ELSEVIER

Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 241 (2007) 160–183

Journal of Differential Equations

www.elsevier.com/locate/jde

# Heteroclinic connections for fully non-linear non-autonomous second-order differential equations

Cristina Marcelli *, Francesca Papalini

*Department of Mathematical Sciences, Technical University of Marche, Via Brecce Bianche, 60131 Ancona, Italy*

Received 4 November 2006

Available online 27 June 2007

**Abstract**

We investigate the solvability of the following strongly non-linear non-autonomous boundary value problem

$$\text{(P)}\qquad \begin{cases} \big(a\big(x(t)\big)x'(t)\big)' = f\big(t, x(t), x'(t)\big) & \text{a.e. } t \in \mathbb{R}, \\ x(-\infty) = \nu^-, \quad x(+\infty) = \nu^+ \end{cases}$$

with $\nu^- < \nu^+$ given constants, where $a(x)$ is a generic continuous positive function and $f$ is a Carathéodory non-linear function. We show that the solvability of (P) is strictly connected to a sharp relation between the behaviors of $f(t, x, \cdot)$ as $|x'| \to 0$ and $f(\cdot, x, x')$ as $|t| \to +\infty$. Such a relation is optimal for a wide class of problems, for which we prove that (P) is not solvable when it does not hold.


*MSC:* primary 34B40; secondary 34B15, 34L30

*Keywords:* Boundary value problems; Unbounded domains; Heteroclinic solutions; Upper and lower solutions; Non-linear differential operators

## 1. Introduction

In this paper we investigate the existence of heteroclinic solutions for the following general second-order non-autonomous boundary value problem on the whole real line

* Corresponding author. Fax: (+39) 071 2204870.
*E-mail addresses:* marcelli@dipmat.univpm.it (C. Marcelli), papalini@dipmat.univpm.it (F. Papalini).

0022-0396/$ – see front matter 
doi:10.1016/j.jde.2007.05.038

$$\text{(P)} \qquad \begin{cases} \big(a\big(x(t)\big)x'(t)\big)' = f\big(t, x(t), x'(t)\big) & \text{a.e. } t \in \mathbb{R}, \\ x(-\infty) = \nu^-, \quad x(+\infty) = \nu^+ \end{cases}$$

with $\nu^- < \nu^+$ given constants. The usual second-order operator $x \mapsto x''$ is here generalized by $x \mapsto (a(x)x')'$ where $a(x)$ is a non-linear continuous positive function. The right-hand side $f : \mathbb{R}^3 \to \mathbb{R}$ is a generic non-linear Carathéodory function, i.e. $f(\cdot, x, y)$ is measurable and $f(t, \cdot, \cdot)$ is continuous.

The study of heteroclinic connections for boundary value problems on the whole real line had a certain impulse in recent years, motivated by applications in various biological, physical and chemical models, such as phase-transition, physical processes in which the variable transits from an unstable equilibrium to a stable one, or front-propagation in reaction–diffusion equations. Indeed, heteroclinic solutions are often called transitional solutions.

Contrary to the case of boundary value problems in compact domains, for which a very wide literature has been produced, in the framework of unbounded intervals many questions are still open and the theory presents some critical aspects. One of the main difficulty consists in the lack of good a priori estimates and appropriate compact embedding theorems for the usual Sobolev spaces.

In particular, the study of heteroclinic connections on the whole real line has been carried out mainly for autonomous quasilinear problems, i.e. for right-hand side of the type $f(x, x') = h(x)x' + g(x)$, owing to the relevant applications to traveling wave solutions for reaction–diffusion equations (see e.g. the monograph [8] for a rather exhaustive literature).

In recent years an increasing attention has been devoted to non-autonomous problems. In [18] Volpert and Suhov considered equations of the type

$$x'' - cx' + g(t, x) = 0$$

for source terms $g(t, x)$ satisfying $g(t, 0) = g(t, 1) \equiv 0$, $g(t, x) > 0$ for $x \in (0, 1)$ (we mention also a contribution by Sanchez [17] for $g(t, x) = a(t)b(x)$). Such equations appear when searching for stationary non-constant solutions of semilinear parabolic equations describing a chemical reaction. The studies in this setting have been recently extended to equations having the general quasi-linear structure

$$x'' = h_1(t, x)x' + h_2(t, x)$$

in [13], in which existence, multiplicity and non-existence results have been proved.

Further recent results concerning the theory of existence and multiplicity of solutions for more general second-order equations with boundary conditions at infinity have been obtained also in different contexts, see [1–5,7,9,11,12,15,19,20] and the book [16].

In [14] we began a study of fully non-linear second-order equations $x'' = f(t, x, x')$ which are specially applicable to right-hand side having the product structure $f(t, x, x') = c(t, x)b(x, x')$, providing some sufficient conditions for the existence of heteroclinic solutions. This research has been developed in [6] for second-order differential operators of the type $(\phi(x'))'$ where $\phi$ is a monotone function which generalizes the one-dimensional $p$-Laplacian operator.

In this paper we investigate problem (P) in presence of a second-order operator of the type $(a(x)x')'$. This operator naturally arises in reaction–diffusion equations with non-constant diffusivity or in porous media equations (see [8]).

Problem (P) was already considered in the framework of compact intervals in [10]. Here we investigate this problem in the whole real line and we give some general existence results (see Theorems 2 and 3) which emphasize the relevance of a precise link between the behaviors of $f(t,x,\cdot)$ as $y \to 0$ and $f(\cdot,x,y)$ as $|t| \to +\infty$. The conditions guaranteeing the existence of solutions are very sharp, in the sense that they are not improvable, since for a rather wide class of problems they are also necessary for the existence of solutions (see the non-existence Theorems 4 and 5).

Our results find immediate and simple applications to right-hand sides having the product structure

$$f(t,x,y) = b(x,y)c(t,x)$$

which are presented in Section 4 (see Theorems 6–9). We here wish just to mention criteria for the particular case $f(t,x,y) = h(t)g(x)b(y)$ with $h \in L^p_{\mathrm{loc}}(\mathbb{R})$, for some $1 \leqslant p \leqslant \infty$, satisfying $t \cdot h(t) \leqslant 0$ for every $t$; $g$ positive in $[\nu^-, \nu^+]$, $b(y)$ satisfying $b(0) = 0$, $0 < b(y) < K|y|^{2-1/p}$ for $y \neq 0$. We have (see Corollary 2):

if

$$\lim_{|t|\to+\infty} \bigl|h(t)\bigr||t|^{-\delta} = \ell_1 \in (0,+\infty) \quad \text{and} \quad \lim_{|y|\to 0} b(y)|y|^{-\gamma} = \ell_2 \in (0,+\infty)$$

for some $\delta > -1, \gamma > 0$, then

$$\text{(P) admits solutions} \quad \Leftrightarrow \quad \gamma < 2 + \delta.$$

This result, which is a necessary and sufficient condition for the existence of solutions, shows the crucial relation between the infinitesimal order $\gamma$ of $b(y)$ as $|y| \to 0$ and the rate $\delta$ of $h(t)$ as $|t| \to +\infty$. Moreover, observe that in this case the behavior of both the right-hand side $f$ and the differential operator $a(x)x'$ with respect to $x$ does not influence the solvability of (P), which results to be completely independent on $a(x)$ and $g(x)$.

A different situation occurs when $h(t) \sim |t|^{-1}$ as $|t| \to +\infty$. In this case we prove that if $\gamma > 1$ problem (P) does not admit solutions. Instead when $\gamma = 1$ a relevant role is assumed by the functions $a(x)$ and $g(x)$. Indeed we prove that (see Corollary 3) if

$$\lim_{|t|\to+\infty} \bigl|th(t)\bigr| = h_1 \in (0,+\infty), \qquad \lim_{|y|\to 0} b(y)|y|^{-1} = h_2 \in (0,+\infty) \quad \text{and}$$
$$b(y) \geqslant h_2|y| \quad \text{for every } y \in \mathbb{R},$$

then

$$h_1 h_2 \cdot \min g(x) > \max a(x) \quad \Rightarrow \quad \text{(P) admits solutions;}$$
$$h_1 h_2 \cdot \max g(x) < \min a(x) \quad \Rightarrow \quad \text{(P) does not admit solutions,}$$

where the maxima and the minima are intended in the interval $[\nu^-, \nu^+]$.

Our approach is based on a suitable combination between fixed point techniques and upper and lower solutions method. In Section 2 we give some preliminary results concerning the solvability and the convergence of solutions of certain auxiliary functional problems in bounded intervals.

Section 3 contains the general existence and non-existence results, while Section 4 is devoted to the applications to right-hand side having the product structure $f(t,x,y)=c(t,x)b(x,y)$.

## 2. Auxiliary results

In this section we will study an auxiliary two-points problem for a functional differential equation in a compact interval, for which we will provide an existence and a convergence result, that will be used in the next section.

Let $I=[a,b]\subset\mathbb{R}$ be a compact interval and let $A: C^1(I)\to C(I)$, $x\mapsto A_x$, and $F: C^1(I)\to L^1(I)$, $x\mapsto F_x$, be two continuous functionals. Let us consider the following functional boundary value problem on $[a,b]$

$$(\mathrm{Q})\qquad \begin{cases} \big(A_u(t)u'(t)\big)'=F_u(t), & \text{a.e. on } I,\\ u(a)=\nu_1, \quad u(b)=\nu_2, \end{cases}$$

where $\nu_1,\nu_2\in\mathbb{R}$ are given.

Throughout this section we assume the following hypotheses on the functionals $A$ and $F$:

(F1) there exist $m, M>0$ such that $m\leqslant A_x(t)\leqslant M$ for every $x\in C^1(I)$, $t\in I$;
(F2) $A$ maps bounded sets of $C^1(I)$ into uniformly continuous sets in $C(I)$, i.e. for every bounded set $D\subset C^1(I)$ and every $\varepsilon>0$ there exists a real $\rho=\rho(\varepsilon)>0$ such that

$$\big|A_x(t_1)-A_x(t_2)\big|<\varepsilon \quad \text{for every } x\in D \text{ and } t_1,t_2\in I \text{ with } |t_1-t_2|<\rho;$$

(F3) there exists $\eta\in L^1_+(I)$: $|F_x(t)|\leqslant\eta(t)$, a.e. on $I$, for every $x\in C^1(I)$.

The following theorem provides an existence result for problem (Q).

**Theorem 1.** *Under the assumptions* (F1)–(F3), *for every $\nu_1,\nu_2\in\mathbb{R}$ there exists a function $u\in C^1(I)$ such that $A_u\cdot u'\in W^{1,1}(I)$ and*

$$\begin{cases} \big(A_u(t)u'(t)\big)'=F_u(t), & \textit{a.e. on } I,\\ u(a)=\nu_1, \quad u(b)=\nu_2, \end{cases}$$

*i.e. $u$ is a solution of problem* (Q).

**Proof.** For every $x\in C^1(I)$, put

$$I_x:=\frac{\nu_2-\nu_1-\int_a^b\big(\frac{1}{A_x(\tau)}\int_a^\tau F_x(s)\,\mathrm{d}s\big)\,\mathrm{d}\tau}{\int_a^b\frac{1}{A_x(\tau)}\,\mathrm{d}\tau}\in\mathbb{R}$$

and observe that

$$\int_a^b\frac{1}{A_x(\tau)}\left(I_x+\int_a^\tau F_x(s)\,\mathrm{d}s\right)\mathrm{d}\tau=\nu_2-\nu_1. \tag{1}$$

Let us consider the operator $\Gamma : C^1(I) \to C^1(I)$, $x \mapsto \Gamma_x$, defined by

$$\Gamma_x(t) := \nu_1 + \int_a^t \frac{1}{A_x(\tau)} \left( I_x + \int_a^\tau F_x(s)\,\mathrm{d}s \right) \mathrm{d}\tau.$$

It is well defined and by (1) it is immediate to check that if $\Gamma$ has a fixed point, then this is a solution to (Q).

Of course, if $(x_k)_k$ converges to $x$ in the space $C^1(I)$, then by the continuity of the functionals $A$ and $F$ we have that $(A_{x_k})_k$ converges to $A_x$ in $C(I)$ and $(F_{x_k})_k$ converges to $F_x$ in $L^1(I)$. Moreover, since

$$\frac{1}{A_{x_k}(\tau)} \left| \int_a^\tau F_{x_k}(s)\,\mathrm{d}s \right| \leqslant \frac{1}{m} \left( \int_a^b \eta(s)\,\mathrm{d}s \right) \quad \text{for every } \tau \in I,$$

by applying the dominate convergence theorem we obtain that $I_{x_k}$ converges to $I_x$.

Let us now show that $(\Gamma'_{x_k})_k$ uniformly converges to $\Gamma'_x$ in $I$. To this aim, note that

$$\begin{aligned}
\left|\Gamma'_{x_k}(t) - \Gamma'_x(t)\right| &= \frac{1}{A_{x_k}(t)A_x(t)} \left| A_x(t)\left( I_{x_k} + \int_a^t F_{x_k}(s)\,\mathrm{d}s \right) - A_{x_k}(t)\left( I_x + \int_a^t F_x(s)\,\mathrm{d}s \right) \right| \\
&\leqslant \frac{1}{m^2} \Bigg( A_x(t)|I_{x_k} - I_x| + |I_x|\left|A_{x_k}(t) - A_x(t)\right| \\
&\quad + A_x(t) \left| \int_a^t \left(F_{x_k}(s) - F_x(s)\right)\mathrm{d}s \right| + \left| \int_a^t F_x(s)\,\mathrm{d}s \right| \left|A_{x_k}(t) - A_x(t)\right| \Bigg),
\end{aligned}$$

for every $t \in I$. So,

$$\begin{aligned}
\|\Gamma'_{x_k} - \Gamma'_x\|_{C(I)} \leqslant \frac{1}{m^2} \Bigg[ & M|I_{x_k} - I_x| + \left( |I_x| + \int_a^b \left|F_x(t)\right|\mathrm{d}t \right) \|A_{x_k} - A_x\|_{C(I)} \\
& + M\|F_{x_k} - F_x\|_{L^1(I)} \Bigg],
\end{aligned}$$

hence $(\Gamma'_{x_k})_k$ uniformly converges to $\Gamma'_x$ in $I$. Then, since

$$\|\Gamma_{x_k} - \Gamma_x\|_{C(I)} \leqslant \int_a^b \left|\Gamma'_{x_k}(t) - \Gamma'_x(t)\right|\mathrm{d}t,$$

we finally deduce that $\Gamma_{x_k} \to \Gamma_x$ in $C^1(I)$, i.e. $\Gamma$ is a continuous operator.

Observe now that

$$|I_x| \leqslant \frac{M}{b-a}\left[|\nu_2-\nu_1| + \frac{b-a}{m}\|\eta\|_{L^1(I)}\right] = \frac{M}{b-a}|\nu_2-\nu_1| + \frac{M}{m}\|\eta\|_{L^1(I)} \tag{2}$$

for every $x \in C^1(I)$. Therefore,

$$\left|\Gamma_x'(t)\right| \leqslant \frac{1}{m}\left[\frac{M}{b-a}|\nu_2-\nu_1| + \frac{M+m}{m}\|\eta\|_{L^1(I)}\right] \quad \text{for every } x \in C^1(I) \text{ and } t \in I,$$

and this implies that $\Gamma(C^1(I))$ is bounded in $C^1(I)$, say $\|\Gamma_x\|_{C^1(I)} \leqslant S$ for every $x \in C^1(I)$. So, put

$$D := \left\{x \in C^1(I)\colon\ \|x\|_{C^1(I)} \leqslant S\right\},$$

$\Gamma$ is a continuous operator mapping $D$ onto itself.

Our goal is to show that $\Gamma(D)$ is relatively compact, and in order to do this it suffices to prove that $\Gamma(D)$ is equicontinuous. To this aim, fixed $\epsilon > 0$, from (F2) there exists $\rho = \rho(\epsilon) > 0$ such that $\forall x \in D$ we have

$$\left|A_x(t_1) - A_x(t_2)\right| < \epsilon \quad \text{and} \quad \left|\int_{t_1}^{t_2} \eta(s)\,\mathrm{d}s\right| < \epsilon \quad \text{for every } t_1, t_2 \in I \text{ with } |t_1-t_2| < \rho.$$

Hence, if $|t_1-t_2| < \rho$, by (2) for every $x \in D$ we have

$$\begin{aligned}
\left|\Gamma_x'(t_1) - \Gamma_x'(t_2)\right| &= \frac{1}{A_x(t_1)A_x(t_2)}\left|A_x(t_2)\left(I_x + \int_a^{t_1} F_x(s)\,\mathrm{d}s\right) - A_x(t_1)\left(I_x + \int_a^{t_2} F_x(s)\,\mathrm{d}s\right)\right| \\
&\leqslant \frac{1}{m^2}\left[\left|A_x(t_2) - A_x(t_1)\right|\left(\left|\int_a^{t_2} F_x(s)\,\mathrm{d}s\right| + |I_x|\right) + A_x(t_2)\left|\int_{t_2}^{t_1} F_x(s)\,\mathrm{d}s\right|\right] \\
&\leqslant \frac{1}{m^2}\left[\left|A_x(t_2) - A_x(t_1)\right|\left(\int_a^{b} \left|F_x(s)\right|\mathrm{d}s + |I_x|\right) + M\left|\int_{t_1}^{t_2} F_x(s)\,\mathrm{d}s\right|\right] \\
&\leqslant \frac{\epsilon}{m^2}\left[\frac{M}{b-a}|\nu_2-\nu_1| + \frac{M+m}{m}\|\eta\|_{L^1(I)} + M\right]
\end{aligned}$$

that is $\Gamma(D)$ is equicontinuous.

Summarizing, $\Gamma : D \to D$ is a continuous, compact operator defined in a closed, bounded, convex set. Hence it admits a fixed point and then problem (Q) admits at least a solution. □

Let us now consider the equation

$$\text{(E)} \quad \left(a\big(x(t)\big)x'(t)\right)' = f\big(t, x(t), x'(t)\big) \quad \text{a.e. } t \in \mathbb{R},$$

where $f : \mathbb{R}^3 \to \mathbb{R}$ is a given Carathéodory function and $a : \mathbb{R} \to \mathbb{R}$ is a positive continuous function.

The following result concerns the convergence of sequences of functions related, in a certain sense, to solutions of the previous equation.

For all $n \in \mathbb{N}$ let $I_n = [-n, n]$ and $u_n \in C^1(I_n)$ be such that $a(u_n)u_n' \in W^{1,1}(I_n)$ and

$$\big(a\big(u_n(t)\big)u_n'(t)\big)' = f\big(t, u_n(t), u_n'(t)\big) \quad \text{a.e. } t \in I_n.$$

Consider the following sequences of functions $(y_n)_n$, $(z_n)_n$, $(x_n)_n$ defined by

$$y_n(t) := \begin{cases} u_n'(t) & \text{for } t \in I_n, \\ 0 & \text{elsewhere in } \mathbb{R}, \end{cases} \qquad z_n(t) := \begin{cases} (a(u_n(t))u_n'(t))' & \text{for a.e. } t \in I_n, \\ 0 & \text{elsewhere in } \mathbb{R}, \end{cases}$$

$$x_n(t) := u_n(0) + \int_0^t y_n(s)\,\mathrm{d}s.$$

**Lemma 1.** *Let $J \subset \mathbb{R}$ be a given interval* (*not necessarily bounded*). *Assume that*:

(i) *the sequences $(u_n(0))_n$ and $(u_n'(0))_n$ are bounded*;
(ii) *there exist two functions $H, \gamma \in L^1(J)$ such that $|y_n(t)| \leqslant H(t)$ and $|z_n(t)| \leqslant \gamma(t)$ a.e. on $J$ and for all $n \in \mathbb{N}$.*

*Then there exist three subsequences $(y_{n_k})_k$, $(z_{n_k})_k$, $(x_{n_k})_k$ and a function $x \in C^1(J)$, with $a(x)x' \in W^{1,1}(J)$, such that*:

(a) *$x_{n_k} \to x$ uniformly on $J$*;
(b) *$y_{n_k} \to x'$ in $L^1(J)$ and pointwise on $J$*;
(c) *$z_{n_k} \rightharpoonup (a(x)x')'$ weakly in $L^1(J)$*;
(d) *$(a(x(t))x'(t))' = f(t, x(t), x'(t))$ a.e. on $J$.*

**Proof.** From assumption (ii) it follows that the sequences $(y_n)_n$ and $(z_n)_n$ are uniformly integrable. Thus, by applying the Dunford–Pettis theorem, we deduce the existence of two subsequences $(y_{n_k})_k$, $(z_{n_k})_k$ such that $y_{n_k} \rightharpoonup g$ and $z_{n_k} \rightharpoonup h$ weakly in $L^1(J)$, for some $g, h \in L^1(J)$. Therefore, for every measurable subset $A \subset J$ we have

$$\int_A y_{n_k}(t)\,\mathrm{d}t \to \int_A g(t)\,\mathrm{d}t \quad \text{and} \quad \int_A z_{n_k}(t)\,\mathrm{d}t \to \int_A h(t)\,\mathrm{d}t, \quad \text{as } k \to +\infty.$$

By assumption (i) we can assume that $u_{n_k}(0) \to u_0$ and $u_{n_k}'(0) \to y_0$, for some $u_0, y_0 \in \mathbb{R}$. So, putting $x(t) := u_0 + \int_0^t g(s)\,\mathrm{d}s$, we have that $x_{n_k}(t) \to x(t)$ as $k \to +\infty$; moreover $x \in C(J)$ and $x'(t) = g(t)$ a.e. on $J$.

In $I_{n_k}$ we have $u_{n_k}'(t) = y_{n_k}(t)$ and then $u_{n_k}(t) = x_{n_k}(t)$, so

$$a\big(x_{n_k}(t)\big)y_{n_k}(t) = a\big(u_{n_k}(t)\big)u_{n_k}'(t) = a\big(x_{n_k}(0)\big)y_{n_k}(0) + \int_0^t z_{n_k}(s)\,\mathrm{d}s.$$

Then, for every fixed $t \in J$ we have

$$\lim_{k\to+\infty} y_{n_k}(t) = \frac{1}{a(x(t))}\left(a(u_0)y_0 + \int_0^t h(s)\,\mathrm{d}s\right)$$

hence the right-hand side coincides with $g(t)$. Consequently $g$ is continuous, $x \in C^1(J)$, $y_{n_k} \to x'$ in $L^1(J)$ and pointwise on $J$. Moreover, we have

$$a\big(x(t)\big)x'(t) = a\big(x(t)\big)g(t) = a(u_0)y_0 + \int_0^t h(s)\,\mathrm{d}s,$$

so $a(x(t))x'(t) \in W^{1,1}(\mathbb{R})$, with $(a(x(t))x'(t))' = h(t)$ a.e. $t \in J$. Therefore $z_{n_k} \rightharpoonup (a(x)x')'$ weakly in $L^1(J)$.

Furthermore, note that for every $t \in J$

$$\big|x_{n_k}(t) - x(t)\big| \leqslant \big|u_{n_k}(0) - u_0\big| + \left|\int_0^t \big|y_{n_k}(s) - g(s)\big|\,\mathrm{d}s\right| \leqslant \big|u_{n_k}(0) - u_0\big| + \|y_{n_k} - g\|_{L^1(J)}$$

then the sequence $(x_{n_k})_k$ uniformly converges to $x$ in $J$.

Finally, for a.e. $t \in I_{n_k}$ we have $z_{n_k}(t) = (a(x_{n_k}(t))y_{n_k}(t))' = f(t, x_{n_k}(t), x'_{n_k}(t))$ hence

$$\big(a\big(x(t)\big)x'(t)\big)' = f\big(t, x(t), x'(t)\big) \quad \text{a.e. in } J,$$

from the continuity of $f(t,\cdot,\cdot)$. □

**Remark 1.** If there exists $L > 0$ such that $u_n'(t) \geqslant 0$ for every $|t| > L$ and $n \in \mathbb{N}$, then $x$ is definitively increasing, since $x'(t) \geqslant 0$ for every $|t| > L$.

**Remark 2.** If there exist $\alpha, \beta \in C(J)$ such that $\alpha(t) \leqslant u_n(t) \leqslant \beta(t)$ for every $t \in I_n$, $n \in \mathbb{N}$, then $\alpha(t) \leqslant x(t) \leqslant \beta(t)$ for every $t \in J$.

The uniform convergence of the sequence $(x_{n_k})_k$ to $x$ leads to the following result concerning the attaining of boundary conditions.

**Corollary 1.** *Let $J = \mathbb{R}$. Under the same assumptions as in Lemma* 1*, if we suppose that*

$$\lim_{n\to+\infty} u_n(-n) = u^-, \qquad \lim_{n\to+\infty} u_n(n) = u^+,$$

*then we have that*

$$\lim_{t\to-\infty} x(t) = u^-, \qquad \lim_{t\to+\infty} x(t) = u^+.$$

## 3. Existence and non-existence theorems

In this section we investigate the existence of heteroclinic solutions to equation (E). Our approach is based on fixed point techniques suitably combined to the method of upper and lower solutions, according to the following definition.

**Definition 1.** A lower (upper) solution for equation (E) is a bounded function $\alpha \in C^1(\mathbb{R})$ such that $a(\alpha)\alpha' \in W^{1,1}(\mathbb{R})$ and

$$\big(a\big(\alpha(t)\big)\alpha'(t)\big)' \geqslant (\leqslant) f\big(t, \alpha(t), \alpha'(t)\big) \quad \text{for a.e. } t \in \mathbb{R}.$$

Throughout the paper we will assume the existence of an ordered pair of lower and upper solutions $\alpha, \beta$, i.e. satisfying $\alpha(t) \leqslant \beta(t)$ for every $t \in \mathbb{R}$, and we will adopt the following notations:

$$\mathcal{I} := \Big[\inf_{t\in\mathbb{R}} \alpha(t), \sup_{t\in\mathbb{R}} \beta(t)\Big], \qquad \nu := |\mathcal{I}| = \sup_{t\in\mathbb{R}} \beta(t) - \inf_{t\in\mathbb{R}} \alpha(t),$$

$$m := \min_{x\in\mathcal{I}} a(x) > 0, \qquad M := \max_{x\in\mathcal{I}} a(x), \qquad d := \max\big\{\big|\alpha'(t)\big| + \big|\beta'(t)\big|:\ t \in \mathbb{R}\big\}.$$

Note that the value $d$ is well defined, in fact

$$\lim_{|t|\to+\infty} \alpha'(t) = \lim_{|t|\to+\infty} \beta'(t) = 0,$$

since $a(\alpha)\alpha' \in W^{1,1}(\mathbb{R})$ and $m > 0$ (the same argument holds for $\beta'$).

Moreover, in what follows, $x^+$ and $x^-$ will denote the positive and negative parts of the real number $x$, respectively, and we will put $x \wedge y := \min\{x, y\}$, $x \vee y := \max\{x, y\}$.

Our main result is the following existence theorem.

**Theorem 2.** *Assume that there exists a pair of lower and upper solutions $\alpha, \beta \in C^1(\mathbb{R})$ of the equation* (E)*, satisfying $\alpha(t) \leqslant \beta(t)$, for every $t \in \mathbb{R}$, with $\alpha$ increasing in $(-\infty, -\Lambda)$, $\beta$ increasing in $(\Lambda, +\infty)$, for some constant $\Lambda \in \mathbb{R}$. Moreover, assume that there exist two constants $L > \Lambda$, $H > \frac{\nu}{2L}$, a continuous function $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ and a function $\lambda \in L^p([-L, L])$ with $1 \leqslant p \leqslant \infty$, such that*

$$\int^{+\infty} \frac{r^{1-\frac{1}{p}}}{\theta(r)}\, \mathrm{d}r = +\infty, \tag{3}$$

$$\big|f(t, x, y)\big| \leqslant \lambda(t)\theta\big(a(x)|y|\big) \quad \textit{for a.e. } |t| \leqslant L,\ \textit{every } x \in \mathcal{I},\ \textit{every } |y| \geqslant H. \tag{4}$$

*Finally, suppose that there exists a constant $\gamma > 1$ such that for every $C > 0$ there exist a function $\eta_C \in L^1(\mathbb{R})$ and a function $K_C \in W^{1,1}_{\mathrm{loc}}([0, +\infty))$, null in $[0, L]$ and strictly increasing in $[L, +\infty)$, such that*:

$$\int^{\infty} \big(K_C(t)\big)^{\frac{1}{1-\gamma}}\, \mathrm{d}t < +\infty, \tag{5}$$

$$\begin{cases} f(t,x,y) \leqslant -K_C'(t)|y|^\gamma, \\ f(-t,x,y) \geqslant K_C'(t)|y|^\gamma \end{cases} \quad \text{for a.a. } t \geqslant L, \text{ every } x \in \mathcal{I}, \ |y| \leqslant 2\left(\frac{M}{m} C + d\right) \tag{6}$$

*and*

$$\big|f\big(t,x,u(t)\big)\big| \leqslant \eta_C(t) \quad \begin{array}{l} \text{for a.a. } t \in \mathbb{R}, \text{ every } x \in \mathcal{I}, \text{ and } u \in W^{1,1}(\mathbb{R}) \\ \text{such that } \big|u(t)\big| \leqslant 2N_C(t) + \big|\alpha'(t)\big| + \big|\beta'(t)\big|, \end{array} \tag{7}$$

*where*

$$N_C(t) := \frac{M}{m}\left(C^{1-\gamma} + \frac{\gamma-1}{M} K_C\big(|t|\big)\right)^{1/(1-\gamma)}.$$

*Then, there exists a function $x \in C^1(\mathbb{R})$, with $a(x)x' \in W^{1,1}(\mathbb{R})$, such that*

$$\begin{cases} \big(a\big(x(t)\big)x'(t)\big)' = f\big(t, x(t), x'(t)\big) \quad \text{a.e. } t \in \mathbb{R}, \\ \alpha(t) \leqslant x(t) \leqslant \beta(t) \quad \text{for every } t \in \mathbb{R}, \\ x(-\infty) = \alpha(-\infty), \quad x(+\infty) = \beta(+\infty). \end{cases}$$

**Proof.** By (3), there exists a constant $C > \frac{M}{m} H \geqslant H$ such that

$$\int_{MH}^{mC} \frac{r^{1-1/p}}{\theta(r)} \,\mathrm{d}r > (M\nu)^{1-\frac{1}{p}} \|\lambda\|_p. \tag{8}$$

Let us fix an integer $n \in \mathbb{N}$ and put $I_n := [-n, n]$. Let us introduce the truncation operator $T : W^{1,1}(I_n) \to W^{1,1}(I_n)$ defined by

$$T(x) := T_x \quad \text{where } T_x(t) := \big[\beta(t) \wedge x(t)\big] \vee \alpha(t). \tag{9}$$

Of course, $T$ is well defined and $T_x'(t) = x'(t)$ for a.a. $t \in I_n$ such that $\alpha(t) < x(t) < \beta(t)$, whereas $T_x'(t) = \alpha'(t)$ for a.e. $t$ such that $x(t) \leqslant \alpha(t)$, $T_x'(t) = \beta'(t)$ for a.e. $t$ such that $x(t) \geqslant \beta(t)$.

Moreover, consider the penalty function $u : \mathbb{R}^2 \to \mathbb{R}$ defined by $u(t,x) := [x - \beta(t)]^+ - [x - \alpha(t)]^-$. Of course, $u(t,x) = 0$ if $\alpha(t) \leqslant x \leqslant \beta(t)$.

For every $x \in W^{1,1}_{\mathrm{loc}}(\mathbb{R})$, put

$$Q_x(t) := -\big(2N_C(t) + \big|\alpha'(t)\big| + \big|\beta'(t)\big|\big) \vee \big[T_x'(t) \wedge \big(2N_C(t) + \big|\alpha'(t)\big| + \big|\beta'(t)\big|\big)\big]$$

and let us consider the following auxiliary boundary value problem on the compact interval $I_n = [-n, n]$, for $n > L$:

$$(\mathrm{P}_n^*) \quad \begin{cases} \big(a\big(T_x(t)\big)x'(t)\big)' = f\big(t, T_x(t), Q_x(t)\big) + \arctan\big(u\big(t, x(t)\big)\big), \quad \text{a.e. } t \in I_n, \\ x(-n) = \alpha(-n), \quad x(n) = \beta(n). \end{cases}$$

*Step* 1. Let us now prove that if $x \in C^1(I_n)$ is a solution of problem $(\mathrm{P}_n^*)$, then $\alpha(t) \leqslant x(t) \leqslant \beta(t)$ for all $t \in I_n$, hence $T_x(t) \equiv x(t)$ and $u(t, x(t)) \equiv 0$.

First we show that $\alpha(t) \leqslant x(t)$ for every $t \in I_n$. If $t_0$ is such that $x(t_0) - \alpha(t_0) := \min(x(t) - \alpha(t)) < 0$, then $t_0$ belongs to a compact interval $[t_1, t_2] \subset I_n$ satisfying $x(t_1) - \alpha(t_1) = x(t_2) - \alpha(t_2) = 0$ and $x(t) - \alpha(t) < 0$ for every $t \in (t_1, t_2)$. Hence, $T_x(t) \equiv \alpha(t)$ and $Q_x(t) \equiv \alpha'(t)$ in $[t_1, t_2]$, then

$$\big(a\big(\alpha(t)\big)x'(t)\big)' = f\big(t, \alpha(t), \alpha'(t)\big) + \arctan\big(x(t) - \alpha(t)\big) < \big(a\big(\alpha(t)\big)\alpha'(t)\big)' \quad \text{a.e. in } (t_1, t_2).$$

Thus, the function $a(\alpha(t))(x'(t) - \alpha'(t))$ is strictly decreasing in $(t_1, t_2)$, so we have $a(\alpha(t))(x'(t) - \alpha'(t)) < a(\alpha(t_0))(x'(t_0) - \alpha'(t_0)) = 0$ for $t \in (t_0, t_2)$, then also $x'(t) - \alpha'(t) < 0$ in $(t_0, t_2)$, a contradiction. Similarly one can show that $x(t) \leqslant \beta(t)$ for every $t \in I_n$.

*Step* 2. Now we prove that if $x \in C^1(I_n)$ is a solution of problem $(\mathrm{P}_n^*)$, then $|x'(t)| \leqslant N_C(t)$ for every $t \in I_n$ and $x'(t) \geqslant 0$ in $I_n \setminus [-L, L]$.

Since $x \in C^1([-L, L])$ and $x([-L, L]) \subset \mathcal{I}$, we can apply Lagrange theorem to deduce that for some $\tau_0 \in I_n$ we have

$$\big|x'(\tau_0)\big| = \frac{1}{2L}\big|x(L) - x(-L)\big| \leqslant \frac{\sup\beta - \inf\alpha}{2L} < H < C.$$

We start by proving that $|x'(t)| < C$ for every $t \in [-L, L]$. To this end, assume, by contradiction, the existence of an interval $J = (\tau_1, \tau_2) \subset (-L, L)$, such that $H < |x'(t)| < C$ in $J$ and $|x'(\tau_1)| = H$, $|x'(\tau_2)| = C$ or vice versa. Of course, $x'(t)$ keeps constant sign in $J$; assume now $x'(t) > 0$ in $J$ (the proof will proceed similarly if $x'(t) < 0$).

Since $x'(t) < C$ for every $t \in J$, by the definition of $(\mathrm{P}_n^*)$ and assumption (4), for a.e. $t \in J$ it results

$$\big|\big(a\big(x(t)\big)x'(t)\big)'\big| = \big|\big(a\big(T_x(t)\big)x'(t)\big)'\big| = \big|f\big(t, x(t), x'(t)\big)\big| \leqslant \lambda(t)\theta\big(a\big(x(t)\big)x'(t)\big).$$

Therefore, by Hölder inequality, if $q$ is the conjugate exponent of $p$, we deduce

$$\int_{MH}^{mC} \frac{r^{1/q}}{\theta(r)}\,\mathrm{d}r \leqslant \int_{\tau_1}^{\tau_2} \frac{(a(x(t))x'(t))^{1/q}}{\theta(a(x(t))x'(t))}\big|\big(a\big(x(t)\big)x'(t)\big)'\big|\,\mathrm{d}t \leqslant \int_{\tau_1}^{\tau_2} \big(a\big(x(t)\big)x'(t)\big)^{1/q}\lambda(t)\,\mathrm{d}t$$
$$\leqslant \left(\int_{\tau_1}^{\tau_2} \big(a\big(x(t)\big)x'(t)\big)\,\mathrm{d}t\right)^{\frac{1}{q}} \|\lambda\|_p \leqslant M^{\frac{1}{q}}\|\lambda\|_p \left(\int_{\tau_1}^{\tau_2} x'(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \leqslant (M\nu)^{\frac{1}{q}}\|\lambda\|_p$$

in contradiction with (8). Thus, we get $|x'(t)| < C \leqslant \frac{M}{m}C = N_C(t)$ for every $t \in [-L, L]$.

By assumption (6) and the definition of $Q_x$, we deduce that $(a(x(t))x'(t))' \leqslant 0$ for a.e. $t \geqslant L$. So, if $x'(\bar{t}) < 0$ for some $\bar{t} \in [L, n)$ we have $a(x(t))x'(t) \leqslant a(x(\bar{t}))x'(\bar{t}) < 0$ for every $t \in [\bar{t}, n]$ and then $x'(t) < 0$ for every $t \in [\bar{t}, n]$, hence $x(n) < x(\bar{t}) \leqslant \beta(\bar{t}) \leqslant \beta(n)$, a contradiction. Similarly we can show that $x'(t) \geqslant 0$ for every $t \in [-n, -L]$.

Observe now that if $x'(t_0) = 0$ for some $t_0 \in [L, n)$, then $x'(t) = 0$ for every $t \in [t_0, n]$. Indeed, since $x'(t) < N_C(t)$ in a right neighborhood $[t_0, t_0 + \delta]$, by (6) we deduce $(a(x(t))x'(t))' = f(t, x(t), x'(t)) \leqslant 0$, i.e. $a(x(t))x'(t)$ is decreasing in $[t_0, t_0 + \delta]$. This implies that $x'(t) \leqslant 0$ in $[t_0, t_0 + \delta]$ and being $x'(t) \geqslant 0$ in $[L, n]$, we conclude $x'(t) = 0$ in $[t_0, t_0 + \delta]$. Therefore, $\sup\{t \leqslant n\colon\ x'(t) \equiv 0 \text{ in } [t_0, t]\} = n$.

Let us now prove that $|x'(t)| \leqslant N_C(t)$ in $I_n \setminus [-L, L]$. To this aim, let $\hat{t} := \sup\{t > L\colon x'(\tau) \leqslant N_C(\tau) \text{ in } [L, t]\}$. If $\hat{t} < n$ then $x'(\hat{t}) = N_C(\hat{t}) > 0$ and $0 < x'(t) < 2N_C(t)$ in a right neighborhood $[\hat{t}, \hat{t} + \delta]$. Moreover, by virtue of what observed above, we have $x'(t) > 0$ in $[L, \hat{t} + \delta]$. Hence, we have $(a(x(t))x'(t))' = f(t, x(t), x'(t))$ a.e. in $[L, \hat{t} + \delta]$ and by applying assumption (6) we have

$$\big(a\big(x(t)\big)x'(t)\big)' \leqslant -K_C'(t)\big|x'(t)\big|^\gamma \leqslant -\frac{K_C'(t)}{M^\gamma}\big(a\big(x(t)\big)x'(t)\big)^\gamma, \quad \text{a.e. in } [L, \hat{t} + \delta].$$

Then

$$\frac{1}{1-\gamma}\big[\big(a\big(x(t)\big)x'(t)\big)^{1-\gamma} - \big(a\big(x(L)\big)x'(L)\big)^{1-\gamma}\big] = \int_L^t \frac{(a(x(s))x'(s))'}{(a(x(s))x'(s))^\gamma}\,\mathrm{d}s \leqslant -\frac{K_C(t)}{M^\gamma},$$

for every $t \in [L, \bar{t} + \delta]$. Hence,

$$\big(a\big(x(t)\big)x'(t)\big)^{1-\gamma} \geqslant \big(a\big(x(L)\big)x'(L)\big)^{1-\gamma} + \frac{\gamma - 1}{M^\gamma}K_C(t) \geqslant (MC)^{1-\gamma} + \frac{\gamma - 1}{M^\gamma}K_C(t),$$

i.e. $x'(t) \leqslant N_C(t)$ for every $t \in [L, \hat{t} + \delta]$, in contradiction with the definition of $\hat{t}$. The same argument works in the interval $[-n, -L]$.

*Step* 3. Let us now prove that problem $(\mathrm{P}_n^*)$ admits solutions for every $n > L$. To this aim, let $A : C^1(I_n) \to C(I_n)$, $x \mapsto A_x$, and $F : C^1(I_n) \to L^1(I_n)$, $x \mapsto F_x$, be the functionals defined by

$$A_x(t) := a\big(T_x(t)\big), \qquad F_x(t) := f\big(t, T_x(t), Q_x(t)\big) + \arctan\big(u\big(t, x(t)\big)\big).$$

As it is easy to check, by (7) the functionals are well defined and continuous. Moreover, if $D$ is a bounded subset of $C^1(I_n)$, i.e. there exists $S > 0$ such that $\|x\|_{C^1(I)} \leqslant S$, then for fixed $\epsilon > 0$, by the uniform continuity of $a(\cdot)$ in $\mathcal{I}$, there exists $\delta = \delta(\epsilon) > 0$ such that $|a(\xi_1) - a(\xi_2)| < \epsilon$ whenever $|\xi_1 - \xi_2| < \delta$. Therefore, putting $\rho = \frac{\delta}{S}$, if $|t_1 - t_2| < \rho$ we have

$$\big|T_x(t_1) - T_x(t_2)\big| \leqslant \left|\int_{t_1}^{t_2} \big|x'(\tau)\big|\,\mathrm{d}\tau\right| \leqslant S|t_1 - t_2| < \delta \quad \text{for every } x \in D$$

and consequently $|A_x(t_1) - A_x(t_2)| < \epsilon$ for every $x \in D$, whenever $|t_1 - t_2| < \rho$.

Therefore, the functionals $A$ and $F$ satisfy the hypotheses (F1)–(F3) of Theorem 1. So, by applying such a result with $\nu_1 = \alpha(-n)$ and $\nu_2 = \beta(n)$, we obtain the existence of a function $u_n \in C^1(I_n)$ such that $a(u_n)u_n' \in W^{1,1}(I_n)$ which is a solution of the problem $(\mathrm{P}_n^*)$. Moreover, taking into account the properties proved in Steps 1 and 2, we infer that

$$\big(a\big(u_n(t)\big)u_n'(t)\big)' = f\big(t, u_n(t), u_n'(t)\big) \quad \text{a.e. } t \in I_n$$

for every $n \in \mathbb{N}$.

Observe now that the sequence of solutions $(u_n)_n$ satisfies all the assumptions of Lemma 1, with $J = \mathbb{R}$, $H(t) = N_C$ and $\gamma(t) = \eta_C(t)$, for $t \in \mathbb{R}$, where $C$ is the constant fixed at the beginning of the proof of Theorem 2. So, by assertion (d) of such a lemma, we deduce the existence of a solution $x$ of equation (E) and from Corollary 1 we deduce the assertion. □

**Remark 3.** In view of the previous proof (see (8)), note that condition (3) can be weakened as follows:

$$\int_{HM}^{+\infty} \frac{r^{1-\frac{1}{p}}}{\theta(r)}\,\mathrm{d}r > (M\nu)^{1-\frac{1}{p}} \|\lambda\|_p. \tag{10}$$

The following existence result is analogous to the previous one and covers the case when (6) is fulfilled for $\gamma = 1$.

**Theorem 3.** *Let all the assumptions of Theorem* 2 *be satisfied for* $\gamma = 1$, *with assumption* (5) *replaced by the following one*

$$\int^{\infty} e^{-\frac{1}{M}K_C(t)}\,\mathrm{d}t < +\infty \tag{11}$$

*and the function* $N_C(t)$ *replaced by*

$$N_C^*(t) := \frac{M}{m} C e^{-\frac{1}{M}K_C(|t|)}. \tag{12}$$

*Then the assertion of Theorem* 2 *holds.*

**Proof.** The proof is the same of that of Theorem 2. The sole difference is in proving that $x'(t) \leqslant N_C^*(t)$ in $I_n \setminus [-L, L]$. Indeed, defining as before $\hat{t} := \sup\{t > L\colon\ x'(\tau) \leqslant N_C^*(\tau) \text{ in } [L, t]\}$, we deduce again $(a(x(t))x'(t))' = f(t, x(t), x'(t))$ in a right neighborhood $[\hat{t}, \hat{t} + \delta]$ and $x'(t) > 0$ in $[L, \hat{t} + \delta]$. Since $\gamma = 1$ in this case we get

$$\big(a\big(x(t)\big)x'(t)\big)' \leqslant -K_C'(t)\big|x'(t)\big| \leqslant -\frac{K_C'(t)}{M}\big(a\big(x(t)\big)x'(t)\big), \quad \text{a.e. in } [L, \hat{t} + \delta],$$

then

$$\frac{a(x(t))x'(t)}{a(x(L))x'(L)} = e^{\int_L^t \frac{(a(x(s))x'(s))'}{(a(x(s))x'(s))}\,\mathrm{d}s} \leqslant e^{-\frac{1}{M}K_C(t)}$$

implying

$$a\big(x(t)\big)x'(t) \leqslant a\big(x(L)\big)x'(L)e^{-\frac{1}{M}K_C(t)} \leqslant MCe^{-\frac{1}{M}K_C(t)},$$

i.e. $x'(t) \leqslant N_C^*(t)$ for every $t \in [\hat{t}, \hat{t} + \delta]$, in contradiction with the definition of $\hat{t}$. The same argument works in the interval $[-n, -L]$. □

**Remark 4.** Actually, as regards the asymptotic behavior of the solutions $x$ found in the previous two existence theorems, note that we showed it is constant or strictly increasing for $|t|$ sufficiently large.

As already mentioned above, the assumptions of the previous existence theorems are not improvable in the sense that if the right-hand side $f$ satisfies assumption (6) with the reversed inequalities and the summability condition (5) (or (11)) does not hold, then problem (P) does not admit solutions, as the following results state.

**Theorem 4.** *Assume that there exist three constants $L \geqslant 0$, $\rho > 0$, $\gamma > 1$ and a positive strictly increasing function $K \in W^{1,1}_{\mathrm{loc}}([L, +\infty))$ satisfying*

$$\int^{\infty} \big(K(t)\big)^{\frac{1}{1-\gamma}}\, \mathrm{d}t = +\infty \tag{13}$$

*such that one of the following pair of conditions holds*:

$$f(t, x, y) \geqslant -K'(t)|y|^{\gamma} \quad \textit{for a.e. } t \geqslant L,\ \textit{every } x \in \big[\nu^-, \nu^+\big],\ |y| < \rho, \tag{14}$$

*or*

$$f(t, x, y) \leqslant K'(-t)|y|^{\gamma} \quad \textit{for a.e. } t \leqslant -L,\ \textit{every } x \in \big[\nu^-, \nu^+\big],\ |y| < \rho. \tag{15}$$

*Moreover, assume that*

$$tf(t, x, y) \leqslant 0 \quad \textit{for a.e. } |t| \geqslant L,\ \textit{every } x \in \mathbb{R},\ |y| < \rho. \tag{16}$$

*Then, problem* (P) *can only admits solutions which are constant in $[L, +\infty)$ (when* (14) *holds) or constant in $(-\infty, -L]$ (when* (15) *holds). Therefore, if both* (14) *and* (15) *hold and $L = 0$, then problem* (P) *does not admit solutions. More precisely, no function $x \in C^1(\mathbb{R})$, with $a(x)x'$ almost everywhere differentiable, exists satisfying the boundary conditions and the differential equation in* (P).

**Proof.** Suppose that (14) holds (the proof is the same if (15) holds).

Let $x \in C^1(\mathbb{R})$, with $a(x)x'$ almost everywhere differentiable (not necessarily belonging to $W^{1,1}(\mathbb{R})$), be a solution of problem (P). First of all, let us prove that $\lim_{t\to+\infty} x'(t) = 0$.

Indeed, since $x(+\infty) = \nu^+ \in \mathbb{R}$, we have $\limsup_{t\to+\infty} x'(t) \geqslant 0$ and $\liminf_{t\to+\infty} x'(t) \leqslant 0$. If $\liminf_{t\to+\infty} x'(t) < 0$, then there exists an interval $[t_1, t_2] \subset [L, +\infty)$ such that $-\rho < x'(t) < 0$ in $[t_1, t_2]$, $x'(t_2) > \frac{m}{M} x'(t_1)$. But by virtue of assumption (16) we deduce that $a(x(t))x'(t)$ is decreasing in $[t_1, t_2]$ and then

$$x'(t_2) \leqslant \frac{1}{M} a\big(x(t_2)\big)x'(t_2) \leqslant \frac{1}{M} a\big(x(t_1)\big)x'(t_1) \leqslant \frac{m}{M} x'(t_1),$$

a contradiction. So, necessarily $\liminf_{t\to+\infty} x'(t) = 0$.

Similarly, if $\limsup_{t\to+\infty} x'(t) > 0$, then there exists an interval $[t_1, t_2] \subset [L, +\infty)$ such that $0 \leqslant x'(t) < \rho$ in $[t_1, t_2]$, $x'(t_1) < \frac{m}{M} x'(t_2)$. Since $a(x(t))x'(t)$ is decreasing in $[t_1, t_2]$ we get

$$x'(t_1) \geqslant \frac{1}{M} a\big(x(t_1)\big)x'(t_1) \geqslant \frac{1}{M} a\big(x(t_2)\big)x'(t_2) \geqslant \frac{m}{M} x'(t_2),$$

a contradiction. So, $\lim_{t\to+\infty} x'(t) = 0$ and we can define $t^* := \inf\{t \geqslant L\colon\ |x'(t)| < \rho \text{ in } [t,+\infty)\}$.

Let us now prove that $x'(t) \geqslant 0$ for every $t \geqslant t^*$. Indeed, if $x'(\hat{t}) < 0$ for some $\hat{t} \geqslant t^*$, being $a(x(t))x'(t)$ decreasing in $[t^*, +\infty)$, we get

$$x'(t) \leqslant \frac{1}{M} a\big(x(t)\big)x'(t) \leqslant \frac{1}{M} a\big(x(\hat{t})\big)x'(\hat{t}) \leqslant \frac{m}{M} x'(\hat{t}) < 0,$$

for every $t \geqslant \hat{t}$, in contradiction with the boundedness of $x$.

Let us define $\tilde{t} := \inf\{t \geqslant t^*\colon\ x(\tau) \geqslant \nu^- \text{ in } [t,+\infty)\} \geqslant t^*$. Let us assume by contradiction that $x'(\bar{t}) > 0$ for some $\bar{t} \geqslant \tilde{t}$. Put $T := \sup\{t \geqslant \bar{t}\colon\ x'(\tau) > 0 \text{ in } [\bar{t}, t]\}$, observe that $T = +\infty$. Indeed, if $T < +\infty$, since $0 < x'(t) < \rho$ in $[\bar{t}, T]$, by (14) we have

$$\big(a\big(x(t)\big)x'(t)\big)' = f\big(t, x(t), x'(t)\big) \geqslant -K'(t)\big(x'(t)\big)^{\gamma} \quad \text{for a.e. } t \in [\bar{t}, T]. \tag{17}$$

So, assuming without restriction $\rho \leqslant 1$, being $\gamma > 1$ we get

$$\big(a\big(x(t)\big)x'(t)\big)' \geqslant -K'(t)x'(t) \geqslant -\frac{K'(t)}{\tilde{m}} a\big(x(t)\big)x'(t)$$

where $\tilde{m} := \min_{\xi\in[x(\bar{t}),x(T)]} a(\xi)$. Then, integrating in $[t, T]$ with $t < T$ we obtain (taking into account that $x'(T) = 0$)

$$a\big(x(t)\big)x'(t) \leqslant \int_t^T \frac{K'(\tau)}{\tilde{m}} a\big(x(\tau)\big)x'(\tau)\,\mathrm{d}\tau \quad \text{for every } t \in (\bar{t}, T],$$

so by the Gronwall's inequality we deduce $a(x(t))x'(t) \leqslant 0$, i.e. $x'(t) \leqslant 0$ in the same interval, in contradiction with the definition of $T$. Hence $T = +\infty$.

Observe now that by (17) we get

$$\frac{1}{1-\gamma}\Big[\big(a\big(x(t)\big)x'(t)\big)^{1-\gamma} - \big(a\big(x(\bar{t})\big)x'(\bar{t})\big)^{1-\gamma}\Big] = \int_{\bar{t}}^{t} \frac{(a(x(s))x'(s))'}{(a(x(s))x'(s))^{\gamma}}\,\mathrm{d}s \geqslant \frac{1}{\tilde{m}^{\gamma}}\big(K(\bar{t}) - K(t)\big)$$

therefore, putting $\tilde{M} := \max_{\xi\in[x(\bar{t}),x(T)]} a(\xi)$, for a.e. $t \geqslant \bar{t}$ we have

$$\tilde{M}^{1-\gamma} x'(t)^{1-\gamma} \leqslant \big(a\big(x(t)\big)x'(t)\big)^{1-\gamma} \leqslant \big(a\big(x(\bar{t})\big)x'(\bar{t})\big)^{1-\gamma} + \frac{(\gamma-1)}{\tilde{m}^{\gamma}}\big(K(t) - K(\bar{t})\big)$$

then

$$x'(t) \geqslant \frac{1}{\tilde{M}}\bigg(\big(a\big(x(\bar{t})\big)x'(\bar{t})\big)^{1-\gamma} + \frac{(\gamma-1)}{\tilde{m}^{\gamma}}\big(K(t) - K(\bar{t})\big)\bigg)^{\frac{1}{1-\gamma}}.$$

By virtue of (13) we deduce that $x(+\infty) - x(\bar{t}) = \int_{\bar{t}}^{+\infty} x'(t)\,\mathrm{d}t = +\infty$, in contradiction with the boundedness of $x$.

Therefore, $x'(t) \equiv 0$ in $[\tilde{t}, +\infty)$ and by the definition of $\tilde{t}$ this implies $\tilde{t} = t^*$. So, $x'(t) \equiv 0$ in $[t^*, +\infty)$ and by the definition of $t^*$ this implies $t^* = L$. □

**Theorem 5.** *Let all the assumptions of Theorem* 4 *be satisfied, with* $\gamma = 1$ *and* (13) *replaced by*

$$\int^{\infty} e^{-\frac{1}{\tilde{m}}K(t)}\,\mathrm{d}t = +\infty, \tag{18}$$

*where* $\tilde{m} := \min_{x\in[\nu^-,\nu^+]} a(x)$. *Then, the same assertion of Theorem* 4 *holds.*

**Proof.** The proof proceeds exactly as that of Theorem 4 till the definition of $T$ and the observation that $T = +\infty$. Since $0 < x'(t) < \rho$ and $\nu^- \leqslant x(t) \leqslant \nu^+$ in $[\bar{t}, +\infty)$, we have

$$\big(a\big(x(t)\big)x'(t)\big)' = f\big(t, x(t), x'(t)\big) \geqslant -K'(t)\big(x'(t)\big) \quad \text{for a.e. } t \geqslant \bar{t}$$

and then

$$\log\frac{a(x(t))x'(t)}{a(x(\bar{t}))x'(\bar{t})} = \int_{\bar{t}}^{t} \frac{(a(x(s))x'(s))'}{a(x(s))x'(s)}\,\mathrm{d}s \geqslant \frac{1}{\tilde{m}}\big(K(\bar{t}) - K(t)\big)$$

therefore, for a.e. $t \geqslant \bar{t}$ we have

$$x'(t) \geqslant \frac{1}{\tilde{M}} a\big(x(\bar{t})\big)x'(\bar{t})e^{\frac{1}{\tilde{m}}(K(\bar{t})-K(t))}$$

where $\tilde{M} := \max_{x\in[\nu^-,\nu^+]} a(x)$. By virtue of (18) we deduce that $x(+\infty) - x(\bar{t}) = \int_{\bar{t}}^{+\infty} x'(t)\,\mathrm{d}t = +\infty$, in contradiction with the boundedness of $x$. Therefore, $x'(t) \equiv 0$ in $[\tilde{t}, +\infty)$ and by the definition of $\tilde{t}$ this implies $\tilde{t} = t^*$. So, $x'(t) \equiv 0$ in $[t^*, +\infty)$ and by the definition of $t^*$ this implies $t^* = L$. □

**Remark 5.** Note that in the case $\gamma > 1$ the behavior of the non-linear function $a(x)$ entering in the differential operator of equation (E) does not influence the existence or non-existence of solutions. Instead, in the case $\gamma = 1$ the maximum and the minimum attained by $a(x)$ in $[\nu^-, \nu^+]$ are crucial parameters for the existence or the non-existence of solutions.

**Remark 6.** If the sign condition in (16) is satisfied with the reversed inequality, i.e. if

$$tf(t, x, y) \geqslant 0 \quad \text{for a.e. } |t| \geqslant L, \text{ every } x \in \mathbb{R}, |y| < \rho, \tag{19}$$

then also in this case it is possible to prove that $\lim_{x\to\pm\infty} x'(t) = 0$ and $x'(t) \leqslant 0$ for $|t| \geqslant L$. So, since $\nu^- < \nu^+$, when $L = 0$ problem (P) does not admit solutions.

## 4. Criteria for right-hand side of the type $f(t,x,y) = c(t,x)b(x,y)$

In this section we present some operative criteria useful when the right-hand side has the following product structure

$$f(t,x,y) = c(t,x)b(x,y).$$

As we will show, there is a strict link between the local behaviors of $b(x,\cdot)$ at $y=0$ and $c(\cdot,x)$ at infinity which plays a key role for the existence or non-existence of heteroclinic solutions.

In what follows we assume that $b$ is a continuous function satisfying

$$b(x,y) > 0 \quad \text{for every } y \neq 0, \quad b(\nu^-,0) = b(\nu^+,0) = 0$$

and $c$ is a Carathéodory function satisfying

$$t \cdot c(t,x) \leqslant 0 \quad \text{for a.e. } |t| \geqslant \bar{t}, \text{ every } x \in [\nu^-,\nu^+] \tag{20}$$

for some $\bar{t} \geqslant 0$.

We investigate the solvability of the following boundary value problem

$$\text{(P)} \quad \begin{cases} \big(a(x(t))x'(t)\big)' = f\big(t,x(t),x'(t)\big) & \text{a.e. } t \in \mathbb{R}, \\ x(-\infty) = \nu^-, \quad x(+\infty) = \nu^+ \end{cases}$$

with functions $x \in C^1(\mathbb{R})$, such that $a(x)x' \in W^{1,1}(\mathbb{R})$.

We split the discussion into two subsections, according to the value of the exponent $\delta$ such that $|c(t,x)| \geqslant h|t|^\delta$ for some constant $h > 0$, as $|t| \to +\infty$.

### 4.1. Case $\delta > -1$

The next result provides sufficient conditions for the existence, whereas the following one states sufficient conditions for the non-existence of solutions.

**Theorem 6.** *Suppose that there exists a function $\lambda \in L^p_{\mathrm{loc}}(\mathbb{R})$, $1 \leqslant p \leqslant +\infty$, such that*

$$\big|c(t,x)\big| \leqslant \lambda(t) \quad \textit{for a.e. } t \in \mathbb{R}, \textit{ every } x \in [\nu^-,\nu^+]. \tag{21}$$

*Assume that there exist real constants $-1 < \delta_1 \leqslant \delta_2$, $0 < \gamma_2 \leqslant \gamma_1$, such that*

$$1 < \gamma_1 < \delta_1 + 2 \quad \textit{and} \quad \gamma_2(\delta_1 + 1) > (\gamma_1 - 1)(\delta_2 + 1) \tag{22}$$

*and for every $x \in [\nu^-,\nu^+]$ we have*

$$h_1|t|^{\delta_1} \leqslant \big|c(t,x)\big| \leqslant h_2|t|^{\delta_2}, \quad \textit{a.e. } |t| > L, \tag{23}$$

$$k_1|y|^{\gamma_1} \leqslant b(x,y) \leqslant k_2|y|^{\gamma_2}, \quad \textit{whenever } |y| < \rho, \tag{24}$$

$$b(x,y) \leqslant k_2|y|^{2-\frac{1}{p}} \quad \textit{whenever } |y| > L, \tag{25}$$

*for certain positive constants $h_1, h_2, k_1, k_2, \rho, L > \bar{t}$.*

*Then, problem* (P) *admits solutions.*

**Proof.** We will show that under these conditions, all the assumptions of Theorem 2 hold. First observe that in this case the constant functions $\alpha(t) = \nu^-$ and $\beta(t) = \nu^+$, $t \in \mathbb{R}$, are a pair of monotone, ordered lower and upper solutions, so assumption (H1) holds.

Fixing $H > \max\{L, \frac{\nu^+ - \nu^-}{2L}\}$ and putting $\theta(r) := k_2(\frac{r}{m})^{2-\frac{1}{p}}$ for $r > 0$, it is immediate to verify the validity of conditions (3) and (4).

Let us now fix a constant $C > 0$ and put $\bar{C} = 2\frac{M}{m}C$. Since $b(x,y) > 0$ for $y \neq 0$, denoted by $\hat{m}_C := \min\{b(x,y)\colon\ x \in [\nu^-, \nu^+],\ \rho \leqslant |y| \leqslant \bar{C}\}$, we have $\hat{m}_C > 0$. So, put

$$\mu_C := \min\left\{\frac{\hat{m}_C}{C^{\gamma_1}}, k_1\right\}$$

and

$$K_C(t) := \begin{cases} \mu_C \int_L^t \min\{\min_{x\in[\nu^-,\nu^+]} c(-\tau, x), -\max_{x\in[\nu^-,\nu^+]} c(\tau, x)\}\, \mathrm{d}\tau & \text{for } t \geqslant L, \\ 0 & \text{for } 0 \leqslant t \leqslant L. \end{cases}$$

By condition (21), we have $K_C \in W^{1,1}_{\mathrm{loc}}([0, +\infty))$ and due to (20) it is strictly increasing for $t \geqslant L$.

Observe that by the definition of $\mu_C$ and (24), it follows that

$$b(x,y) \geqslant \mu_C |y|^{\gamma_1} \quad \text{for every } x \in \left[\nu^-, \nu^+\right],\ y \in [-\bar{C}, \bar{C}].$$

Therefore, by (20) we obtain

$$f(t,x,y) = c(t,x)b(x,y) \leqslant \mu_C c(t,x)|y|^{\gamma_1} \leqslant -K_C'(t)|y|^{\gamma_1}$$

and

$$f(-t,x,y) = c(-t,x)b(x,y) \geqslant \mu_C c(-t,x)|y|^{\gamma_1} \geqslant K_C'(t)|y|^{\gamma_1}$$

for a.a. $t \geqslant L$, every $x \in [\nu^-, \nu^+]$ and every $y \in [-\bar{C}, \bar{C}]$. Then, condition (6) of Theorem 2 holds.

Now, from (23) it follows that $h_1\mu_C|t|^{\delta_1} \leqslant K_C'(t)$ for a.a. $|t| \geqslant L$. As a consequence, we have

$$K_C(t) \geqslant \frac{h_1\mu_C}{\delta_1 + 1}\left(|t|^{\delta_1+1} - L^{\delta_1+1}\right). \tag{26}$$

Hence, for $|t| \geqslant L$ we obtain

$$\int^{\infty} \left(K_C(t)\right)^{\frac{1}{1-\gamma}} \mathrm{d}t \leqslant \left(\frac{h_1\mu_C}{\delta_1 + 1}\right)^{1/(1-\gamma_1)} \int^{\infty} \left(|t|^{\delta_1+1} - L^{\delta_1+1}\right)^{1/(1-\gamma_1)} \mathrm{d}t < +\infty$$

since by (22) we have $\delta_1 + 1 > \gamma_1 - 1$. Moreover, since $\lim_{|t|\to+\infty} N_C(t) = 0$, a constant $L_C^* > L$ exists such that $2N_C(t) \leqslant \rho$ for every $|t| \geqslant L_C^*$.

Let us define

$$\eta_C(t) := \begin{cases} \max_{x\in[\nu^-,\nu^+]} |c(t,x)| \cdot \max_{(x,y)\in[\nu^-,\nu^+]\times[-\bar{C},\bar{C}]} b(x,y) & \text{if } |t| \leqslant L_C^*, \\ h_2 k_2 |t|^{\delta_2} (2N_C(t))^{\gamma_2} & \text{if } |t| > L_C^*. \end{cases}$$

By (24) and (23), for every $y \in W^{1,1}(\mathbb{R})$ such that $|y(t)| \leqslant 2N_C(t)$ for a.a. $t \in \mathbb{R}$ and every $x \in [\nu^-, \nu^+]$, it results

$$\left|f\big(t,x,y(t)\big)\right| = \left|c(t,x)\right| b\big(x,y(t)\big) \leqslant h_2|t|^{\delta_2} b\big(x,y(t)\big) \leqslant \eta_C(t),$$

so it remains to prove that $\eta_C \in L^1(\mathbb{R})$.

Obviously $\eta_C \in L^1([-L_C^*, L_C^*])$. Moreover, when $|t| > L_C^*$, by (26) we have

$$\begin{aligned} 0 < \eta_C(t) &= h_2 k_2 |t|^{\delta_2} \left( \frac{2}{m} \left( (MC)^{1-\gamma_1} + \frac{(\gamma_1 - 1)}{M^{\gamma_1}} \int_L^{|t|} \left|k_C(s)\right| \mathrm{d}s \right)^{1/(1-\gamma_1)} \right)^{\gamma_2} \\ &\leqslant h_2 k_2 |t|^{\delta_2} \left(\frac{2}{m}\right)^{\gamma_2} \left( (MC)^{1-\gamma_1} + \frac{(\gamma_1 - 1) h_1 \mu_C}{M^{\gamma_1}(\delta_1 + 1)} \big(|t|^{\delta_1+1} - L^{\delta_1+1}\big) \right)^{\gamma_2/(1-\gamma_1)} \\ &= C_1 |t|^{\delta_2} \big(C_2 + C_3 |t|^{\delta_1+1}\big)^{\frac{-\gamma_2}{\gamma_1-1}} \end{aligned}$$

where $C_i$, $i = 1, 2, 3$, are constants and $C_1, C_3 > 0$. Therefore, there exists a positive constant $C_4$ such that

$$\int_{L_C^*}^{+\infty} \eta_C(t)\,\mathrm{d}t \leqslant C_4 \int_{L_C^*}^{+\infty} t^{\delta_2 - \frac{\gamma_2(\delta_1+1)}{\gamma_1-1}}\,\mathrm{d}t < +\infty$$

since by (22) we have $\frac{\gamma_2(\delta_1+1)}{\gamma_1-1} - \delta_2 > 1$. An analogous argument holds for $\int_{-\infty}^{L_C^*} \eta_C(t)\,\mathrm{d}t$, then $\eta_C \in L^1(\mathbb{R})$. Therefore, Theorem 2 applies and guarantees the assertion of the present result. □

**Theorem 7.** *Suppose that condition* (20) *holds for* $\bar{t} = 0$ *and let there exist real constants* $\delta > -1$, $\gamma > 0$, $\Lambda \geqslant 0$, *and a positive function* $\ell(t) \in L^1([0, \Lambda])$ *such that*

$$\gamma \geqslant \delta + 2, \tag{27}$$

$$\left|c(t,x)\right| \leqslant \lambda_1 |t|^{\delta} \quad \text{for every } x \in \left[\nu^-, \nu^+\right],\ \text{a.e. } |t| > \Lambda, \tag{28}$$

$$b(x,y) \leqslant \lambda_2 |y|^{\gamma} \quad \text{for every } x \in \left[\nu^-, \nu^+\right],\ |y| < \rho, \tag{29}$$

$$\left|c(t,x)\right| \leqslant \ell\big(|t|\big) \quad \text{for a.e. } |t| \leqslant \Lambda, x \in \left[\nu^-, \nu^+\right], \tag{30}$$

*for some positive constants* $\lambda_1, \lambda_2, \rho$.

*Then, problem* (P) *does not admit solutions.*

**Proof.** Put $g(t) := \ell(|t|)$ for $|t| \leqslant \Lambda$ and $g(t) := \lambda_1 |t|^\delta$ for $|t| > \Lambda$. Moreover, set $K(t) := \lambda_2 \int_0^t g(\tau)\,\mathrm{d}\tau$ for $t \geqslant 0$. Of course, $K(t)$ is a strictly increasing function belonging to $W^{1,1}_{\mathrm{loc}}([0,+\infty))$. Moreover, for every $t \geqslant \Lambda$ we have

$$K(t) = \lambda_2 \int_0^\Lambda \ell(\tau)\,\mathrm{d}\tau + \lambda_1\lambda_2 \int_\Lambda^t \tau^\delta\,\mathrm{d}\tau = \lambda_2 \int_0^\Lambda \ell(\tau)\,\mathrm{d}\tau + \frac{\lambda_1\lambda_2}{\delta+1}\bigl(t^{\delta+1} - \Lambda^{\delta+1}\bigr).$$

So, since $\gamma > 1$ and $\delta + 1 > 0$, we get

$$\int^\infty \bigl(K(t)\bigr)^{\frac{1}{1-\gamma}}\,\mathrm{d}t = +\infty \quad \Leftrightarrow \quad \int^\infty t^{\frac{\delta+1}{1-\gamma}}\,\mathrm{d}t = +\infty$$

and the last integral diverges owing to (27).

Moreover from (28) and (29) we obtain that (14) and (15) of Theorem 4 hold for $L = 0$ and this concludes the proof. □

**Remark 7.** In most of the concrete situations, the link between the local behaviors of $b(x,\cdot)$ at $y = 0$ and $c(\cdot, x)$ at infinity, emphasized in Theorems 6 and 7 (see conditions (23), (24), (29), (28)) can be simplified.

Indeed, when the function $c(t,x)$ has precise rate of growth (when $\delta > 0$) or infinitesimal order (when $-1 < \delta < 0$) as $|t| \to +\infty$, we can assume $\delta_1 = \delta_2 = \delta$ in condition (23). In this case, the second inequality in (22) reduces to $\gamma_2 > \gamma_1 - 1$.

So, if the infinitesimal order of $b(x,y)$ as $|y| \to 0$ is $\gamma$, for some $\gamma < \delta + 2$, then put $\gamma_2 := \gamma$, it is possible to find $\gamma_1$ in such a way that conditions in (22) are satisfied and Theorem 6 applies. On the other hand, when the infinitesimal order of $b(x,y)$ as $|y| \to 0$ is greater or equal than $\delta + 2$, then by Theorem 7 we deduce the non-existence of bounded solutions. In other words, $\delta + 2$ is a threshold value for the infinitesimal order of $b(x,y)$ as $|y| \to 0$, in order to have bounded solutions.

Such considerations are precisely stated in the next result and the following remark.

**Corollary 2.** *Let $f(t,x,y) = h(t)g(x)b(y)$, where $h \in L^p_{\mathrm{loc}}(\mathbb{R})$, for some $1 \leqslant p \leqslant \infty$, $b$ is continuous in $\mathbb{R}$ and $g$ is continuous and positive in $[\nu^-, \nu^+]$.*

*Assume that $b(0) = 0$, $b(y) > 0$ for $y \neq 0$; $t \cdot h(t) \leqslant 0$ for every $t$. Moreover, assume that*

$$\lim_{|t|\to+\infty} \bigl|h(t)\bigr||t|^{-\delta} = \ell_1 \in (0,+\infty) \quad \textit{for some } \delta > -1, \tag{31}$$

$$\lim_{|y|\to 0} b(y)|y|^{-\gamma} = \ell_2 \in (0,+\infty) \quad \textit{for some } \gamma > 0, \tag{32}$$

$$\lim_{|y|\to+\infty} b(y)|y|^{\frac{1}{p}-2} = \ell_3 \in [0,+\infty). \tag{33}$$

*Then, problem* (P) *admits solutions if and only if $\gamma < 2 + \delta$.*

**Proof.** Put $c(t,x) := h(t)g(x)$. It is immediate to verify that assumptions (31) and (32) imply the validity of (28) and (29) for suitable positive constants $\lambda_1, \lambda_2, \Lambda, \rho$. So, when $\gamma \geqslant \delta + 2$ problem (P) does not admit solutions, as a consequence of Theorem 7.

Assume now $\gamma < \delta + 2$ and prove that all the assumptions of Theorem 6 hold. To this aim, first observe that both the conditions (21) and (25) are satisfied.

Moreover, observe that assumption (31) implies that condition (23) holds for $\delta_1 = \delta_2 = \delta$ and suitable positive constants $h_1, h_2, L$.

Put $\gamma_2 := \gamma$, since $\gamma_2 > 0$ and $\gamma_2 < \delta + 2$, we can choose a value $\gamma_1 \in (1, \delta + 2) \cap (\gamma_2, \gamma_2 + 1)$ in such a way that the inequalities in (22) are satisfied and condition (24) holds. So, by applying Theorem 6 we deduce that (P) admits solutions. □

**Remark 8.** The previous result remains true (as regards both the existence and the non-existence) even if we replace assumption (32) with the following one

$$\lim_{|y|\to 0} b(y)|y|^{-\gamma} = 0 \quad \text{and} \quad \lim_{|y|\to 0} b(y)|y|^{-(\gamma+\epsilon)} = +\infty \quad \text{for every } \epsilon > 0 \tag{34}$$

with $\gamma > 0$. Indeed, if $\gamma < \delta + 2$ we can put $\gamma_2 := \gamma$ and for every $\gamma_1 \in (1, \delta + 2) \cap (\gamma_2, \gamma_2 + 1)$ conditions (22) and (24) hold. Moreover, also condition (29) is satisfied, so (P) admits solutions if and only if $\gamma < \delta + 2$.

Similarly, we can replace assumption (32) with the following one:

$$\lim_{|y|\to 0} b(y)|y|^{-\gamma} = +\infty \quad \text{and} \quad \lim_{|y|\to 0} b(y)|y|^{-\gamma+\epsilon} = 0 \quad \text{for every } \epsilon > 0. \tag{35}$$

In this case, if $\gamma < \delta + 2$ we can put $\gamma_1 := \gamma$ and $\gamma_2 := \gamma - \epsilon$ for $\epsilon < \min\{1, \gamma\}$. In this way, conditions (22) and (24) are satisfied, and (P) is solvable. Instead, if $\gamma > \delta + 2$, chosen $\epsilon$ in such a way that $\gamma - \epsilon > \delta + 2$, condition (29) is verified with $\gamma - \epsilon$ and problem (P) does not admit solutions.

Finally, we wish just to mention that analogous modifications could be made in assumption (31) concerning the rate of growth (or infinitesimal order, according to the sign of $\delta$) of $h(t)$ as $|t| \to +\infty$. We avoid to present them in detail, since the argument is very similar to that above developed.

**Example 1.** Let $f(t,x,y) := -\frac{1}{t} g(x)|t|^{\alpha}|y|^{\gamma}\sqrt{1+|y|^{\beta}}$, with $\alpha, \beta, \gamma > 0$ and $g$ a generic continuous positive function. Note that assumption (31) is satisfied with $\delta = \alpha - 1$. Moreover, when $\alpha \geqslant 1$ then all the other assumptions of Corollary 2 hold with $p = \infty$ and $\gamma + \beta \leqslant 2$. Hence, there exist solutions if and only if $\gamma < \alpha + 1$.

Instead, when $\alpha < 1$ then we have to take $p < \frac{1}{1-\alpha}$ and so condition (33) holds if $\gamma + \beta < \alpha + 1$. In this case, being $\gamma < \gamma + \beta$ there exist solutions.

**Example 2.** Let $f(t,x,y) := -\frac{1}{t}\sqrt{|t|}|y|^{\alpha}|\log(1+|y|)|$, with $\alpha < \frac{3}{2}$. Chosen $p \in (\frac{1}{2-\alpha}, 2)$ all the assumptions of Corollary 2 are satisfied, with $\delta = -\frac{1}{2}$, $\gamma = \alpha + 1$ and $\ell_3 = 0$. So, we deduce that if $\alpha < \frac{1}{2}$ there exist solutions, otherwise if $\frac{1}{2} \leqslant \alpha < \frac{3}{2}$ do not exist solutions.

The discussion in the previous examples holds whatever $g(x)$ and $a(x)$ may be.

*4.2. Case $\delta = -1$*

In order to treat such situations we will make use of Theorems 3 and 5. Contrary to the previous case ($\delta_1 > -1$) in which the existence of solutions does not depend on the behavior of the function $a(x)$ appearing in the differential operator of equation (E), now a crucial role is played by the values

$$M = \max_{x\in[\nu^-,\nu^+]} a(x), \qquad m = \min_{x\in[\nu^-,\nu^+]} a(x).$$

**Theorem 8.** *Suppose that there exists a function $\lambda \in L^p_{\mathrm{loc}}(\mathbb{R})$, $1 \leqslant p \leqslant +\infty$, such that* (21) *holds. Moreover, assume that there exist positive real constants $h_1, k_1, \mu$ with $\mu \leqslant 1$, $h_1k_1 > M$, $\delta \in [-1, -1 + \frac{h_1k_1}{M}\mu)$, such that for every $x \in [\nu^-, \nu^+]$ we have*

$$k_1|y| \leqslant b(x,y) \leqslant k_2|y|^{2-\frac{1}{p}}, \quad \textit{for every } |y| \geqslant \rho; \tag{36}$$

$$k_1|y| \leqslant b(x,y) \leqslant k_3|y|^{\mu}, \quad \textit{whenever } |y| < \rho; \tag{37}$$

$$h_1|t|^{-1} \leqslant \left|c(t,x)\right| \leqslant h_2|t|^{\delta}, \quad \textit{a.e. } |t| > L \tag{38}$$

*for certain positive constants $h_2, k_2, k_3, \rho, L > \bar{t}$.*

*Then, problem* (P) *admits solutions.*

**Proof.** As in the proof of Theorem 6, let us define $\theta(r) = k_2(\frac{r}{m})^{2-\frac{1}{p}}$,

$$K_C(t) := \begin{cases} k_1 \int_L^t \min\{\min_{x\in[\nu^-,\nu^+]} c(-\tau,x), -\max_{x\in[\nu^-,\nu^+]} c(\tau,x)\}\,\mathrm{d}\tau & \text{for } t \geqslant L,\\ 0 & \text{for } 0 \leqslant t \leqslant L,\end{cases}$$

and put

$$\eta_C(t) := \begin{cases} \max_{x\in[\nu^-,\nu^+]} |c(t,x)| \cdot \max_{(x,y)\in[\nu^-,\nu^+]\times[-\bar{C},\bar{C}]} b(x,y) & \text{if } |t| \leqslant L_C^*,\\ 2h_2k_3|t|^{\delta}(\tilde{N}_C(t))^{\mu} & \text{if } |t| > L_C^*,\end{cases} \tag{39}$$

where $\tilde{N}_C(t)$ is the function defined in (12) (see Theorem 3) and $L_C^*$ is such that $2\tilde{N}_C(t) \leqslant \rho$ for $|t| \geqslant L_C^*$.

Also in this case it is easy to verify that assumptions (21), (36) and (37) imply the validity of conditions (3), (4), (6) for $\gamma = 1$, and (7). Moreover, note that since $K_C(t) \geqslant h_1k_1 \log\frac{t}{L}$ and $h_1k_1 > M$, we have

$$\int^{\infty} e^{-\frac{1}{M}K_C(t)}\,\mathrm{d}t \leqslant L^{\frac{h_1k_1}{M}} \int^{\infty} t^{-\frac{h_1k_1}{M}}\,\mathrm{d}t < +\infty,$$

then condition (11) is satisfied. So, it remains to prove the summability of $\eta_C(t)$ in $\mathbb{R}$.

To this aim, when $|t|$ is sufficiently large we have

$$0 < \eta_C(t) = 2h_2k_3|t|^{\delta}\left(\frac{M}{m}C\right)^{\mu} e^{-\frac{\mu}{M}K_C(|t|)} \leqslant \text{const.}|t|^{\delta-\frac{h_1k_1}{M}\mu}.$$

Since $\delta < \frac{h_1k_1}{M}\mu - 1$ we get $\eta_C \in L^1(\mathbb{R})$ and this concludes the proof. □

In the previous result the requirement $h_1k_1 > M$ is not merely technical, but it is essential, as it will be clarified by the following non-existence result.

**Theorem 9.** *Suppose that condition* (20) *holds for* $\bar{t} = 0$ *and assumption* (30) *is satisfied. Moreover, assume that there exist positive constants* $h_1, k_1$ *with* $h_1k_1 \leqslant m$ *such that*

$$\left|c(t,x)\right| \leqslant h_1|t|^{-1}, \quad \text{for every } x \in \mathbb{R},\ \text{a.e. } |t| > \Lambda, \tag{40}$$

$$b(x,y) \leqslant k_1 y, \quad \text{for every } x \in \mathbb{R},\ 0 < y < \rho, \tag{41}$$

*for some positive constants* $\Lambda, \rho$.

*Then problem* (P) *does not admit solutions.*

**Proof.** In this case, put $K(t) := k_1\int_0^t \ell(\tau)\,\mathrm{d}\tau$ for $t \in [0,\Lambda]$ and $K(t) := \int_0^\Lambda \ell(\tau)\,\mathrm{d}\tau + h_1k_1(\log t - \log L)$ for $t \geqslant \Lambda$. Note that assumptions (14) and (15) are satisfied with $\gamma = 1$ and $L = 0$. Moreover,

$$\int^{\infty} e^{-\frac{1}{m}K(t)}\,\mathrm{d}t = +\infty \quad \Leftrightarrow \quad \int^{\infty} t^{-\frac{1}{m}h_1k_1}\,\mathrm{d}t = +\infty,$$

and the last equality holds since $h_1k_1 \leqslant m$. So, the assertion follows from Theorem 5. □

The next criterium is an immediate consequence of Theorems 8 and 9.

**Corollary 3.** *Let* $f(t,x,y) = h(t)g(x)b(y)$, *with* $h \in L^p_{\mathrm{loc}}(\mathbb{R})$, *for* $1 \leqslant p \leqslant \infty$, *b continuous in* $\mathbb{R}$ *and g continuous and positive in* $[\nu^-, \nu^+]$.

*Assume that* $b(0) = 0$, $b(y) > 0$ *for* $y \neq 0$ *and* $th(t) \leqslant 0$ *for every t. Moreover, suppose that* $b(y) \geqslant k_1|y|$ *for every* $y \in \mathbb{R}$ *and*

$$\lim_{|t|\to+\infty}\left|th(t)\right| = h_1, \qquad \lim_{|y|\to 0}\frac{b(y)}{|y|} = h_2, \qquad \limsup_{|y|\to+\infty}\frac{b(y)}{|y|^{2-\frac{1}{p}}} < +\infty$$

*for some positive real constants* $h_1, h_2$.

*Then if* $h_1k_1 \cdot \min_{x\in[\nu^-,\nu^+]} g(x) > M$ *then problem* (P) *admits solutions*; *instead if* $h_1h_2 \cdot \max_{x\in[\nu^-,\nu^+]} g(x) < m$ *then* (P) *does not admit solutions.*

**Remark 9.** Analogous considerations to those made in Remark 8 hold also in this case.

We present now some simple examples in which the previous corollary applies.

**Example 3.** Let

$$f(t,x,y) := -htg(x)|y|\sqrt{\frac{1+y^2}{1+t^4}}$$

where $h > 0$ and $g$ is a generic continuous function, positive in $[\nu^-, \nu^+]$. Take $a(x) \equiv 1$ for every $x \in \mathbb{R}$. In this case put $c(t) := -t(1+t^4)^{\frac{1}{2}}$ and $b(y) := |y|\sqrt{1+y^2}$, it is immediate to check that all the assumptions of Corollary 3 are satisfied for $p = +\infty$, $h_1 = h$, $k_1 = h_2 = 1$ and $\mu = 2$. Then, if $\min_{x\in[\nu^-,\nu^+]} g(x) > \frac{1}{h}$ problem (P) admits solutions, instead if $\max_{x\in[\nu^-,\nu^+]} g(x) < \frac{1}{h}$ then problem (P) does not admit solutions.

**Example 4.** Let $f(t,x,y) = f(t,y) := -\frac{t}{t^2+1}\frac{ky^2}{\arctan|y|}$ for $y \neq 0$, defined by continuity at $y = 0$. Also in this case all the assumptions of Corollary 3 hold for $p = \infty$, $h_1 = 1$, $k_1 = h_2 = k$ and $\mu = 2$. Hence, if $k > 1$ problem (P) admits solutions, instead if $k < 1$ problem (P) does not admit solutions.

## References


[1] R.P. Agarwal, M. Bohner, D. O'Regan, Time scale systems on infinite intervals, Nonlinear Anal. 47 (2) (2001) 837–848.

[2] R.P. Agarwal, M. Bohner, D. O'Regan, Time scale boundary value problems on infinite intervals, J. Comput. Appl. Math. 141 (2002) 27–34.

[3] R.P. Agarwal, D. O'Regan, Infinite Interval Problems for Differential, Difference and Integral Equations, Kluwer Academic, Dordrecht, 2001.

[4] A. Ambrosetti, Z.-Q. Wang, Positive solutions to a class of quasilinear elliptic equations on $\mathbb{R}$, Discrete Contin. Dyn. Syst. 9 (1) (2003) 575–590.

[5] C. Avramescu, C. Vladimirescu, Existence of solutions to second order ordinary differential equations having finite limits at $\pm\infty$, Electron. J. Differential Equations 2004 (18) (2004) 1–12.

[6] B. Bianconi, F. Papalini, Non-autonomous boundary value problems on the real line, Discrete Contin. Dyn. Syst. 15 (3) (2006) 759–776.

[7] P.W. Eloe, L.J. Grimm, J.D. Mashburn, A boundary value problem on an unbounded domain, Differential Equations Dynam. Systems 8 (2000) 125–140.

[8] B.H. Gilding, R. Kersner, Travelling Waves in Nonlinear Diffusion–Convection–Reaction, Birkhäuser, Basel, 2004.

[9] G.S. Guseinov, I. Yaslan, Boundary value problems for second order nonlinear differential equations on infinite intervals, J. Math. Anal. Appl. 290 (2004) 620–638.

[10] S. T Krytsi, N. Matzakos, N.S. Papageorgiou, Nonlinear boundary value problems for second order differential equations, Czechoslovak Math. J. 55 (3) (2005) 545–579.

[11] Y. Liu, Boundary value problems for second-order differential equations on unbounded domains in a Banach space, Appl. Math. Comput. 135 (2003) 569–583.

[12] R. Ma, Existence of positive solutions for second-order boundary value problems on infinity intervals, Appl. Math. Lett. 16 (1) (2003) 33–39.

[13] L. Malaguti, C. Marcelli, N. Partsvania, On transitional solutions of second order nonlinear differential equations, J. Math. Anal. Appl. 303 (2005) 258–273.

[14] C. Marcelli, F. Papalini, Heteroclinic solutions for second order non-autonomous boundary value problems on the real line, Differential Equations Dynam. Systems 11 (2003) 333–353.

[15] J. Mawhin, J.R. Ward, Guiding-like functions for periodic or bounded solutions of ordinary differential equations, Discrete Contin. Dyn. Syst. 8 (2002) 39–54.

[16] D. O'Regan, Existence Theory for Nonlinear Ordinary Differential Equations, Kluwer Academic, Dordrecht, 1997.

[17] L. Sanchez, A note on nonautonomous ODE related to the Fisher equation, J. Comput. Appl. Math. 113 (2000) 201–209.

[18] V.A. Volpert, Y.M. Suhov, Stationary solutions of non-autonomous Kolmogorov–Petrovsky–Piskunov equations, Ergodic Theory Dynam. Systems 19 (1999) 800–835.

[19] B. Yan, Multiple unbounded solutions of boundary value problems for second order differential equations on the half-line, Nonlinear Anal. 51A (6) (2002) 1031–1044.

[20] B. Yan, Y. Liu, Unbounded solutions of the singular boundary value problems for second order differential equations on the half-line, Appl. Math. Comput. 147 (3) (2004) 629–644.