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Subnormal Weighted Translation Semigroups

MARY R. EMBRY AND ALAN LAMBERT*

Mathematics Department, The University of North Carolina, Charlotte, North Carolina 28223

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A weighted translation semigroup $\{S_t\}$ on $L^2(\mathbb{R}_+)$ is defined by $(S_t f)(x) = (\phi(x)/\phi(x-t)) f(x-t)$ for $x \geqslant t$ and 0 otherwise, where ϕ is a continuous nonzero scalar-valued function on $\mathbb{R}_+$. It is shown that $\{S_t\}$ is subnormal if and only if ϕ^2 is the product of an exponential function and the Laplace–Stieltjes transform of an increasing function of total variation one. A necessary and sufficient condition for similarity of weighted translation semigroups is developed.

1. INTRODUCTION

In [3] the authors initiated the study of a special class of semigroups of operators on $L^2(\mathbb{R}_+)$, called weighted translation semigroups. Such a semigroup is defined by $(S_t f)(x) = (\phi(x)/\phi(x-t)) f(x-t)$ if $x \geqslant t$ and 0 otherwise, where ϕ is a continuous nonzero complex-valued function on $\mathbb{R}_+$. These semigroups appear to be the natural continuous analogs of weighted shifts. It was shown in [3] that the following conditions on $\{S_t\}$ are equivalent: (i) $\{S_t\}$ is hyponormal, (ii) the infinitesimal generator of $\{S_t\}$ is hyponormal, and (iii) $\log |\phi|$ is convex.

The question arose of equivalent conditions that $\{S_t\}$ be subnormal. It follows from results in [6] that $\{S_t\}$ is subnormal if and only if its infinitesimal generator is subnormal. (This is true for arbitrary semigroups as well as weighted translation semigroups.) The problem of finding a condition for subnormal weighted translation semigroups analogous to (iii) for the hyponormals was challenging.

Even in the case of a single operator A it may be exceedingly difficult to determine whether A is subnormal. The best known methods are the following: (1) write down a normal extension of A, or (2) show that A satisfies the Halmos–Bram criterion for subnormality [1], $\sum_{i,j=0}^{n} (A^i x_j, A^j x_i) \geqslant 0$ for each finite collection $x_0, \ldots, x_n$, or (3) in case A is a weighted shift, show that A satisfies the

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Berger condition [4, pp. 895–897] that the sequence of products of the squares of the first n weights be a moment sequence, or (4) in case A is a weighted shift, show that the weights of A satisfy Stampfli's iterated condition [8]. Stampfli's condition allows one to write down the normal extension of a subnormal weighted shift.

One of the main results of this paper (Theorems 2.2 and 2.3) is that a weighted translation semigroup $\{S_t\}$ is subnormal if and only if its defining function ϕ is a Laplace–Stieltjes transform of a probability measure. This characterization is the continuous analog of the Berger condition.

In Section 3 several methods for constructing subnormal weighted translation semigroups are indicated. Furthermore a criterion is developed to show when two such semigroups are similar.

Throughout the paper we shall use the notation $\{S_t\} \sim \phi$ to mean that $\{S_t\}$ is the semigroup on L^2 $(= L^2(\mathbb{R}_+))$ defined by $(S_t f)(x) = \phi(x)/\phi(x-t) f(x-t)$ for $x \geq t \geq 0$, and 0 otherwise. We call ϕ the *symbol* for $\{S_t\}$. To avoid needless difficulties, we assume that ϕ is continuous and nonzero. To insure the strong continuity of $\{S_t\}$ [3, Lemma 2.1], we assume that ess $\sup_{x \in \mathbb{R}_+} |\phi(x+t)/\phi(x)| \leq Me^{wt}$ for all t and some constants M and w. Since $\{S_t\} \sim \phi$ and $\{T_t\} \sim \rho$ are unitarily equivalent [3, Theorem 2.5] if and only if $|\phi/\rho|$ is constant, we can and do assume that ϕ is positive-valued.

An operator A on a Hilbert space X is subnormal if and only if A is the restriction of a normal operator to an invariant subspace. A semigroup $\{S_t\}$ of operators on X is subnormal if and only if each S_t is subnormal on X. By a theorem of Ito [6] this is equivalent to the seemingly stronger assertion that there exists a normal semigroup $\{N_t\}$ (on a larger space) such that $\{S_t\}$ is the restriction of $\{N_t\}$ to X. By an arbitrary semigroup $\{S_t\}$ of operators on a Hilbert space X we mean a set $\{S_t\}$ such that $S_0 = I$, the identity operator, and for all s and t in $\mathbb{R}_+$, $S_t S_s = S_{t+s}$. $\{S_t\}$ is a strongly continuous semigroup if $\lim_{t \to 0} S_t f = f$ for all f in X.

2. A Characterization of Subnormal Weighted Translation Semigroups

Our major objective in this section is to characterize the subnormal weighted translation semigroups $\{S_t\}$ with symbol ϕ. This we do in Theorem 2.2 in which we show that $\{S_t\}$ is subnormal if and only if ϕ^2 has an integral representation

$$\phi^2(x) = \int_0^a s^x \, d\rho(s),$$

where ρ is a probability measure on $[0, a]$. This is analogous to Berger's result on subnormal weighted shifts [4]; S is a subnormal weighted shift with real weights $\{\lambda_0, \lambda_1, \ldots\}$ if and only if $\beta_n{}^2 = (\lambda_0, \ldots, \lambda_n)^2$ is a moment sequence, $\beta_n{}^2 = \int_0^a s^n \, d\rho(s)$, ρ a probability measure.

Before proving our characterization of subnormal weighted translation semigroups we consider arbitrary subnormal semigroups. In [2] Embry proved that a single operator S is subnormal if and only if $\{S^{*n}S^n\}$ is a Hausdorff moment sequence: $S^{*n}S^n = \int_0^a s^n \, d\rho(s)$, where ρ is an operator measure, of total variation I. The following theorem is the continuous analog for semigroups.

THEOREM 2.1. *Let $\{S_t\}$ be a strongly continuous semigroup. $\{S_t\}$ is subnormal if and only if there exists an integral representation of $\{S_t{}^*S_t\}$ of the form*

$$S_t{}^*S_t = \int_0^a s^t \, d\rho,$$

where ρ is an operator measure of total variation I, *and*

$$a = \lim_{t\to\infty} \| S_t{}^*S_t \|^{1/t}.$$

Proof. Assume first that $\{S_t\}$ is subnormal and that $\{N_t\}$ on Y is the minimal normal extension of $\{S_t\}$. Then $\{N_t{}^*N_t\}$ is a semigroup of self-adjoint operators and by [5, Theorem 22.3.1, p. 588] there exists a spectral resolution σ such that $N_t{}^*N_t = \int_{-\infty}^{w_0} e^{ts} \, d\sigma(s)$ and $w_0 = \lim_{t\to\infty} t^{-1} \log \| N_t{}^*N_t \|$. Let P be the projection of Y onto X and define $\tau = P\sigma P$. We have

$$\begin{aligned} S_t{}^*S_t &= PN_t{}^*N_tP \\ &= \int_{-\infty}^{w_0} e^{ts} \, d\tau(s) \\ &= \int_0^{e^{w_0}} s^t \, d\rho(s), \end{aligned}$$

where $\rho = \tau \circ \log$. Finally $e^{w_0} = \lim_{t\to\infty} \| N_t{}^*N_t \|^{1/t} = \lim_{t\to\infty} \| S_t{}^*S_t \|^{1/t}$, since $\{N_t\}$ is the minimal normal extension of $\{S_t\}$.

Now assume that $\{S_t\}$ has the stated integral representation. Then for fixed t, $S_t^{*n}S_t{}^n = S_{nt}^*S_{nt} = \int_0^a (s^t)^n \, d\sigma$, so that $\{S_t^{*n}S_t{}^n\}$ is a Hausdorff moment sequence. By [2], this implies that S_t is subnormal. Consequently, by [6], $\{S_t\}$ is subnormal and the proof is complete.

THEOREM 2.2. *The weighted translation semigroup with symbol ϕ is subnormal if and only if there exists a probability measure ρ on an interval* $[0, a]$ *such that, on* $\mathbb{R}_+$,

$$\phi^2(x) = \int_0^a s^x \, d\rho(s). \tag{1}$$

Proof. Note that if $\{S_t\}$ is the weighted translation semigroup with symbol ϕ, then for each f in L^2, $(S_t{}^*S_tf)(x) = (\phi^2(x + t)/\phi^2(x))f(x)$ a.e. dx.

Assume first that $\phi^2(x) = \int_0^a s^x \, d\rho(s)$, ρ a probability measure on $[0, a]$. Define

$$\rho(s, x) = (1/\phi^2(x)) \int_0^s t^x \, d\rho(t)$$

for s in $[0, a]$ and x in $\mathbb{R}_+$. For each x in $\mathbb{R}_+$, $\rho(s, x)$ is a nondecreasing function of s and $\rho(a, x) = 1$ and $\rho(0, x) = 0$. Thus for each x in $\mathbb{R}_+$, $\rho(\ , x)$ defines a Borel measure on $[0, a]$. Furthermore,

$$\int_0^a s^t \, d\rho(s, x) = (1/\phi^2(x)) \int_0^a s^t \cdot s^x \, d\rho(s)$$
$$= \phi^2(x + t)/\phi^2(x),$$

for all t and x in $\mathbb{R}^+$. We conclude first that $\phi^2(x + t)/\phi^2(x) \leqslant a^t$, so that the semigroup $\{S_t\}$ with symbol ϕ is strongly continuous [3]. Second, since $(S_t^* S_t f)(x) = (\phi^2(x + t)/\phi^2(x)) f(x) = \int_0^a s^t \, d\rho(s, x) f(x)$, we conclude, by applying Theorem 2.1, that $\{S_t\}$ is subnormal.

Conversely, we assume that $\{S_t\}$ is subnormal and $S_t^* S_t = \int_0^a s^t \, d\rho$. We shall consider ρ to be a nondecreasing operator-valued function on $[0, a]$, continuous from the right on $(0, a)$ with $\rho(a) = I$ and $\rho(0) = 0$. Integrating by parts, we have

$$S_t^* S_t = a^t - \int_0^a \rho(s) \, ds^t.$$

Now if E is a measurable subset of $\mathbb{R}_+$ and f is an element of L^2 such that $f = 0$ a.e. on E, then

$$0 = (\phi^2(x + t)/\phi^2(x)) f(x) = a^x f(x) - \int_0^a (\rho(s) f)(x) \, ds^t$$

a.e. on E. Thus $\int_0^a (\rho(s) f)(x) \, g(s) \, dx = 0$ a.e. on E for all continuous g on $\mathbb{R}_+$. We conclude that $(\rho(s) f)(x) = 0$ for all s and almost all x in E. Thus $L^2(E)$ is invariant under $\rho(s)$ for each measurable E, implying that $\rho(s)$ is a multiplication operator on L^2 and that there exist functions $\rho(s, x)$ such that, for f in L^2, $(\rho(s) f)(x) = \rho(s, x) f(x)$. Consequently,

$$(\phi^2(x + t)/\phi^2(x)) f(x) = \int_0^a s^t \, d\rho(s, x) f(x).$$

Choosing f to be the characteristic function of $[0, 1]$ and $x = 0$, we have

$$\phi^2(t) = \int_0^a s^t \, d\rho(s, 0).$$

It remains to be shown that $\rho(s, 0)$ is nondecreasing and of total variation 1. Since $\rho(s)$ is nondecreasing, it follows from the definition of $\rho(s, x)$ that for each pair s and t of real numbers ($s \leqslant t$), $\rho(s, x) \leqslant \rho(t, x)$ for almost all x. We conclude that, except on a set E of measure zero, $\rho(s, x) \leqslant \rho(t, x)$ for all x and all rational s and t. Since $\rho(s)$ is continuous from the right, $\rho(s, x) \leqslant \rho(t, x)$ for all s and t and all $x \notin E$. Finally, since $\rho(a) = I$ and $\rho(0) = 0$, we conclude that $\rho(s, 0)$ generates a probability measure; the proof is complete.

The integral representation (1) for the symbol ϕ of a subnormal semigroup leads easily to

Corollary 2.3. *$\{S_t\}$ is subnormal if and only if ϕ^2 is the product of an exponential and a Laplace–Stieltjes transform of a monotone function σ of total variation* 1,

$$\phi^2(x) = a^x \int_0^\infty e^{-sx}\, d\sigma(s). \tag{2}$$

Proof. In (1), first substitute $t = s/a$. Then substitute $u = -\log t$ and $\sigma(u) = 1 - \rho(ae^{-u})$.

For a general discussion of Laplace–Stieltjes transforms we refer the reader to [9]. Several facts concerning the symbols ϕ associated with subnormal semigroups are easily obtained. For example, if we define $\phi^2(z) = \int_0^\infty e^{-sz}\, d\sigma(s)$ for $\operatorname{Re} z \geqslant 0$, then ϕ^2 is well-defined (Theorem 3.1, p. 47) and is analytic for $\operatorname{Re} z > 0$ (Theorem 5a, p. 57). An immediate result of this fact is that if $\{S_t\} \sim \phi$ is subnormal, then ϕ is C^∞ on $(0, \infty)$. In particular, if ϕ is not differentiable on $(0, \infty)$, then $\{S_t\} \sim \phi$ is not subnormal. We note that any ϕ for which $\log \phi$ is convex, but ϕ is not, C^∞ gives rise to a hyponormal, nonsubnormal weighted translation semigroup.

A second observation is that if ϕ_1 and ϕ_2 are symbols for subnormal semigroups and $\phi_1(x) = \phi_2(x)$ for each x in an infinite subset of $\mathbb{R}_+$ with a cluster point in $\mathbb{R}_+\backslash\{0\}$, then $\phi_1 = \phi_2$. In particular, if $\phi_1(x) = Ae^{bx}$ on some interval, then $\phi_1(x) = Ae^{bx}$ for all x and $\{S_t\} \sim \phi$ is quasinormal. This is analogous to a result of Stampfli [7] concerning subnormal weighted shifts: If $\{\lambda_0, \lambda_1, \ldots\}$ is the weight sequence for a subnormal weighted shift and $\lambda_k = \lambda_{k+1}$ for some integer k, then $\lambda_k = \lambda_1$ for all $k \geqslant 1$.

3. Examples of Subnormal Weighted Translation Semigroups

In this section we examine some methods in which Theorem 2.2 may be used to construct subnormal weighted translation semigroups.

(i) Perhaps the simplest application of Theorem 2.2 is when $d\rho$ is a point mass, that is, ρ is defined on an interval $[0, c]$ by

$$\begin{aligned} \rho(s) &= 0 \quad \text{for} \quad 0 \leqslant s < d, \\ &= 1 \quad \text{for} \quad d \leqslant s \leqslant c. \end{aligned}$$

It then follows that $\phi^2(x) = \int_0^c s^x \, d\rho(s) = d^x$. Conversely, we can show that if $\phi(x) = a^x$, then $d\rho$ is a point mass. It was shown in [3] that the semigroups arising in this way are precisely the quasinormal weighted translation semigroups.

(ii) Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of nonnegative real numbers such that $\sum a_n = 1$. Let $g(s) = \sum_{n=0}^{\infty} (n+1) a_n s^n$, $0 \leqslant s < 1$. Then g is increasing and integrable over $[0, 1)$. Let $d\rho(s) = g(s)\, ds$. Then set $\phi^2(x) = \int_0^1 s^x g(s)\, ds = \sum_{n=0}^{\infty} a_n \cdot (n+1)/(n+1+x)$. It follows that ϕ is the symbol for a subnormal weighted translation semigroup. If we choose $a_0 = 1$ and $a_{n+1} = 0$ for all $n \geqslant 0$, we see that $(1+x)^{-1/2}$ is such a symbol. This last gives us an example of a subnormal, nonquasinormal weighted translation semigroup.

(iii) Since a convex combination of increasing functions of total variation one is again such a function, we see that the set $\{\phi^2 : \phi > 0$ and $\{S_t\} \sim \phi$ is subnormal$\}$ is convex.

(iv) Since the product of two absolutely convergent Laplace–Stieltjes transforms is again a Laplace–Stieltjes transform [5, p. 216], we see that $\{\phi: \{S_t\} \sim \phi$ is subnormal$\}$ is a semigroup under multiplication in $C^{\infty}(\mathbb{R}_+)$.

Remarks. (a) Categories (iii) and (iv) above indicate that the set of ϕ's which are the symbols for subnormal semigroups has a reasonable algebraic structure in $C^{\infty}(\mathbb{R}_+)$. Further development of this structure will be of interest.

(b) As noted in [3], if $\{S_t\} \sim \phi$ and if for each $t > 0$ we set M_t = closed linear span $\{\phi X_{[nt,(n+1)t]} : n \geqslant 0\}$, where $X_{(a,b)}$ is the characteristic function of the interval (a, b), then M_t is invariant for S_t and the restriction of S_t to M_t is a weighted shift with nth weight $(\int_{nt}^{(n+1)t} \phi^2(x)\, dx / \int_{(n-1)t}^{nt} \phi^2(x)\, dx)^{1/2}$. It was shown, moreover, that $\{S_t\}$ is subnormal if and only if each of these shifts is subnormal. Thus, Theorem 2.2 enables one to construct large numbers of subnormal weighted shifts without resorting to Berger's theorem a correspondingly large number of times.

(c) A slightly different connection between Berger's representation for subnormal weighted shifts and our representation for subnormal weighted translation semigroups is arrived at as follows. If $\{S_t\} \sim \phi$ is subnormal, then $\phi^2(x) = \int_0^a s^x \, d\rho(x)$, for some probability measure ρ. In particular, $\phi^2(n) = \int_0^a s^n \, d\rho(s)$. Thus $\{\phi^2(n)\}$ is a moment sequence and, by Berger's theorem, $\{\phi(n)/\phi(n-1)\}$ are the weights of a subnormal weighted shift.

In [3] it was shown that if $\{S_t\} \sim \phi$ and $\{T_t\} \sim \rho$, then $\{S_t\}$ and $\{T_t\}$ are unitarily equivalent if and only if $|\phi/\rho|$ is constant. We conclude this paper by establishing a necessary and sufficient condition for similarity of weighted translation semigroups. Before doing so, however, it is necessary for us to review some of the basic properties of invertibly weighted shifts and show how these properties relate to weighted translation semigroups. In this regard we wish to express our thanks to Warren Wogen for his helpful suggestions.

If $\{A_n\}_{n=1}^{\infty}$ is a sequence of invertible operators on a Hilbert space H such that $\sup_n \| A_n \| < \infty$, then the operator S defined on $l_2(H) = H \oplus H \oplus \cdots$ by

$$S\langle x_0, x_1, \ldots\rangle = \langle 0, A_1 x_0, A_2 x_1, \ldots\rangle$$

is called the invertibly weighted shift with weight sequence $\{A_n\}$. Let S and T be invertibly weighted shifts with weight sequences $\{A_n\}$ and $\{B_n\}$, respectively, and define $S_n = A_n A_{n-1} \cdots A_1$, $S_0 = \mathrm{I}$, and define $\{T_n\}$ similarly in terms of $\{B_n\}$. Let X be an operator on $l_2(H)$ with matrix $[X_{ij}]_{i,j=0}^{\infty}$ relative to the decomposition $H \oplus H \oplus \cdots$ of $l_2(H)$. It is shown in [7] that $SX = XT$ if and only if

$$\begin{aligned} X_{ij} &= 0, & i < j, \\ &= S_i S_{i-j}^{-1} X_{i-j,0} T_j^{-1}, & i \geq j. \end{aligned}$$

Now, if $\{S_t\} \sim \phi$, then each S_t is unitarily equivalent to an invertibly weighted shift. To see this fix $t > 0$ and for each $f \in L^2(\mathbb{R}_+)$ write $f = \sum_{n=0}^{\infty} \oplus f_n$, where $f_n \in L^2(0, t)$ is given by $f_n(x) = f(x + nt)$. This establishes a unitary equivalence between $L^2(\mathbb{R}_+)$ and $l_2(L^2(0, t))$. Moreover, this equivalence identifies S_t with the invertibly weighted shift S with weight sequence {multiplication by $\phi_n/\phi_{n-1}\}_{n=1}^{\infty}$.

THEOREM 3.1. *The weighted translation semigroups* $\{S_t\} \sim \phi$ *and* $\{T_t\} \sim \rho$ *are similar if and only if* ϕ/ρ *and* ρ/ϕ *are bounded.*

Proof. From [3, Lemma 2.1] we see that there is no loss in generality in assuming that ϕ and ρ are positive valued. Suppose there exist constants a and b so that, for each x in $\mathbb{R}_+$, $0 < a \leq \phi(x)/\rho(x) \leq b < \infty$. Then one easily verifies that the operator A given by multiplication by ϕ/ρ is invertible and satisfies $AT_t = S_t A$ for each $t \geq 0$. Conversely, suppose $\{S_t\}$ and $\{T_t\}$ are similar. Let X be an invertible operator satisfying $S_t X = X T_t$ for all $t \geq 0$. Fix $t > 0$. By the remarks concerning invertibly weighted shifts preceding this theorem we have, for $i \geq j$,

$$X_{ij}^{(t)} = S_i^{(t)} (S_{i-j}^{(t)})^{-1} X_{i-j,0}^{(t)} (T_j^{(t)})^{-1},$$

where the t superscript is used as a reminder that all decompositions, matrices, etc. are with respect to $L^2(\mathbb{R}_+) = l_2(L^2(0, t))$. Also $S_n^{(t)}$ is the operator on $L^2(0, t)$ given by

$$(S_n^{(t)} f)(x) = (\phi(x + nt)/\phi(x))\, f(x),$$

and similarly for $\{T_n^{(t)}\}$. Note that $\| S_n^{(t)} \| = \sup_{0 \leq x \leq t} (\phi(x + nt)/\phi(x))$. Now we see that for each $n \geq 0$, $X_{nn}^{(t)} T_n^{(t)} = S_{00}^{(t)} X_{00}^{(t)}$. Also, since X is invertible and

$T_t X^{-1} = X^{-1} S_t$, X^{-1} has a lower triangular matrix with respect to the decomposition in question, and $(X^{-1})_{00}^{(t)} = (X_{00}^{(t)})^{-1}$. Thus

$$\| S_n^{(t)} \| = \| X_{nn}^{(t)} T_n^{(t)} (X_{00}^{(t)})^{-1} \|$$
$$\leqslant (\| X \| \| X^{-1} \|) \| T_n^{(t)} \|.$$

Letting $c = \| X \| \| X^{-1} \|$ we see that, for every $t > 0$ and every positive integer n, $\sup_{0 \leqslant x \leqslant t} (\phi(x + nt)/\phi(x)) \leqslant c \sup_{0 \leqslant x \leqslant t} (\rho(x + nt)/\rho(x))$. Now, since ϕ and ρ are bounded above and away from 0 on $[0, 1]$, there is a constant d so that, for all $n \geqslant 0$ and all $t \in [0, 1]$, $\sup_{0 \leqslant x \leqslant t} \phi(x + nt) \leqslant d \cdot \sup_{0 \leqslant x \leqslant t} \rho(x + nt)$. We shall show that $\phi(y) \leqslant d \cdot \rho(y)$ for all $y \geqslant 0$. Assume the contrary. Then, for some $y > 0$, $\phi(y) > d \cdot \rho(y)$. By continuity this strict inequality must hold on some interval (a, b). But then there is a number t in $(0, 1]$ (any $t < (b - a)/2$ will do) and an integer n so that $(nt, (n + 1)t) \subset (a, b)$. But then we would have $\sup_{0 \leqslant x \leqslant t} \phi(x + nt) > d \cdot \sup_{0 \leqslant x \leqslant t} \rho(x + nt)$, which is impossible. Thus $\phi(y) \leqslant d \cdot \rho(y)$ for all y. By reversing the roles of ϕ and ρ, we see that both ϕ/ρ and ρ/ϕ are bounded, completing the proof.

Remark. Theorem 3.1 shows that a weighted translation semigroup may be similar to a subnormal weighted translation semigroup without its symbol enjoying any of the smoothness properties associated, via Theorem 2.2, with the symbol of a subnormal weighted translation semigroup. Indeed, even using the isometric semigroup $\{U_t\}$ associated with $\phi(x) = 1$, we see that if $\{T_t\} \sim \rho$ then $\{T_t\}$ and $\{U_t\}$ are similar if and only if ρ and $1/\rho$ are bounded; that is, the graph of ρ lies in a finite horizontal strip which is bounded away from the x-axis.

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