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Discrete Mathematics 249 (2002) 65–70

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

Arithmetic progressions of cycles in outer-planar graphs

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Received 1 September 1999; revised 20 February 2000; accepted 26 March 2001

Abstract

A question of Erdős asks if every graph with minimum degree 3 must contain a pair of cycles whose lengths differ by 1 or 2. Some recent work of Häggkvist and Scott (see Arithmetic progressions of cycles in graphs, preprint), whilst proving this, also shows that minimum degree $500k^2$ guarantees the existence of cycles whose lengths are $m, m+2, m+4, \ldots, m+2k$ for some m—an arithmetic progression of cycles. In like vein, we prove that an outer-planar graph of order n, with bounded internal face size, and outer face a cycle, must contain a sequence of cycles whose lengths form an arithmetic progression of length $\exp((c\log n)^{1/3} - \log\log n)$. Using this we give an answer for outer-planar graphs to a question of Erdős concerning the number of different sets which can be achieved as cycle spectra. © 2002 Elsevier Science B.V. All rights reserved.

1. Introduction

At a meeting in Memphis in 1996 just a few months before his death, Paul Erdős posed the following problem:

Problem. Given a graph G, let $\mathscr{C}(G)$, the *cycle spectrum of* G, be the set of lengths of all cycles in G. Now let $\mathscr{C}(n)$ be the total number of distinct cycle spectra over all possible graphs on n vertices. How does $\mathscr{C}(n)$ behave? Is $\mathscr{C}(n) = o(2^n)$ or is $\mathscr{C}(n) = \Theta(2^n)$?

Little progress seems to have been made on the problem in its most general sense. But the same problem still seems to be of interest if we introduce $\mathscr{C}_{\mathscr{G}}(n)$ as the number of distinct sets which can be achieved as cycle spectra by graphs of order n from a specific class of graphs $\mathscr{G}$.

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PII: S0012-365X(01)00234-5

In [2], we show that $\mathscr{C}_{\mathscr{G}}(n)=\mathrm{o}(2^n)$ if $\mathscr{G}$ consists of all cubic hamiltonian graphs. Here we do the same when $\mathscr{G}$ is the set of all outer-planar graphs with bounded face size.

The idea behind the proofs of all of these results is the same. We show that every member of $\mathscr{G}$ contains a *long* sequence of cycles whose lengths form an arithmetic progression, where for us *long* will mean of length more than $2\log_2 n$. Since only $\mathrm{o}(2^n)$ subsets of $\{1,2,\dots,n\}$ contain such an arithmetic progression it follows that there can be only $\mathrm{o}(2^n)$ different cycle spectra for that class.

2. The results

As we mentioned in the introduction, our arguments rely on the following lemma.

Lemma 1. *There are at most* $2^n/(2\log_2 n)$ *subsets of* $\{1,2,\dots,n\}$ *which contain an arithmetic progression of length at least* $2\log_2 n$.

Proof. We shall put a probability distribution on the subsets of $\{1,2,\dots,n\}$, by making each element i, for $i=1,\dots,n$ a member of a random set A with probability $\frac{1}{2}$. Of course, this model produces the uniform distribution on the set of all subsets of $\{1,2,\dots,n\}$.

Now, let X be a random variable which counts the number of arithmetic progressions of length $2\log_2 n$ in a randomly chosen set. Let us calculate $E(X)$.

$$E(X)=(\#\text{AP's of length } 2\log_2 n)\times P\text{ (a set contains such an AP)}.$$

There are at most $n^2/(2\log_2 n)$ such arithmetic progressions, since there are at most $n/(2\log_2 n)$ possible differences for the progression. Hence,

$$E(X)\leqslant \frac{n^2}{2\log_2 n}\left(\frac{1}{2}\right)^{2\log_2 n}=\frac{1}{2\log_2 n}.$$

Thus, there are at most $2^n/(2\log_2 n)$ sets which contain long arithmetic progressions. $\square$

To proceed then, we need to find a way to produce long arithmetic progressions of cycles in our graphs. For this we need several lemmas.

Lemma 2. *Let $d\geqslant 3$ be given. Let G be an outer-planner graph of order n in which every face, other than the outer face has size at most d, and in which the outer face is a cycle in G. Then G^*, the weak planar dual of G, is a tree of order at least $\lceil n/d\rceil$.*

Proof. Consider the weak dual of the graph G. Since G is outer-planar its weak dual must be a forest (see Proposition 7.1.15 in [4]). Indeed, the outer face of G is a cycle and so the weak dual is actually a tree.

It remains only to show that this tree has the required number of vertices. Let G have n^* faces. Then

$$d(n^* - 1) + n \geqslant \sum_{F \text{ a face of } G} l(F) = 2e(G) \geqslant 2n,$$

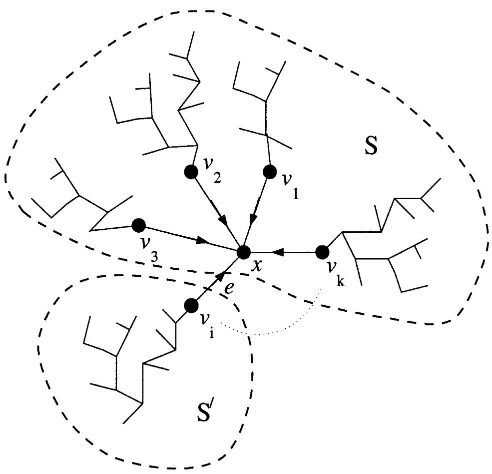
where $l(F)$ is the number of edges in the face F. Hence, this tree has order $n^* - 1 \geqslant \lceil n/d \rceil$. □

Lemma 3. *Let T be a tree with maximum degree $d \geqslant 2$. Then there is an edge e so that $T - e$ consists of two trees S and S' for which*

$$\frac{|S| - 1}{d - 1} \leqslant |S'| \leqslant |S|.$$

Proof. We may assume that there is no edge e whose deletion creates two trees of equal order, for such an edge would satisfy the conclusion of the theorem.

Clearly, the deletion of any edge e in T creates two subtrees T_1 and T_2, which we can envisage as being rooted at the ends of e. We orient the edges of T so that each edge is directed from the root of the smaller subtree to the root of the larger. Observe that each vertex must have out-degree 1 or 0. For suppose that a vertex x has two edges directed out, $\vec{xy}$ and $\vec{xz}$. Then, y is the root of a tree T_y which has larger order than the subtree rooted at x—indeed T_y contains more than half of the vertices of T. Similarly, T_z is a subtree which has larger order than another subtree rooted at x and also contains more than half of the vertices of T. However, T_y and T_z are disjoint, which is a contradiction.



Clearly, each leaf of T has out degree 1. It is easy to see by induction that this implies the existence of an internal vertex x with out-degree 0. Consider the subtrees $T_1, T_2, \ldots, T_k$ rooted at the neighbours of x, $v_1, v_2, \ldots, v_k$ and choose e to be the edge joining x to v_i, where T_i has the largest order amongst these trees.

Then, clearly,

$$\frac{1}{k-1}\sum_{j\neq i} v(T_j) \leqslant v(T_i) \leqslant 1 + \sum_{j\neq i} v(T_j).$$

Hence e divides the tree into two subtrees with the required properties. □

We are now in a position to use this structure to show that all outer-planar graphs, with bounded internal face size and outer face a cycle, have long arithmetic progressions of cycles. For this we need a result of Bourgain (see [1]).

Theorem A (Bourgain [1]). *Let A and B be sets of integers in $\{1,2,\ldots,N\}$ with densities $\alpha = |A|/N$ and $\beta = |B|/N$. Then, provided that N is large enough, $A+B = \{a+b\colon a\in A, b\in B\}$ contains an arithmetic progression of length L whenever*

$$L < \exp[c(\alpha\beta\log N)^{1/3} - \log\log N]$$

for some constant c.

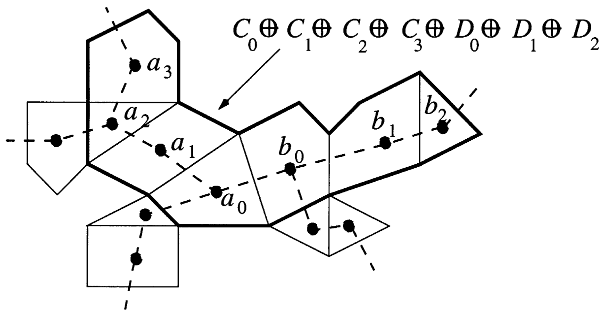
Theorem 4. *Let $d \geqslant 3$ be a natural number. There is a constant $c > 0$ such that if n is large enough every outer-planar graph of order n in which every internal face has size at most d and the outer face is a cycle, contains a sequence of $\exp[c(\log n)^{1/3} - \log\log n]$ cycles whose lengths form an arithmetic progression.*

Proof. Consider such an outer-planar graph G and let T be the tree of size at least $\lceil n/d\rceil$ guaranteed by Lemma 2. Since every face of G contains at most d vertices, T must have maximum degree at most d. Hence, Lemma 3 gives a partition of T into two trees S and S', divided by an edge $e = a_0b_0$, so that both S and S' each have at least n/d^2 vertices (in fact, the lemma gives more, but this will be enough for our purposes).

We shall now use these two trees S and S' to form the two subsets A and B which we shall use in Bourgain's result.

Consider the subtree S, rooted at a vertex a_0. Then in G the vertex a_0 is a face of G, bounded by a cycle C_0. Let $a_0, a_1, \ldots, a_k$ be the vertices of S, numbered as in a depth first search from a_0, and let C_i be the cycle in G which bounds the face corresponding to vertex a_i. Then C_0, $C_0 \oplus C_1$, $C_0 \oplus C_1 \oplus C_2, \ldots, C_0 \oplus \cdots \oplus C_k$ are k cycles in G, each of different lengths. These lengths will be the elements of A.

We could similarly consider the subtree S', rooted at b_0, and let B be the different lengths of the cycles formed from the bounding cycles $D_0, D_1, \ldots, D_l$ of its face vertices.



To complete the proof we need to observe some facts about A and B. First observe that A and B are both subsets of $\{1, \ldots, n\}$ of size at least n/d^2, and so their densities do not vary with n, as required in Theorem A. Finally, notice that each member of the set $A + B$ corresponds to a length one greater than the length of a cycle of the form $C_0 \oplus \cdots \oplus C_s \oplus D_0 \oplus \cdots \oplus D_t$. The result follows. □

To conclude we can use Theorem 4 to contribute to Erdős' question, at least for the class or outer-planar graphs.

Corollary 5. *Let $d \geqslant 3$ be given. Let $\mathscr{A}$ be the set of outer-planar graphs whose internal faces are of size at most d. Then $\mathscr{C}_{\mathscr{A}}(n) = \mathrm{o}(2^n)$.*

Proof. First, suppose that G contains a block of order at least $n - \log_2 n$. Then this block alone is an outer-planar graph whose external face is a cycle. Hence, by combining Theorem 4 with Lemma 1 we see that such graphs only contribute $\mathrm{o}(2^n)$ distinct cycle spectra.

However, if every block of G has order less than $n - \log_2 n$ then every cycle in G must have length less than $n - \log_2 n$. Hence, the cycle spectrum of G must be a subset of $\{1, 2, \ldots, n - \log_2 n\}$, of which there are only $2^{n - \log_2 n} = 2^n/n$. The result follows. □

3. Uncited reference

[3]

Acknowledgements

The author would like to send his heartiest thanks to the referees for their observations, suggestions and corrections during the writing of this paper.

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