Controllability of Linear Stochastic Systems in Hilbert Spaces

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Submitted by Jerald P. Dauer

Received November 8, 1999

The classical theory of controllability for deterministic systems is extended to linear stochastic systems defined on infinite-dimensional Hilbert spaces. Three types of stochastic controllability are studied: approximate, complete, and S-controllability. Tests for complete, approximate, and S-controllabilities are proved and the relation between the controllability of linear stochastic systems and the controllability of the corresponding deterministic systems is studied.

Key Words: controllability; stochastic controllability; linear system.

1. INTRODUCTION

The present paper systematically studies the notions of controllability for stochastic dynamical systems, described by linear stochastic differential equations in Hilbert spaces.

For finite-dimensional systems, the notion of controllability was introduced by Kalman [12]. Later, the notion was extended to infinite-dimensional systems (see Fattorini [11] and Russel [17]) and many interesting results were obtained (see for example [5, 8, 20]). Afterward, many works appeared concerning controllability of stochastic systems. We mention a few here [1–4, 9, 10, 13, 16, 18, 19].

In a stochastic setting there are at least three concepts of controllability: complete controllability, approximate controllability, and S-controllability. These notions are introduced in Section 2. One of these three concepts, S-controllability of partially observed linear stochastic systems, was studied by Bashirov and this author (see [2, 3]).
With reference to infinite-dimensional Hilbert spaces, three types of stochastic controllability are studied, approximate, complete, and $S$-controllability, all of which generalize the classical ones of deterministic systems in infinite-dimensional spaces. Our analysis then shows the following:

(i) The complete controllability of the linear stochastic system at time $T$, the small time complete controllability of the linear stochastic system, and the small time complete controllability of the corresponding deterministic system are equivalent (Section 3).

(ii) The approximate controllability of the linear stochastic system at time $T$, the small time approximate controllability of the linear stochastic system, and the small time approximate controllability of the corresponding deterministic system are equivalent (Section 4).

(iii) The small time approximate controllability of the stochastic system is equivalent to the small time $S$-controllability of the same system with the gaussian control set (Section 5; see also [1, 3, 4]).

It is known that criteria for controllability in infinite-dimensional systems can be obtained by considering the operator $L_T^0(u) = \int_0^T \mathcal{A}(t) - t)Bu(t) \, dt$ and studying the positiveness of $L_T^0(L_T^0)^* = \Gamma_0^T$. To investigate controllability concepts we define the stochastic analogue of $L_T^0$, $(L_T^0)^*$, and study the relationship between the controllability operator $\Gamma_0^T$ and its stochastic analogue $\Pi_0^T$. We also find a formula for the control which transfers the linear stochastic system from an arbitrary $x_0$ to an arbitrary $h$ under natural conditions.

2. PRELIMINARIES

Notations

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $\{\mathcal{F}_t : 0 \leq t \leq T\}$; $X$, $H$, $E$, and $U$ are separable Hilbert spaces. Let $w(t)$ be a Wiener process on $E$ with covariance operator $Q$ and $w$ be an $H$-valued Gaussian variable with mean $x_0$ and covariance operator $P_0$, $w_1(t)$ is a vector-valued Wiener process on $R_k$ with covariance matrix $Q_1$. We assume that $\xi, w, w_1$ are mutually independent. $\mathcal{L}_2(Q^{1/2}E, H)$ is the space of all Hilbert–Schmidt operators from $Q^{1/2}E$ into $H$ endowed with the Hilbert–Schmidt norm $|| \cdot ||_2$. $\mathcal{L}(U, H)$ is the space of all linear bounded operators from $U$ to $H$. $L_2^0(0, T; X)$ is the space of the $\mathcal{F}$-adapted, $X$-valued measurable process $\rho(t, \omega)$ on $[0, T]$ such that $E\int_0^T || \varphi(t, \omega) ||^2 \, dt < +\infty$. $\mathcal{D}(\Delta; \mathcal{L}(R^k, U))$ is the space of all $\mathcal{L}(R^k, U)$-valued functions on $\Delta = \{(t, s) : 0 \leq t < s \leq T\}$.
\[ \leq s \leq t \leq T \} \] that are strongly measurable and square integrable with respect to the Lebesgue measure on \( \Delta \). \( L_2(\mathcal{F}_T, H) := L_2(\Omega, \mathcal{F}_T, H) \). \( D(R) \) is a domain and \( \text{Im} \; R \) is an image of the linear operator \( R \). A self-adjoint operator \( R \) on a Hilbert space \( H \) is nonnegative (\( R \geq 0 \)) if \( \langle Rz, z \rangle \geq 0 \) for all \( z \in D(R) \); \( R \) is positive (\( R > 0 \)) if \( \langle Rz, z \rangle > 0 \) for all nonzero \( z \in D(R) \); and \( R \) is coercive (\( R - \gamma I \geq 0 \)) if there exists a \( \gamma > 0 \) such that \( \langle Rz, z \rangle \geq \gamma \|z\|^2 \) for all \( z \in D(R) \). Let \( f \) be a \( \mathbb{P} \)-measurable function on \( \Omega \) to \( L_2(0, T; Y) \) and \( Y \) a Hilbert space. For \( 0 < t \leq T \), denote by \( f_t \) the restriction of \( f \) to \([0, t] \times \Omega \) and by \( \sigma(f_t) \) denote the \( \sigma \)-algebra generated by \( f_t(s), \) \( 0 \leq s \leq t \). Let \( \mathbf{X}_{f_t} = L_2(\Omega, \sigma(f_t), \mathbb{P}, Y) \). The set

\[ \int \mathbf{X}_{f_t} \; dt = \{ x \in L_2(0, T; X) : x_t \in \mathbf{X}_{f_t} \text{ for a.e. } t \in [0, T] \} \]

is the Hilbertian sum of subspaces \( \mathbf{X}_{f_t} \).

**Partially Observable Systems**

We consider the class of infinite-dimensional partially observable stochastic system in the interval \([0, T]\),

\[
\begin{align*}
\mathbf{d}x(t) &= [A\mathbf{x}(t) + B\mathbf{u}(t)] \; dt + \Sigma \; dw(t), \\
\mathbf{d}z(t) &= C\mathbf{x}(t) + F \; dw(t),
\end{align*}
\]

(1)

where \( A \) is the infinitesimal generator of a \( C_0 \)-semigroup \( \mathcal{S}(t) \) on \( H, B \in \mathcal{L}(U, H), \Sigma \in \mathcal{L}(\mathcal{Q}^{1/2} E, H), C \in \mathcal{L}(H, \mathcal{R}^k), Q_1 \) and \( F \) are invertible, and \( Q_1^{-1}, F, F^{-1} \in \mathcal{L}(\mathcal{R}^k) \). Following Bensoussan and Viot [6], to avoid the problem of the dependence of \( \mathcal{U}_{z_i} \) on the control process, we take the class of admissible controls

\[
\mathbf{U}_{ad} = \int \mathcal{U}_{z_i} \; dt \cap \int \mathcal{U}_{z_i} \; dt,
\]

where \( z_i(\cdot) \) is the observation process under zero control. For these admissible controls, \( u \in \mathbf{U}_{ad} \); it was shown in [6] that \( \mathcal{U}_{z_i} = \mathcal{U}_{z_i} \) and so is independent of the particular control chosen. So for \( u \in \mathbf{U}_{ad} \), (1) is a well-defined stochastic process and \( u \) is actually “feedback” in the sense of feedback control laws \( u(t) = \psi(t, z_i) \), where \( \psi \) is measurable, nonanticipative, and satisfies a uniform Lipschitz condition are admissible.

Let \( I(t) = [z^0(t) - \int_0^t C E(x^0(s) | Z^0_s) \; ds] \) be an innovation process and let \( Z_i = \sigma(z_i), Z_i^0 = \sigma(z_i^0), \) and \( \mathcal{S} = \sigma(I) \) be \( \sigma \)-algebras generated by \( z_i(\cdot), z_i^0(\cdot), \) and \( I_i(\cdot) \), respectively. Introduce an important class of admissi-
ble controls, denoted by $\mathcal{U}_{ad}^c$, defined by a linear feedback on the observation

$$
\mathcal{U}_{ad}^c = \left\{ u(t) = \int_0^t K(t, s) \, dz(s) + v(t) : v(\cdot) \in L_2([0, T]; U), K(\cdot, \cdot) \in \mathcal{B}_2(\Delta; \mathcal{L}(R^k, U)) \right\}. \quad (2)
$$

**Lemma 2.1** (see [19]). $\mathcal{U}_{ad}^c \subset \mathcal{U}_{ad}$. In other words, there exist $v_1(\cdot) \in L_2([0, T]; U)$ and $K(\cdot, \cdot) \in \mathcal{B}_2(\Delta; \mathcal{L}(R^k, U))$ such that

$$
u(t) = \int_0^t R(t, s) \, d\hat{w}(s) + v_1(t).$$

The innovation process $I(t)$ is a Wiener process relative to $\mathcal{F}_t$ and $\hat{w}(t) = F^{-1}I(t)$, where $\hat{w}(t)$ is a $k$-dimensional Wiener process with covariance matrix $Q_1$.

**Lemma 2.2** (see [7]). If $u(\cdot) \in \mathcal{U}_{ad}$ then the Kalman filter

$$
\hat{x}(t) = E\{x(t) | \mathcal{F}_t\} = E\{x(t) | \mathcal{F}_0\}
$$

is the mild solution of

$$
d\hat{x}(t) = \left[ A\hat{x}(t) + Bu(t) \right] dt + D(t) \, d\hat{w}(t), \quad \hat{x}(0) = x_0, \quad (3)
$$

where $D(\cdot) = P(\cdot)C^*(FQ_1^{-1}F)^{-1}F$ and $P(\cdot)$ is a solution of the Riccati equation

$$
\frac{d}{dt} \langle P(t)h, k \rangle - \langle P(t)h, A^*k \rangle - \langle A^*h, P(t)k \rangle - \langle \Sigma Q \Sigma^* h, k \rangle
$$

$$
+ \langle P(t)C^*(FQ_1^{-1}F)^{-1}CP(t)h, k \rangle = 0,
$$

$$
P(0) = P_0, \quad h, k \in D(A^*).$$

The mild solution of the linear stochastic differential equation (3) can be written as

$$
\hat{x}(t; x_0, u) = \mathcal{T}(t) x_0 + \int_0^t \mathcal{T}(t-s) Bu(s) \, ds
$$

$$
+ \int_0^t \mathcal{T}(t-s) D(s) \, d\hat{w}(s). \quad (4)
$$
We also consider the deterministic system corresponding to (4),

\[ y(t; y_0, v) = \mathcal{A}(t)y_0 + \int_0^t \mathcal{A}(t - s)Bv(s) \, ds, \quad (5) \]

for \( y_0 \in H \) and \( v(\cdot) \in L_2([0, T]; U) \).

**The Stochastic Controllability Operator**

Define the operator \( L^T_0: \mathcal{U} \rightarrow L_2(\mathcal{Z}_T, H) \), the controllability operator \( \Pi^T_s: L_2(\mathcal{Z}_T, H) \rightarrow L_2(\mathcal{Z}_T, H) \) associated with (4), and the controllability operator \( \Gamma^T_s \) associated with (5) as

\[ L^T_0u = \int_0^T \mathcal{A}(T - s)Bu(s) \, ds, \quad (6) \]

\[ \Pi^T_s\{\cdot\} = \int_s^T \mathcal{A}(T - t)BB^*\mathcal{A}^*(T - t)\mathcal{E}\{\cdot|\mathcal{Z}_s\} \, dt, \quad (7) \]

and

\[ \Gamma^T_s = \int_s^T \mathcal{A}(T - t)BB^*\mathcal{A}^*(T - t) \, dt, \quad (8) \]

where \( \mathcal{U} = L_2^2(0, T; U) \).

It is straightforward that the operators \( L^T_0, \Pi^T_s, \Gamma^T_s \) are linear bounded operators, and the adjoint \( (L^T_0)^* \in \mathcal{L}(L_2(\mathcal{Z}_T, H), \mathcal{U}) \) of \( L^T_0 \) is defined by

\[ (L^T_0)^* z = B^*\mathcal{A}^*(T - t)\mathcal{E}\{z|\mathcal{Z}_s\} \]

and

\[ \Pi^T_s = L^T_0(L^T_0)^* . \]

Before studying the stochastic control problem, let us first investigate the relationship between \( \Pi^T_s \) and \( \Gamma^T_r \), \( s \leq r < T \), and \( R(\lambda, \Pi^T_r) = (\mathcal{A} + \Pi^T_r)^{-1} \) and \( R(\lambda, \Gamma^T_r) = (\mathcal{A} + \Gamma^T_r)^{-1} \), \( s \leq r < T \), for \( \lambda > 0 \), respectively.

**Lemma 2.3.** For every \( z \in L_2(\mathcal{Z}_T, H) \) there exists \( \varphi(\cdot) \in L_2^2(0, T; \mathcal{L}(\mathcal{Z}, H)) \) such that

(a) \( \mathcal{E}\{z|\mathcal{Z}_z\} = \mathcal{E}(z) + \int_0^T \varphi(s) \, d\mathcal{W}(s) \)

(b) \( \Pi^T_s z = \Gamma^T_s \mathcal{E}z + \int_s^T \Gamma^T_r \varphi(r) \, d\mathcal{W}(r) \)

(c) \( R(\lambda, \Pi^T_r)z = R(\lambda, \Gamma^T_r)\mathcal{E}\{z|\mathcal{Z}_s\} + \int_s^T R(\lambda, \Pi^T_r) \varphi(r) \, d\mathcal{W}(r) \)

**Proof.** (a) We consider an orthonormal basis of \( H \), denoted by \( \{h_i\} \). Let \( H_n = [h_1, \ldots, h_n] \) be the finite-dimensional subspace generated by \( h_1, \ldots, h_n \). Let \( \mathcal{P}_n: H \rightarrow H_n \) projection and let \( z^n = \mathcal{P}_n z \). Then \( z^n \in \mathcal{Z}_n \).
$L_2(\mathcal{Z}_T, H_n)$ and by [15, Theorem 5.6] there exists $\varphi^n(\cdot) \in L_2^\mathcal{Z}(0, T; \mathcal{Z}(\mathbb{R}^k, H_n))$ such that

$$\mathbf{E}\{z^n | \mathcal{Z}_T\} = \mathbf{E}\{z^n\} + \int_0^T \varphi^n(s) \, d\hat{\mathbf{w}}(s). \quad (9)$$

Since

$$\mathbf{E}\|\mathbf{E}\{z^n | \mathcal{Z}_T\} - \mathbf{E}\{z | \mathcal{Z}_T\}\|^2 \leq \mathbf{E}\|z^n - z\|^2 \to 0,$$

the sequences $\{\mathbf{E}\{z^n | \mathcal{Z}_T\}\}$ and $\{\mathbf{E}\{z^n\}\}$ are fundamental and by Ito's formula from the equality

$$\mathbf{E}\|\mathbf{E}\{z^n | \mathcal{Z}_T\} - \mathbf{E}\{z^m | \mathcal{Z}_T\}\|^2 = \|\mathbf{E}z^n - \mathbf{E}z^m\|^2$$

$$+ \mathbf{E}\int_0^T \|\varphi^n(s) - \varphi^m(s)\|^2 \, ds$$

we obtain that the sequence $\{\varphi^n(\cdot)\}$ is fundamental. Thus there exists a function $\varphi(\cdot) \in L_2^\mathcal{Z}(0, T; \mathcal{Z}(\mathbb{R}^k, H))$ such that $\varphi^n(\cdot) \to \varphi(\cdot)$ strongly in $L_2^\mathcal{Z}(0, T; \mathcal{Z}(\mathbb{R}^k, H))$. Now, to obtain the desired representation it remains to pass the limit in (9).

(b) Let $z \in L_2(\mathcal{Z}_T, H)$. From Part (a) it follows that there exists $\varphi \in L_2^\mathcal{Z}(0, T; \mathcal{Z}(\mathbb{R}^k, H))$ such that

$$\mathbf{E}\{z | \mathcal{Z}_T\} = \mathbf{E}\{z\} + \int_0^T \varphi(s) \, d\hat{\mathbf{w}}(s).$$

The definition of the operator $\Pi_T^\mathcal{Z}$ and the stochastic Fubini Theorem lead to the desired representation,

$$\Pi_T^\mathcal{Z}z = \int_s^T \mathcal{S}(T - t) BB^* \mathcal{S}^*(T - t) \mathbf{E}\{z | \mathcal{Z}_T\} \, dt$$

$$= \int_s^T \mathcal{S}(T - t) BB^* \mathcal{S}^*(T - t) \left[ \mathbf{E}\{z | \mathcal{Z}_T\} + \int_s^T \varphi(r) \, d\hat{\mathbf{w}}(r) \right] \, dt$$

$$= \Gamma_T^\mathcal{Z} \mathbf{E}\{z | \mathcal{Z}_t\} + \int_s^T \int_t^T \mathcal{S}(T - r) BB^* \mathcal{S}^*(T - r) \varphi(r) \, dt \, d\hat{\mathbf{w}}(r)$$

$$= \Gamma_T^\mathcal{Z} \mathbf{E}\{z | \mathcal{Z}_t\} + \int_s^T \Gamma_T^\mathcal{Z} \varphi(r) \, d\hat{\mathbf{w}}(r).$$

(c) This follows from (a) and (b).
Definitions

Let us introduce the sets
\[ \mathcal{R}(t, x_0) = \{ \hat{x}(t; x_0, u) \mid u \in \mathcal{U}_{ad} \}, \]
\[ \mathcal{A}_\varepsilon^p(t, x_0) = \{ h \in H : \exists u \in \mathcal{U}_{ad} \mathbb{P}(\|\hat{x}(t; x_0, u) - h\|^2 \leq \varepsilon) \geq p \}. \]

First we define the stochastic analogue of complete controllability and approximate controllability concepts.

**Definition 2.1.** The linear stochastic system (4) is said to be
(a) completely controllable on \([0, T]\) if all the points in \(L_2(\mathcal{Z}_T, H)\) can be reached from the point \(x_0\) at time \(T\), that is, if
\[ \mathcal{R}(T, x_0) = L_2(\mathcal{Z}_T, H); \]
(b) approximately controllable on \([0, T]\) if
\[ \mathcal{A}(T, x_0) = L_2(\mathcal{Z}_T, H). \]

That is, it is possible to steer from the point \(x_0\) to within a distance \(\varepsilon > 0\) from all points in the state space \(L_2(\mathcal{Z}_T, H)\) at time \(T\).

To introduce the \(S\)-controllability notion we need the following theorem which shows that we need not introduce separately approximate controllability and complete controllability concepts in the stochastical sense.

Denote
\[ \mathcal{A}(T, x_0) = \bigcap_{\varepsilon > 0, 0 \leq p < 1} \mathcal{A}_\varepsilon^p(T, x_0). \]

**Theorem 2.1.** The following proposition holds:
\[ \mathcal{A}(T, x_0) = \bigcap_{\varepsilon > 0, 0 \leq p < 1} \mathcal{A}_\varepsilon^p(T, x_0) = \mathcal{A}(T, x_0). \]

**Proof.** It is known that
\[ \mathcal{A}(T, x_0) \subset \bigcap_{\varepsilon > 0, 0 \leq p < 1} \mathcal{A}_\varepsilon^p(T, x_0). \]
We will only show that
\[ \bigcap_{\varepsilon > 0, 0 \leq p < 1} \mathcal{A}_\varepsilon^p(T, x_0) \subset \mathcal{A}(T, x_0). \]
Let $h \in \cap_{\epsilon > 0, -1 < \epsilon < 1, 0 < \epsilon} \mathcal{A}(T, x_0)$ and fix $\epsilon_0 > 0$, $0 \leq p_0 < 1$. Then there exist $h_n \in \mathcal{A}(T, x_0)$ and $N(\epsilon_0) > 0$ such that for all $n > N(\epsilon_0)$,
\[
\|h_n - h\|^2 \leq \epsilon_0 / 4,
\]
for all $\epsilon > 0$, $0 \leq p < 1$. On the other hand, since $h_n \in \mathcal{A}(T, x_0)$ then there exists $u \in \mathcal{U}_{ad}$ with
\[
\forall P\left\{\|\hat{x}(T; x_0, u) - h_n\|^2 > \epsilon_0 / 4\right\} \leq 1 - p_0.
\]
Hence, for such $u \in \mathcal{U}_{ad}$ we have
\[
P\left\{\|\hat{x}(T; x_0, u) - h_n\|^2 > \epsilon_0\right\} \\
\leq P\left\{\|\hat{x}(T; x_0, u) - h_n\| + \|h_n - h\| > \sqrt{\epsilon_0}\right\} \\
\leq P\left\{\|\hat{x}(T; x_0, u) - h_n\| + \sqrt{\epsilon_0} / 2\sqrt{\epsilon_0}\right\} \\
\leq 1 - p_0.
\]
From this
\[
P\left\{\|\hat{x}(T; x_0, u) - h\|^2 \leq \epsilon_0\right\} = 1 - P\left\{\|\hat{x}(T; x_0, u) - h\|^2 > \epsilon_0\right\} \\
\geq 1 - (1 - p_0) = p_0.
\]
Thus, $h \in \mathcal{A}(T, x_0)$. As $\epsilon_0 > 0$, $0 \leq p_0 < 1$ are arbitrary, we have $h \in \cap_{\epsilon > 0, -1 < \epsilon < 1} \mathcal{A}(T, x_0)$. 

This theorem motivates us to introduce the following concept.

**Definition 2.2.** The linear stochastic system (4) is said to be $S$-controllable if
\[
\mathcal{A}(T, x_0) = \mathcal{A}(T, x_0) = H;
\]
i.e., given an arbitrary $\epsilon > 0$ it is possible to steer from the point $x_0$ to within a distance $\sqrt{\epsilon}$ from all points in the state space $H$ at time $T$ with a probability arbitrarily close to one.

If $T > 0$ can be arbitrarily small, we add the words “small time” in front of “controllable.” We say small time completely controllable, small time approximately controllable, and small time $S$-controllable.

**Minimum Energy Principle**

We define the linear regulator problem: Minimize
\[
J(u) = E\|\hat{x}(T; x_0, u) - h\|^2 + \lambda E \int_0^T \|u(t)\|^2 dt,
\]
(10)
over all \( u(\cdot) \in \mathcal{U}_{ad} \), where \( \dot{x}(t; x_0, u) \) is a state process, defined by (4); \( h \in L_2(\mathcal{Z}, H) \) and \( \lambda > 0 \) are parameters; and \( h \) has the representation 
\[ h = \mathbf{E}[h] + \int_0^T h(s) \, d\hat{\omega}(s) \] (see Lemma 2.3).

**Lemma 2.4.** There exists a unique optimal control \( u^\lambda(\cdot) \in \mathcal{U}_{ad} \) at which the functional (10) takes on its minimum value and

\[
u^\lambda(t) = -B^*\mathcal{P}^*(T - t) \left\{ R(\lambda, \Gamma_0^T)(\mathcal{P}(T)x_0 - \mathbf{E}h) ight. \\
+ \left. \int_0^T R(\lambda, \Gamma_0^T)[\mathcal{P}(T - s)D(s) - h(s)] \, d\hat{\omega}(s) \right\},
\]

\[
\dot{x}(T; x_0,u^\lambda) - h = \lambda R(\lambda, \Gamma_0^T)(\mathcal{P}(T)x_0 - \mathbf{E}h) \\
+ \int_0^T \lambda R(\lambda, \Gamma_0^T)[\mathcal{P}(T - s)D(s) - h(s)] \, d\hat{\omega}(s).
\]

**Proof.** The proof is similar to that of [14] and hence will be omitted.

The next lemma gives a formula for a control transferring \( x \) to an arbitrary \( h \) for the deterministic analogue of this formula see [4, 5].

**Lemma 2.5.** Assume that for arbitrary \( 0 \leq t \leq T \) the operator \( \Gamma_t^T \) is invertible. Then

(i) for arbitrary \( x_0 \in H \) and \( h \in L_2(\mathcal{Z}, H) \) the control

\[
u^0_0 = -B^*\mathcal{P}^*(T - t) \left\{ (\Gamma_0^T)^{-1}(\mathcal{P}(T)x_0 - \mathbf{E}h) ight. \\
+ \left. \int_0^t (\Gamma_s^T)^{-1}[\mathcal{P}(T - s)D(s) - h(s)] \, d\hat{\omega}(s) \right\}
\]

transfers \( x_0 \) to \( h \) at time \( T \).

(ii) Among all controls \( u(\cdot) \) transferring \( x_0 \) to \( h \) at time \( T \) the control \( u^0(\cdot) \) minimizes the integral

\[
\mathbf{E} \int_0^T \|u(t)\|^2 \, dt.
\]

**Proof.** Combine [20, Theorem 2.3] with Lemma 2.3.

3. THE COMPLETE CONTROLLABILITY

In this section necessary and sufficient conditions for complete controllability are discussed.
Theorem 3.1. The control system (4) is completely controllable on \([0, T]\) if and only if any one of the following conditions holds.

(a) \(\Pi_T^T \geq \gamma I\).

(b) \(R(\lambda, \Pi_T^T)\) converges as \(\lambda \to 0^+\) in uniform operator topology.

(c) \(\lambda R(\lambda, \Pi_T^T)\) converges to the zero operator as \(\lambda \to 0^+\) in uniform operator topology.

Proof. The proof of the theorem follows closely the proof of [16, Theorem 1].

Theorem 3.2. The following four conditions are equivalent.

(a) The stochastic system (4) is completely controllable on \([0, T]\).

(b) The deterministic system (5) is completely controllable on every \([s, T], 0 \leq s < T\).

(c) The deterministic system (5) is small time completely controllable.

(d) The stochastic system (5) is small time completely controllable.

Proof. It is clear that (b) \(\iff\) (c) and (d) \(\iff\) (a). The proof will consist in showing that (a) \(\iff\) (b) and that (c) \(\iff\) (d).

(a) \(\iff\) (b). Assume that the stochastic system (4) is completely controllable on \([0, T]\). Then by Theorem 3.1(a),

\[
E\langle \Pi_T^T z, z \rangle \geq \gamma E\|z\|^2
\]

for some \(\gamma > 0\) and all \(z \in L_2(\mathcal{Z}_T, H)\).

To prove the complete controllability of the deterministic system (5) let us write the left-hand side of the above inequality in terms of \(\Gamma_T^T\). To do this we use Lemma 2.3,

\[
E\langle \Pi_T^T z, z \rangle = E\left( \Gamma_0^T E z + \sum_{j=1}^{k} \int_0^T \Gamma_T^T \varphi_j(s) \, d\hat{w}_j(s),
\right.
\]

\[
E z + \sum_{j=1}^{k} \int_0^T \varphi_j(s) \, d\hat{w}_j(s)
\]

\[
= \langle \Gamma_0^T E z, E z \rangle + E \sum_{j=1}^{k} \alpha_j \int_0^T \langle \Gamma_T^T \varphi_j(s), \varphi_j(s) \rangle \, ds
\]

\[
\geq \gamma \left( \|E z\|^2 + E \sum_{j=1}^{k} \alpha_j \int_0^T \|\varphi_j(s)\|^2 \, ds \right).
\]

If \(E z = 0\) and \(\varphi(s)\) is such that

\[
\varphi_j(s) = \begin{cases} h & \text{if } s \in [r, r + e) \\ 0 & \text{otherwise} \end{cases}
\]
and \( \varphi_j(\tau) = 0, j = 2, 3, \ldots, k, \tau \in [0, T] \), then the last inequality can be rewritten as

\[
\int_r^{r+\varepsilon} \langle \Gamma_i^\varepsilon \varphi_j(s), \varphi_i(s) \rangle \, ds \geq \int_r^{r+\varepsilon} \|\varphi_i(s)\|^2 \, ds.
\]

Dividing through by \( \varepsilon \), and taking the limit as \( \varepsilon \to 0^+ \) one can see that

\[
\langle \Gamma_i^\varepsilon h, h \rangle \geq \gamma \|h\|^2,
\]

for some \( \gamma > 0 \). (13)

That is, the system (5) is completely controllable on each \([r, T]\).

(c) \( \Rightarrow \) (d). If the deterministic system (5) is controllable on every \([s, \tau]\), then the operator \( \Gamma_i^\tau \) is invertible and an operator defined by

\[
\Lambda_0^\tau z = (\Gamma_0^\tau)^{-1} E z + \int_0^\tau (\Gamma_i^\tau)^{-1} \varphi_i(s) \, d\hat{\nu}(s)
\]

is the inverse of \( \Pi_0^\tau \). From the invertibility of \( \Pi_0^\tau \) for all \( \tau > 0 \) we obtain small time complete controllability of the system (4).

4. THE APPROXIMATE CONTROLLABILITY

**Theorem 4.1.** The control system (4) is approximately controllable on \([0, T]\) if and only if any one of the following conditions holds.

(a) \( \Pi_0^T > 0 \).

(b) \( \lambda \mathcal{R}(\lambda, \Pi_0^T) \) converges to the zero operator as \( \lambda \to 0^+ \) in the strong operator topology.

(c) \( \lambda \mathcal{R}(\lambda, \Pi_0^T) \) converges to the zero operator as \( \lambda \to 0^+ \) in the weak operator topology.

**Proof.** The proof is a straightforward adaptation of the proof of [14, Theorem 2].

**Theorem 4.2.** The following four conditions are equivalent.

(a) The stochastic system (4) is approximately controllable on \([0, T]\).

(b) The deterministic system (5) is approximately controllable on every \([s, T]\), \( 0 \leq s < T \).

(c) The deterministic system (5) is small time approximately controllable.

(d) The stochastic system (4) is small time approximately controllable.

**Proof.** It is clear that (b) \( \Leftrightarrow \) (c) and (d) \( \Rightarrow \) (a).
(a) ⇒ (b). Let the stochastic system (4) be an approximately controllable on \([0, T]\). Then
\[
\mathbf{E}\left\| \lambda R(\lambda, \Pi_0^T)z \right\|^2 \to 0.
\]
From this and (12) we have
\[
\mathbf{E}\left\| \lambda R(\lambda, \Pi_0^T)z \right\|^2 = \left\| \lambda R(\lambda, \Gamma_s^T)Ez \right\|^2
\]
\[
+ \mathbf{E} \sum_{j=1}^{k} \alpha_j \int_0^T \left\| \lambda R(\lambda, \Gamma_s^T) \varphi_j(s) \right\|^2 ds \to 0. \tag{14}
\]
From here
\[
\mathbf{E} \sum_{j=1}^{k} \alpha_j \int_0^T \left\| \lambda R(\lambda, \Gamma_s^T) \varphi_j(s) \right\|^2 ds \to 0
\]
for all \(\varphi(\cdot) \in L_2^2(0, T, L_2^2(R^k, H))\), and consequently there is a subsequence \((\lambda_k)\) such that for all \(h \in H\),
\[
\left\| \lambda_k R(\lambda_k, \Gamma_s^T)h \right\| \to 0, \quad \text{almost everywhere on} \ [0, T].
\]
Because of the continuity of \(R(\lambda, \Gamma_s^T)\) this property holds for all \(0 \leq s < T\). The latter means that the deterministic system (5) is approximately controllable on every \([s, T]\), \(0 \leq s < T\).

(c) ⇒ (d). If the deterministic system (5) is approximately controllable on every \([s, \tau]\) then \(\|\lambda R(\lambda, \Gamma_s^T)h\| \to 0\) as \(\lambda \to 0^+\). Since
\[
\sum_{j=1}^{k} \alpha_j \left\| \lambda R(\lambda, \Gamma_s^T) \varphi_j(s) \right\|^2 \leq \sum_{j=1}^{k} \alpha_j \left\| \varphi_j(s) \right\|^2
\]
by the Lebesgue dominated convergence theorem from (14), we get
\[
\mathbf{E}\left\| \lambda R(\lambda, \Pi_0^T)z \right\|^2 \to 0 \quad \text{as} \quad \lambda \to 0^+.
\]
That is, the stochastic system (4) is small time approximately controllable.

\[\Box\]

**Theorem 4.3.** Let \(\mathcal{S}(t)\) be analytic. The following are equivalent.

(a) The stochastic system (4) is approximately controllable on \([0, T]\).

(b) The deterministic system (5) is approximately controllable on \([0, T]\).

(c) The deterministic system (5) is small time approximately controllable.
Proof. It is known that (b) \(\Leftrightarrow\) (c) if \(\mathcal{S}(t)\) is an analytic semigroup. (a) \(\Leftrightarrow\) (c) follows from Theorem 4.2.

5. THE S-CONTROLLABILITY

To compare approximate controllability and \(S\)-controllability concepts let us restrict ourselves to the case when the control set has to be the gaussian set \(\mathcal{U}_{ad}^g\).

Theorem 5.1. The following two conditions are equivalent:

(i) The stochastic system (4) is approximately controllable on \([0, T]\).

(ii) The stochastic system (4) is small time controllable with the control set \(\mathcal{U}_{ad}^g\).

Proof. (i) \(\Rightarrow\) (ii). Since the system (4) is approximately controllable then by Theorem 4.2 it is approximately controllable on every \([s, \tau]\), so \(\lambda R(\lambda, \Gamma_\tau) \to 0\) strongly as \(\lambda \to 0^+\). Lemma 2.4 says that for arbitrary (nonrandom) \(h \in H\) there exists a gaussian control

\[ u^\lambda(t) = -B^*\mathcal{S}^\lambda(t - t)(R(\lambda, \Gamma_\tau^\lambda)(\mathcal{S}(\tau)x_0 - h) \]

\[ + \int_0^t R(\lambda, \Gamma_\tau^s)\mathcal{S}(\tau - s)D(s)\,d\mathcal{W}(s) \]

in \(\mathcal{U}_{ad}^g\) such that

\[ \hat{x}(\tau; x_0, u^\lambda) - h = \lambda R(\lambda, \Gamma_\tau)\mathcal{S}(\tau)x_0 - Eh \]

\[ + \int_0^T \lambda R(\lambda, \Gamma_\tau^s)\mathcal{S}(\tau - s)D(s)\,d\mathcal{W}(s), \]

from which we deduce that

\[ E\|\hat{x}(\tau; x_0, u^\lambda) - h\|^2 \to 0, \quad \text{as} \quad \lambda \to 0^+. \]

Now small time \(S\)-controllability of the system (4) with the control set \(\mathcal{U}_{ad}^g\) follows from Chebyshev's inequality.

(ii) \(\Rightarrow\) (i). The proof of this implication is similar to the proofs given in [1] and [4, Theorem 12]. Suppose that the system (4) is small time \(S\)-controllable. Let \(h \in H\) and look at the sequences

\[ \{\varepsilon_n : \varepsilon_n > 0, \varepsilon_n \to 0\} \quad \text{and} \quad \{p_n : 0 \leq p_n < 1, p_n \to 1\}. \]
Then there exists a sequence \( \{u^n\} \in \mathcal{U}_{ad}^T \) such that
\[
P\{\|\hat{x}(T; x_0, u^n) - h\|^2 \leq \varepsilon_n\} \geq p_n
\]
and, consequently, for arbitrary \( \varepsilon > 0 \) there is a number \( N \) such that
\[
0 < \varepsilon_n < \varepsilon^2
\]
and
\[
P\{\|\hat{x}(T; x_0, u^n) - h\| > \varepsilon\} = 1 - P\{\|\hat{x}(T; x_0, u^n) - h\| \leq \varepsilon\}
\leq 1 - P\{\|\hat{x}(T; x_0, u^n) - h\|^2 \leq \varepsilon_n\}
= P\{\|\hat{x}(T; x_0, u^n) - h\|^2 > \varepsilon_n\}
\leq 1 - p_n,
\]
for all \( n > N \). This inequality implies the convergence of \( \hat{x}(T; x_0, u^n) \) to \( h \) in probability, so that for arbitrary \( \varepsilon > 0 \),
\[
P\{\|\hat{x}(T; x_0, u^n) - h\| > \varepsilon\} \to 0, \quad \text{as } n \to \infty.
\]
Then for all \( x \in H \),
\[
\lim_{n \to \infty} E e^{i\langle \hat{x}(T; x_0, u^n), x \rangle} = E e^{i\langle h, x \rangle}.
\]
On the other hand, since \( \hat{x}(T; x_0, u^n) \) is a Gaussian random variable (as the solution of (4) corresponding to the Gaussian control \( u^n \)), from the above convergence of characteristic functions and the gaussianity of \( \hat{x}(T; x_0, u^n) \) and \( h \) it follows that
\[
E e^{i\langle \hat{x}(T; x_0, u^n), x \rangle} = e^{i\langle m_n, x \rangle - (1/2)\langle \Lambda_n, x, x \rangle}
\]
and
\[
\lim_{n \to \infty} e^{i\langle m_n, x \rangle - (1/2)\langle \Lambda_n, x, x \rangle} = e^{i\langle h, x \rangle},
\]
So for all \( x \in H \),
\[
\langle m_n, x \rangle \to \langle h, x \rangle \text{ and } \langle \Lambda_n, x, x \rangle \to 0 \quad \text{as } n \to \infty,
\]
where \( m_n = E \hat{x}(T; x_0, u^n) \) and \( \Lambda_n = \text{cov}(\hat{x}(T; x_0, u^n)) \). The first convergence means that the sequence \( \{m_n\} \in H \) converges weakly to \( h \in H \) in \( H \). By Mazur's theorem we can construct the sequence
\[
h_n = \sum_{i=1}^{n} c^n_i m_i, \quad c^n_i \geq 0, \quad \sum_{i=1}^{n} c^n_i = 1, \quad i = 1, 2, \ldots,
\]
of convex combinations of \( m_i = \mathbf{E}\hat{x}(T; x_0, u^i), \) \( i = 1, 2, \ldots, n, \) such that \( h_n \) converges to \( h \) in the strong topology of \( H. \) Denote \( \tilde{u}^n = \sum_{i=1}^n e_i u^i, \) \( n = 1, 2, \ldots. \) It is obvious that \( \tilde{u}^n \in \mathcal{U}_{ad}. \) In view of the affineness of the system (4), we have

\[
\begin{align*}
  h_n &= \sum_{i=1}^n c_i^n m_i = \sum_{i=1}^n c_i^n \mathbf{E}\hat{x}(T; x_0, u^i) = \mathbf{E}\hat{x}(T; x_0, \sum_{i=1}^n c_i^n u^i) \\
  &= \mathbf{E}\hat{x}(T; x_0, \tilde{u}^n).
\end{align*}
\]

Now, if \( \bar{u}^n = \mathbf{E}\tilde{u}^n \in V \) then \( h_n = y(T; x_0, \bar{u}^n) \) and, consequently,

\[
\lim_{n \to \infty} \| y(T; x_0, \bar{u}^n) - h \| = \lim_{n \to \infty} \| h_n - h \| = 0,
\]

which means that the system (5) and by Theorem 4.2 the system (4) are small time approximately controllable.

**Remark 5.1.** If \( H \) is finite-dimensional space, then the complete controllability, the approximate controllability, the \( S \)-controllability of the stochastic system (4), and the controllability of the corresponding deterministic system (5) coincide (see [14]).

### 6. APPLICATIONS

In general, it is hard to prove that a system is completely controllable. We consider a simple example here as an application of Theorem 3.1.

Consider the controlled wave equation with a distributed control \( u(t, \cdot) \) \( \in L_2(0, 1), \)

\[
\begin{align*}
  d(\partial/\partial t)y(t, x) &= \left[ (\partial^2/\partial x^2)y(t, x) + u(t, x) \right] dt + dw(t), \\
  y(t, 0) &= y(t, 1) = 0, \\
  y(0, x) &= f(x), \quad (\partial/\partial t)y(0, x) = g(x),
\end{align*}
\]

(15)

where \( w(\cdot) \) is a Wiener process. By proceeding in a way similar to that in [6] we introduce the Hilbert space \( X = D(A_0^{1/2}) \oplus L_2(0, 1), \) endowed with the inner product

\[
\langle w, v \rangle = \left[ \begin{array}{c} w_1 \\ w_2 \end{array} \right] ' \left[ \begin{array}{c} e_1 \\ e_2 \end{array} \right] = \sum_{n=1}^\infty n^2 \pi^2 \langle w_1, e_n \rangle \langle e_n, v_1 \rangle + \langle w_2, e_n \rangle \langle e_n, v_2 \rangle,
\]

where \( e_n(\theta) = \sqrt{2} \sin(n \pi \theta). \) Setting

\[
\begin{align*}
  z &= \left[ \begin{array}{c} y \\ (\partial/\partial t)y \end{array} \right], \quad z(0) = \left[ \begin{array}{c} f \\ g \end{array} \right], \\
  B &= \left[ \begin{array}{c} 0 \\ I \end{array} \right], \\
  D &= \left[ \begin{array}{c} 0 \\ I \end{array} \right],
\end{align*}
\]

we write the problem (15) as

\[ dz = (Az + Bu) dt + Dw, \quad z(0) = \begin{bmatrix} f \\ g \end{bmatrix}, \]

where

\[ A = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}; \]

\[ A_0h = -(d^2/d\theta^2)h \] with domain

\[ D(A_0) = \left\{ h \in L_2(0, 1) \mid h, (d/d\theta)h \text{ are absolutely continuous}, \right. \]
\[ \left. (d^2/d\theta^2)h \in L_2(0, 1), \text{ and } h(0) = 0 = h(1) \right\}; \]

and \( A \) is the infinitesimal generator of a contraction semigroup \( \mathcal{S}(t) \) on \( X \) given by

\[ \mathcal{S}(t) \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \sum_{n=1}^{\infty} \begin{bmatrix} \cos(n\pi t) & (n\pi)^{-1} \sin(n\pi t) \\ -n\pi \sin(n\pi t) & \cos(n\pi t) \end{bmatrix} \begin{bmatrix} z_1^n \\ z_2^n \end{bmatrix} e_n. \]

It is easy to see that \( \mathcal{S}^*(t) = \mathcal{S}(-t) \) and \( B^* = [0, 1] \).

By Theorem 2.4 the system will be completely controllable on \([0, T] \) if \( \|\lambda R(\lambda, \Gamma_0^T)\| \to 0 \) as \( \lambda \to 0^+ \). Calculation of \( R(\lambda, \Gamma_0^T) \) yields.

\[ \Gamma_0^T z = \int_0^T \mathcal{S}(t) BB^* \mathcal{S}^*(t) z dt \]

\[ = \sum_{n=1}^{\infty} \frac{1}{2} \left( T - \frac{\sin(2n\pi T)}{2n\pi} \right) \frac{1}{4n^2\pi^2} (\cos(2n\pi T) - 1) \begin{bmatrix} z_1^n \\ z_2^n \end{bmatrix} e_n, \]

\[ = \sum_{n=1}^{\infty} A_n^T z \begin{bmatrix} z_1^n \\ z_2^n \end{bmatrix} e_n = \sum_{n=1}^{\infty} \begin{bmatrix} a_n^T & b_n^T \\ c_n^0 & d_n^T \end{bmatrix} \begin{bmatrix} z_1^n \\ z_2^n \end{bmatrix} e_n \]

\[ = \sum_{n=1}^{\infty} \begin{bmatrix} \lambda + \frac{T}{2} - \frac{\sin(2n\pi T)}{4n\pi} & -\frac{1}{4n^2\pi^2} \sin^2(n\pi T) \\ -\frac{1}{2} \sin^2(n\pi T) & \lambda + \frac{T}{2} + \frac{\sin(2n\pi T)}{4n\pi} \end{bmatrix} \begin{bmatrix} z_1^n \\ z_2^n \end{bmatrix} e_n, \]
where \( a_n^0 = \lambda + T/2 - \sin(2n\pi T)/4n\pi \), \( c_n^0 = n^2\pi^2 b_n^0 = -\frac{1}{2}\sin^2(n\pi T) \), and
\[ d_n^0 = \lambda + T/2 + \sin(2n\pi T)/4n\pi, \lambda \geq 0. \]

\[
\det A_n^0 = \lambda^2 + \lambda T + \frac{1}{4} \left( T^2 - \frac{\sin^2(n\pi T)}{n^2\pi^2} \right)
\]

\[
= \lambda^2 + \lambda T + \det A_n^0 \geq \det A_n^0
\]

\[
(\lambda I + \Gamma^2_0)^{-1} z = \sum_{n=1}^{\infty} (A_n^0)^{-1} \begin{bmatrix} z_1^n \\ z_2^n \end{bmatrix} e_n
\]

\[
= \sum_{n=1}^{\infty} (\det A_n^0)^{-1} \begin{bmatrix} d_n^0 & -b_n^0 \\ -c_n^0 & a_n^0 \end{bmatrix} \begin{bmatrix} z_1^n \\ z_2^n \end{bmatrix} e_n
\]

\[
\langle (\lambda I + \Gamma^2_0)^{-1} z, z \rangle = \sum_{n=1}^{\infty} (\det A_n^0)^{-1} \begin{bmatrix} d_n^0 & -b_n^0 \\ -c_n^0 & a_n^0 \end{bmatrix} \begin{bmatrix} z_1^n \\ z_2^n \end{bmatrix} e_n, z \rangle
\]

\[
= \sum_{n=1}^{\infty} (\det A_n^0)^{-1} \left[ n^2\pi^2 (d_n^0 |z_1^n|^2 - b_n^0 \overline{z_1^n} z_2^n) \\
+ (-c_n^0 z_1^n \overline{z_2^n} + a_n^0 |z_2^n|^2) \right]
\]

\[
\leq \sum_{n=1}^{\infty} (\det A_n^0)^{-1} \left[ (n^2\pi^2 d_n^0 + c_n^0) |z_1^n|^2 \\
+ (a_n^0 + c_n^0) |z_2^n|^2 \right]
\]

\[
= \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \left[ n^2\pi^2 (\tilde{d}_n^0 + \tilde{b}_n^0) |z_1^n|^2 + (\tilde{a}_n^0 + \tilde{c}_n^0) |z_2^n|^2 \right],
\]

where

\[
\tilde{d}_n^0 = \frac{d_n^0}{\det A_n^0} = \frac{2 \left( T + \frac{\sin(2n\pi T)}{2n\pi} \right)}{T^2 - \frac{\sin^2(n\pi T)}{n^2\pi^2}}
\]
\[
\tilde{a}_n^0 = \frac{a_n^0}{\det A_n^0} = \frac{2 \left( T - \frac{\sin(2n\pi T)}{2n\pi} \right)}{T^2 - \frac{\sin^2(n\pi T)}{n^2\pi^2}} \\
\tilde{c}_n^0 = \frac{c_n^0}{\det A_n^0} = \frac{-2 \sin^2(2n\pi T)}{T^2 - \frac{\sin^2(n\pi T)}{n^2\pi^2}} \\
\tilde{b}_n^0 = \frac{b_n^0}{\det A_n^0} = \frac{-2 \sin^2(2n\pi T)}{n^2\pi^2 \left( T^2 - \frac{\sin^2(n\pi T)}{n^2\pi^2} \right)}
\]

\[
\left\langle \left( \lambda I + \Gamma_0^T \right)^{-1} z, z \right\rangle \leq \sum_{n=1}^{\infty} \left[ n^2\pi^2 \left| \tilde{a}_n^0 z_1^n \right|^2 - 2\tilde{c}_n^0 \Re (z_1^n z_2^n) + \left| \tilde{a}_n^0 z_2^n \right|^2 \right] \\
\leq \sum_{n=1}^{\infty} \left[ n^2\pi^2 \left| \tilde{a}_n^0 + \tilde{c}_n^0 \right| |z_1^n|^2 + \left| \tilde{a}_n^0 + \tilde{c}_n^0 \right| |z_2^n|^2 \right] \\
= \sum_{n=1}^{\infty} \left[ n^2\pi^2 \left( \tilde{a}_n^0 + \tilde{b}_n^0 \right) |z_1^n|^2 + \left( \tilde{a}_n^0 + \tilde{c}_n^0 \right) |z_2^n|^2 \right]. \tag{16}
\]

It is easy to see that \( \tilde{a}_n^0 \to \frac{2}{\pi}, \tilde{a}_n^0 \to \frac{2}{\pi}, \tilde{c}_n^0 \leq \gamma, \tilde{b}_n^0 \to 0 \). So the sequences \( \tilde{d}_n^0, \tilde{a}_n^0, \tilde{c}_n^0, \tilde{b}_n^0 \) are bounded, say,

\[
\max_{n \geq 1} \left( \tilde{d}_n^0, \tilde{a}_n^0, \tilde{c}_n^0, \tilde{b}_n^0 \right) \leq \gamma_0.
\]

Then from (16) we obtain

\[
\left\langle \left( \lambda I + \Gamma_0^T \right)^{-1} z, z \right\rangle \leq \sum_{n=1}^{\infty} \left[ n^2\pi^2 \left( \tilde{a}_n^0 + \tilde{b}_n^0 \right) |z_1^n|^2 + \left( \tilde{a}_n^0 + \tilde{c}_n^0 \right) |z_2^n|^2 \right] \\
\leq \gamma_0 \sum_{n=1}^{\infty} \left[ n^2\pi^2 |z_1^n|^2 + |z_2^n|^2 \right] = \gamma_0 \|z\|^2,
\]

from which it follows that

\[
\| \lambda (\lambda I + \Gamma_0^T)^{-1} \| \to 0, \quad \text{as } \lambda \to 0^+.
\]

That is, the deterministic system corresponding to (15) and consequently the stochastic system (15) are completely controllable.
REFERENCES