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The Perturbation of the Drazin Inverse and Oblique Projection

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Abstract—Let A and E be $n \times n$ matrices and $B = A + E$. Denote the Drazin inverse of A by A^D. We present bounds for $\|B^D\|$, $\|B^DB\|$, $\|B^D - A^D\|/\|A^D\|$, and $\|B^DB - A^DA\|/\|A^DA\|$ under the weakest condition core rank B = core rank A. The hard problem due to Campbell and Meyer in [1] is completely solved.

Keywords—Index, Drazin inverse, Group inverse, Perturbation bound, Core rank.

1. INTRODUCTION

A necessary and sufficient condition for the continuity of the Drazin inverse (to be defined in the next section) was established by Campbell and Meyer in 1975 [1]. They stated the main result: suppose that A_j, $j = 1, 2, \ldots$, and A are $n \times n$ matrices such that $A_j \to A$. Then $A_j^D \to A^D$ (where A_j^D is the Drazin inverse of A_j) if and only if there is a positive integer j_0 such that core rank A_j = core rank A for $j \geq j_0$ (where core rank $A = \operatorname{rank} A^k$, $k = \operatorname{Ind}(A)$, the index of A defined as the smallest integer $k \geq 0$ such that $\operatorname{rank} A^k = \operatorname{rank} A^{k+1}$).

In the same paper, they also indicated two difficulties in establishing norm estimates for the Drazin inverse. First, the Drazin inverse has a weaker type of "cancellation law" and is somewhat harder to work with algebraically than Moore-Penrose inverse. Also complicating things is the fact that the Jordan form is not a continuous function from $C^{n\times n} \to C^{n\times n}$ and the Drazin inverse can be thought of in terms of the Jordan canonical form. Due to these reasons, they thought that it would be difficult to establish norm estimates for the Drazin inverse similar to those for the Moore-Penrose inverse, as was done by Stewart [2].

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In Campbell's 1977 paper [3], he proved the main result: if a matrix X comes to satisfying the definition of the Drazin inverse of A, A^D, then $\|X - A^D\|$ is small. Norm estimates are given which make precise what is close.

In [4], Rong gave an explicit upper bound for $\|B^D - A^D\|/\|A^D\|$ under certain circumstances with the second-order term of $\|E\|$.

In this paper, we shall give another explicit bound for $\|B^D - A^D\|/\|A^D\|$ in terms of A, A^D, and $E(l)(= B^l - A^l$ for any arbitrary positive integer l), provided E is sufficiently small and core rank A = core rank B, i.e., $\operatorname{rank} B^j = \operatorname{rank} A^k$, where $j = \operatorname{Ind}(B)$ and $k = \operatorname{Ind}(A)$. Also, we present bounds for $\|B^D\|$, $\|B^D B\|$, and $\|B^D B - A^D A\|/\|A^D A\|$. We extend the conclusions by several authors. Wei and Wang [5] obtained the simple perturbation bound under the assumptions of $E = AA^D E = EAA^D$, as well as $E = AA^D E$ or $E = EAA^D$ by Wei [6], respectively. One best lower bound for $\|B^D - A^D\|/\|A^D\|$ is presented provided $E = AA^D E = EAA^D$. These results are analogous to those for the Moore-Penrose inverse as was done by Stewart [2], i.e., we have completely solved the hard problem due to Campbell and Meyer in 1975.

2. PRELIMINARIES

Throughout this paper, the following definitions and notations will be used. $\mathcal{C}^n$ stands for the n-dimensional complex space and $\mathcal{C}^{n\times n}$ stands for the set of all $n \times n$ complex matrices. $\mathcal{R}(A)$ and $\mathcal{N}(A)$ denote the range and the null space of A, respectively. Rank A denotes the rank of A. We will write $\|.\|$ for the spectral norm.

Let $A \in \mathcal{C}^{n\times n}$ with $\operatorname{Ind}(A) = k$ and if $X \in \mathcal{C}^{n\times n}$ such that

$$A^{k+1}X = A^k, \qquad XAX = X, \qquad AX = XA, \tag{2.1}$$

then X is called the Drazin inverse of A, and is denoted by $X = A^D$. In particular, when $\operatorname{Ind}(A) = 1$, the matrix X that satisfies (2.1) is called the group inverse, and is denoted by $X = A^\#$.

It is well known that (by the Jordan form) if $A \in \mathcal{C}^{n\times n}$ with $\operatorname{Ind}(A) = k$, then for any $l \geq k$, $A^D = (A^l)^\# A^{l-1}$, and $\operatorname{Ind}(A^l) = 1$, $A^D A = (A^l)^\# A^l = P_{\mathcal{R}(A^l),\mathcal{N}(A^l)}$, the oblique projector along $\mathcal{N}(A^l)$ onto $\mathcal{R}(A^l)$.

The perturbation bound for the group inverse can be found in the literature [6, Theorem 4.2]. The main result is as follows.

LEMMA 2.1. *Let $B = A + E \in \mathcal{C}^{n\times n}$ such that $\operatorname{Ind}(A) = \operatorname{Ind}(B) = 1$ and $\operatorname{rank} A = \operatorname{rank} B$. If $\|A^\#\|\|E\| < 1/(1 + \|A^\# A\|)(\leq 1/2)$, then*

$$\|B^\#\| \leq \|A^\#\| \frac{1 - \|A^\#\|\,\|E\|}{\left[1 - \|A^\#\|\,\|E\|\,(1 + \|A^\# A\|)\right]^2} \tag{2.2}$$

and

$$\|B^\# B\| \leq \|A^\# A\| \frac{1 - \|A^\#\|\,\|E\|\,(1 - \|A^\# A\|)}{1 - \|A^\#\|\,\|E\|\,(1 + \|A^\# A\|)}. \tag{2.3}$$

The following lemmas are needed in what follows.

LEMMA 2.2. *(See [1].) If S, T are subspaces of $\mathcal{C}^n$ and $\dim(S) > \dim(T) > 0$, then for any complementary space P of T, the intersection $S \cap P$ is nontrivial.*

LEMMA 2.3. *(See [8].) For any oblique projector $P \in \mathcal{C}^{n\times n}$, it holds $\|P\| = \|I - P\|$ where $P \neq 0$.*

LEMMA 2.4. *(See [9].) Suppose that $\|F\| < 1$. Then $I + F$ is nonsingular and*

$$\|(I - F)^{-1}\| \leq \frac{1}{1 - \|F\|}. \tag{2.4}$$

For the details of Drazin inverse, see the excellent books by Ben-Israel and Greville [10] and by Campbell and Meyer [9].

3. SPECIAL CASE

In this section, we will prove Banach-type theorem and perturbation bounds for the Drazin inverse in some special cases.

First, we give a necessary and sufficient condition such that B^D has the simple form (3.1), as shown in the following theorem.

THEOREM 3.1. *Let* $B = A + E$ *with* $\mathrm{Ind}(A) = k$ *and* $\mathrm{Ind}(B) = j$. *Let* $l = \max\{\mathrm{Ind}(A), \mathrm{Ind}(B)\}$ *and* $E(l) = B^l - A^l$. *If* $\|EA^D\| < 1$, *then*

$$B^D = \left(I + A^D E\right)^{-1} A^D = A^D \left(I + EA^D\right)^{-1} \tag{3.1}$$

if and only if

$$\text{core rank}\, B = \text{core rank}\, A \qquad \text{and} \qquad AA^D E(l) = E(l) = E(l)AA^D. \tag{3.2}$$

PROOF.

($\Leftarrow$). Suppose that equation (3.2) holds. It is obvious that

$$B^l = A^l + AA^D E(l) = A^l \left[I + \left(A^l\right)^{\#} E(l)\right] = \left[I + E(l) \left(A^l\right)^{\#}\right] A^l.$$

In view of core rank B = core rank A, we obtain directly $\mathcal{R}(B^l) = \mathcal{R}(A^l)$ and $\mathcal{N}(B^l) = \mathcal{N}(A^l)$, i.e., $AA^D = BB^D$. By direct verification [5], we have $B^D - A^D = -B^D EA^D = -A^D EB^D$. Noticing that the assumption $\|EA^D\| < 1$ implies the nonsingularity of $I + A^D E$ and $I + EA^D$. Thus,

$$B^D = \left(I + A^D E\right)^{-1} A^D = A^D \left(I + EA^D\right)^{-1}.$$

($\Rightarrow$). Suppose that equation (3.1) holds. We can deduce that $\mathcal{R}(B^D) = \mathcal{R}(A^D)$ and $\mathcal{N}(B^D) = \mathcal{N}(A^D)$, which reduces to $\mathrm{rank}\, A^k = \mathrm{rank}\, B^j$ and $AA^D = BB^D$, i.e., core rank B = core rank A. By direct computation, we obtain

$$AA^D E(l) = AA^D \left(B^l - A^l\right) = B^l - A^l = E(l) = E(l)AA^D,$$

which complete the proof. ∎

REMARK. In the above theorem, $\mathrm{Ind}(B)$ may be not equal to $\mathrm{Ind}(A)$ although core rank B = core rank A. The condition $\|EA^D\| < 1$ is only to ensure that $I + A^D E$ and $I + EA^D$ are nonsingular. It can be replaced by other conditions, such as the following theorem.

THEOREM 3.2. *Let* $B = A + E$ *with* $\mathrm{Ind}(A) = k$. *Suppose* $AA^D E = E = EAA^D$. *Then* $I + A^D E$ *is invertible if and only if*

$$\mathcal{R}\left(B^i\right) = \mathcal{R}\left(A^i\right) \quad \text{and} \quad \mathcal{N}\left(B^i\right) = \mathcal{N}\left(A^i\right), \qquad i = 1, 2, \ldots, k. \tag{3.3}$$

If (3.3) holds, then $\mathrm{Ind}(B) = \mathrm{Ind}(A) = k$ *and*

$$B^D = \left(I + A^D E\right)^{-1} A^D = A^D \left(I + EA^D\right)^{-1}. \tag{3.4}$$

Furthermore,

$$AA^D E(k) = E(k) = E(k)AA^D. \tag{3.5}$$

PROOF.

($\Leftarrow$). Suppose that equation (3.3) holds. It is evident that $\mathrm{rank}\, B^k = \mathrm{rank}\, A^k$. Since $\mathcal{R}(A^k) \bigoplus \mathcal{N}(A^k) = C^n$, then $\mathcal{R}(B^k) \bigoplus \mathcal{N}(B^k) = C^n$. This implies $\mathrm{Ind}(B) = k$ and $AA^D = BB^D$, so core rank B = core rank A. Following an exact way of the proof of [7, Theorem 3.1], we can show that $I + A^D E$ is invertible.

($\Rightarrow$). Suppose that $I + A^D E$ is invertible. It follows from [5, Theorem 3.1] that $\mathcal{R}(B^i) = \mathcal{R}(A^i)$ and $\mathcal{N}(B^i) = \mathcal{N}(A^i)$, $i = 1, 2, \ldots, k$. If condition (3.3) holds, then equalities (3.4) and (3.5) are obtained by the same argument of proving Theorem 3.1, where $l = \mathrm{Ind}(B) = \mathrm{Ind}(A) = k$. ∎

Combining Theorem 3.1 and Theorem 3.2, we have the following corollary.

COROLLARY 3.3. *(See [7].) Let $B = A + E$ with* $\operatorname{Ind}(A) = 1$. *Then*

$$B^{\#} = \left(I + A^{\#}E\right)^{-1} A^{\#} = A^{\#}\left(I + EA^{\#}\right)^{-1}, \tag{3.6}$$

if and only if

$$\operatorname{rank} B = \operatorname{rank} A \qquad \text{and} \qquad AA^{\#}E = E = EAA^{\#}. \tag{3.7}$$

Next, we give a Banach-type perturbation theorem for the Drazin inverse by applying Theorem 3.1.

THEOREM 3.4. *Let $B = A + E$ with* $\operatorname{Ind}(A) = k$, $\operatorname{Ind}(B) = j$. *Let* $l = \max\{\operatorname{Ind}(A), \operatorname{Ind}(B)\}$ *and* $E(l) = B^l - A^l$. *Assume that condition (3.2) holds. If* $\|EA^D\| < 1$, *then*

$$\frac{\|A^D\|}{1 + \|EA^D\|} \le \|B^D\| \le \frac{\|A^D\|}{1 - \|EA^D\|} \tag{3.8}$$

and

$$\frac{\|EA^D\|}{\mathcal{K}_D(A)\,(1 + \|A^D\|\,\|E\|)} \le \frac{\|B^D - A^D\|}{\|A^D\|} \le \frac{\|EA^D\|}{1 - \|EA^D\|}, \tag{3.9}$$

where $\mathcal{K}_D(A) = \|A^D\|\|A\|$ *is defined as the condition number of* A^D.

PROOF. It follows directly from Theorem 3.1 that

$$\frac{\|A^D\|}{1 + \|EA^D\|} \le \|B^D\| \le \frac{\|A^D\|}{1 - \|EA^D\|}. \tag{3.10}$$

Notice that $B^D - A^D = -B^D E A^D$, then

$$\|B^D - A^D\| \le \|B^D\|\,\|EA^D\| \le \frac{\|A^D\|\,\|EA^D\|}{1 - \|EA^D\|}, \tag{3.11}$$

which leads to the right inequality of (3.9). On the other hand, from $AA^D = BB^D$, we have

$$EA^D = B\left(A^D - B^D\right) = (A + E)\left(A^D - B^D\right).$$

Hence,

$$\|B^D - A^D\| \ge \frac{\|EA^D\|}{\|A\| + \|E\|},$$

i.e,

$$\frac{\|B^D - A^D\|}{\|A^D\|} \ge \frac{\|EA^D\|}{\mathcal{K}_D(A) + \|A^D\|\,\|E\|} \ge \frac{\|EA^D\|}{\mathcal{K}_D(A)\,(1 + \|A^D\|\,\|E\|)},$$

and we complete the proof. ∎

COROLLARY 3.5. *(See [5].) Let $B = A + E$ with* $\operatorname{Ind}(A) = k$. *Suppose* $AA^DE = E = EAA^D$. *If* $\|EA^D\| < 1$, *then*

$$\frac{\|EA^D\|}{\mathcal{K}_D(A)\,(1 + \|EA^D\|)} \le \frac{\|B^D - A^D\|}{\|A^D\|} \le \frac{\|EA^D\|}{1 - \|EA^D\|}. \tag{3.12}$$

PROOF. The upper bound was proved in [5]. We need only to show the lower bound of (3.12). Note that

$$EA^D = B\left(A^D - B^D\right) = \left(I + EA^D\right) A \left(A^D - B^D\right). \tag{3.13}$$

Taking the norms on both sides of (3.13), we obtain

$$\|EA^D\| \le \left(1 + \|EA^D\|\right) \|A\|\,\|A^D - B^D\|.$$

Hence,

$$\frac{\|B^D - A^D\|}{\|A^D\|} \ge \frac{\|EA^D\|}{(1 + \|EA^D\|)\,\mathcal{K}_D(A)}.$$

∎

Before ending this section, we give an example to show that the lower bound of (3.12) is a sharp one.

EXAMPLE. Let

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad E = \begin{bmatrix} \epsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then it holds $\mathrm{Ind}(A) = \mathrm{Ind}(A + E) = 2$ and $E = AA^D E = EAA^D$.

$$A^D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad (A+E)^D = \begin{bmatrix} \frac{1}{1+\epsilon} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is observed that $\|EA^D\| = \epsilon < 1$ and $\|B^D - A^D\|/\|A^D\| = \epsilon/(1+\epsilon) = \|EA^D\|/(1 + \|EA^D\|)\mathcal{K}_D(A)$.

4. GENERAL CASE

Let $A,\ E \in \mathcal{C}^{n\times n}$, $B = A + E$ with $\mathrm{Ind}(A) = k$ and $\mathrm{Ind}(B) = j$. For any arbitrary positive integer p, define $E(p) = B^p - A^p$. In this section, we shall consider the problem of bounding $\|B^D\|$, $\|B^D B\|$, $\|B^D - A^D\|/\|A^D\|$, and $\|B^D B - A^D A\|/\|A^D A\|$ in terms of $\|E\|$, $\|E(l)\|$, and $\|E(l-1)\|$ under the weakest condition $\text{core rank}\, B = \text{core rank}\, A$.

THEOREM 4.1. *Let* $l = \max\{\mathrm{Ind}(A), \mathrm{Ind}(B)\}$, $\text{core rank}\, A = \text{core rank}\, B$. *If* $\|(A^D)^l\|\|E(l)\| < 1/(1 + \|A^D A\|)(\le 1/2)$, *then*

$$\|B^D\| \le \left\|(A^D)^l\right\| \left(\|A^{l-1}\| + \|E(l-1)\|\right) \frac{1 - \left\|(A^D)^l\right\| \|E(l)\|}{\left[1 - \left\|(A^D)^l\right\| \|E(l)\| \left(1 + \|A^D A\|\right)\right]^2} \tag{4.1}$$

and

$$\|B^D B\| \le \|A^D A\| \frac{1 - \left\|(A^D)^l\right\| \|E(l)\| \left(1 - \|A^D A\|\right)}{1 - \left\|(A^D)^l\right\| \|E(l)\| \left(1 + \|A^D A\|\right)}. \tag{4.2}$$

PROOF. Since $l = \max\{\mathrm{Ind}(A), \mathrm{Ind}(B)\}$, it is evident that $\mathrm{Ind}(A^l) = \mathrm{Ind}(B^l) = 1$ and $(A^l)^{\#} A^l = A^D A$, $(B^l)^{\#} B^l = B^D B$. From the fact $B^D = (B^l)^{\#} B^{l-1} = (B^l)^{\#}[A^{l-1} + E(l-1)]$ and Lemma 2.1, we obtain immediately the upper bounds (4.1) and (4.2) for $\|B^D\|$ and $\|B^D B\|$, respectively. ∎

We are now in a position to bound $\|B^D - A^D\|/\|A^D\|$ and $\|B^D B - A^D A\|/\|A^D A\|$.

THEOREM 4.2. *Let* $l = \max\{\mathrm{Ind}(A), \mathrm{Ind}(B)\}$ *and* core rank B = core rank A. *If* $\|(A^D)^l\|\|E(l)\| < 1/(1+\|A^DA\|)$ $(\le 1/2)$, *then*

$$\begin{aligned}\frac{\|B^D - A^D\|}{\|A^D\|} \le\ & \frac{\left\|(A^D)^l\right\| \|E\| \left(\|A^{l-1}\| + \|E(l-1)\|\right)\left(1 - \|(A^D)^l\|\,\|E(l)\|\right)}{\left[1 - \left\|(A^D)^l\right\| \|E(l)\| \left(1 + \|A^DA\|\right)\right]^2} \\ & + \frac{\left\|(A^D)^l\right\|^2 \|A\|\|E(l)\| \left(\|A^{l-1}\| + \|E(l-1)\|\right)\left(1 - \left\|(A^D)^l\right\| \|E(l)\|\right)^2}{\left[1 - \left\|(A^D)^l\right\| \|E(l)\| \left(1 + \|A^DA\|\right)\right]^4} \\ & + \frac{\left\|(A^D)^l\right\| \|E(l)\| \|A^DA\| \left[1 - \left\|(A^D)^l\right\| \|E(l)\| \left(1 - \|A^DA\|\right)\right]}{1 - \left\|(A^D)^l\right\| \|E(l)\| \left(1 + \|A^DA\|\right)}\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}\frac{\|B^DB - A^DA\|}{\|A^DA\|} \le\ & \frac{\left\|(A^D)^l\right\| \|E(l)\| \left(1 - \left\|(A^D)^l\right\| \|E(l)\|\right)}{\left[1 - \left\|(A^D)^l\right\| \|E(l)\| \left(1 + \|A^DA\|\right)\right]^2} \\ & + \frac{\left\|(A^D)^l\right\| \|E(l)\| \left[1 - \left\|(A^D)^l\right\| \|E(l)\| \left(1 - \|A^DA\|\right)\right]}{1 - \left\|(A^D)^l\right\| \|E(l)\| \left(1 + \|A^DA\|\right)}.\end{aligned} \tag{4.4}$$

PROOF. By a direct computation, we have

$$\begin{aligned} B^D - A^D &= -B^DEA^D + B^D\left(I - AA^D\right) - \left(I - B^DB\right)A^D \\ &= -B^DEA^D + \left(B^D\right)^2 B\left(I - AA^D\right) - \left(I - B^DB\right)A\left(A^D\right)^2 \\ &= -B^DEA^D + B^D\left(B^l\right)^{\#}\left(A^l + E(l)\right)\left(I - AA^D\right) - \left(I - B^DB\right)A^l\left(A^l\right)^{\#}A^D \\ &= -B^DEA^D + B^D\left(B^l\right)^{\#}E(l)\left(I - AA^D\right) + \left(I - B^DB\right)E(l)\left(A^l\right)^{\#}A^D \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} B^DB - A^DA &= B^DB\left(I - AA^D\right) - \left(I - BB^D\right)AA^D \\ &= \left(B^l\right)^{\#}E(l)\left(I - AA^D\right) + \left(I - BB^D\right)E(l)\left(A^l\right)^{\#}. \end{aligned} \tag{4.6}$$

Taking norms on both sides of (4.5) and (4.6) and using Lemma 2.1 and 2.3, we arrive at (4.3) and (4.4). ∎

REMARK. Note that $\|E(l)\| \le \sum_{i=0}^{l-1} C_l^i \|A\|^i \|E\|^{l-i}$, where C_l^i is the binomial coefficient. Then, if $\|E\|$ is sufficiently small, the condition $\|(A^D)^l\|\|E(l)\| < 1/(1+\|A^DA\|)$ in Theorem 4.1 and 4.2 can be satisfied.

On the other hand, if core rank $B \ne$ core rank A, we shall find a lower bound for $\|B^D\|$ which tends to infinity as E approaches zero.

THEOREM 4.3. *Let* $B = A + E$ *with* $\mathrm{Ind}(A) = k$ *and* $\mathrm{Ind}(B) = j$. *Let* $l = \max\{\mathrm{Ind}(A), \mathrm{Ind}(B)\}$. *If* core rank $B >$ core rank A, *then*

$$\|(A+E)^D\| \ge \left\{\frac{1}{\|E(l)\|}\right\}^{1/l} \tag{4.7}$$

and

$$\|B^DB - A^DA\| \ge 1. \tag{4.8}$$

PROOF. Note that core rank B > core rank A is equivalent to rank $B^j > \operatorname{rank} A^k$. Because $\mathcal{R}(A^k)\bigoplus\mathcal{N}(A^k) = C^n$, then by Lemma 2.2, there exists a nonzero vector x such that $x \in \mathcal{R}(B^j) \cap \mathcal{N}(A^k)$. Without loss of generality, we assume that $\|x\| = 1$. The proof of (4.7) is analogous to that of [7, Theorem 4.6].

At the same time, we have

$$\begin{aligned} 1 &= x^H B^D B x = x^H \left(B^D B - A^D A\right) x \\ &\le \|x\| \left\|\left(B^D B - A^D A\right) x\right\| \le \left\|B^D B - A^D A\right\| \end{aligned}$$

and arrive at equation (4.8). ∎

REMARK. When core rank $B \neq$ core rank A and $\|E\|$ is sufficiently small, it is easy to see core rank B > core rank A.

As a corollary of Theorem 4.3, we have the following well-known result about the continuity of Drazin inverse.

COROLLARY 4.4. *(See [1].) The necessary and sufficient condition of*

$$\lim_{B\to A} B^D = A^D$$

is that core rank B = core rank A *as* B *approaches* A.

5. CONCLUDING REMARKS

In this paper, we have discussed more thoroughly the norm estimates for $\|B^D\|$, $\|B^D B\|$, $\|B^D - A^D\|/\|A^D\|$, and $\|B^D B - A^D A\|/\|A^D A\|$, i.e., we have answered the hard question of Campbell and Meyer in [1].

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