n \geq 3$, or by Corollary 1.5, if $n = 2$, there is $R \in \mathscr{L}_{as}^{(t; r_1, \ldots, r_{n-1})}({}^{n-1}E; E)$, that is not Hilbert–Schmidt. By Corollary 1.5, there is $S \in \mathscr{L}_{as}^{(s; r_n)}(E; E)$, that is not Hilbert–Schmidt. We define $T \in \mathscr{L}({}^nE; \mathbb{K})$ as in case (1) and have T absolutely $(r; r_1, \ldots, r_n)$-summing, but not Hilbert–Schmidt.

With this example we complete our answer to the Problem of Pietsch.

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