Groups with the same non-commuting graph

M.R. Darafsheh *

School of Mathematics, College of Science, University of Tehran, Tehran, Iran

ARTICLE INFO

Article history:
Received 29 August 2007
Received in revised form 20 May 2008
Accepted 3 June 2008
Available online 14 July 2008

Keywords:
Non-commuting graph
Simple group
Graph isomorphism

ABSTRACT

The non-commuting graph \( \Gamma_G \) of a non-abelian group \( G \) is defined as follows. The vertex set of \( \Gamma_G \) is \( G - Z(G) \) where \( Z(G) \) denotes the center of \( G \) and two vertices \( x \) and \( y \) are adjacent if and only if \( xy \neq yx \). It has been conjectured that if \( G \) and \( H \) are two non-abelian finite groups such that \( \Gamma_G \cong \Gamma_H \), then \( |G| = |H| \) and moreover in the case that \( H \) is a simple group this implies \( G \cong H \). In this paper, our aim is to prove the first part of the conjecture for all the finite non-abelian simple groups \( H \). Then for certain simple groups \( H \), we show that the graph isomorphism \( \Gamma_G \cong \Gamma_H \) implies \( G \cong H \).

© 2008 Elsevier B.V. All rights reserved.

1. Introduction

Let \( G \) be a group and \( Z(G) \) be its center. We will associate a graph \( \Gamma_G \) to \( G \) which is called the non-commuting graph of \( G \). The vertex set \( V(\Gamma_G) \) is \( G - Z(G) \) and the edge set \( E(\Gamma_G) \) consists of \( \{x, y\} \), where \( x \) and \( y \) are distinct non-central elements of \( G \) such that \( xy \neq yx \). It is clear that we are considering simple graphs, i.e., graphs with no loops or directed or repeated edges. According to [16] the non-commuting graph of a finite group \( G \) was first considered by P. Erdös in connection with the following problem. Let \( G \) be a group whose non-commuting graph \( \Gamma_G \) has no infinite complete subgraphs. Is it true that there is a finite bound on the cardinalities of complete subgraphs of \( \Gamma_G \)? In fact Neuman in [16] gave a positive answer to Erdös’s question and this was the origin of many similar research about non-commuting graph of a group.

Recently in [1] some group and graph properties of the non-commuting graph associated to a non-abelian group are studied. In particular the authors put forward the following conjectures.

Conjecture 1. Let \( G \) and \( H \) be two non-abelian finite groups such that \( \Gamma_G \cong \Gamma_H \). Then \( |G| = |H| \).

Conjecture 2. Let \( P \) be a finite non-abelian simple group and \( G \) be a group such that \( \Gamma_G \cong \Gamma_P \). Then \( G \cong P \).

Since non-abelian finite simple groups are known, it is possible to prove Conjecture 1 in the case that \( H \) is a non-abelian finite simple group using a case by case consideration of \( H \). However we prove that Conjecture 2 is related to a conjecture of Thompson which is stated below.

Conjecture 3. If \( G \) is a finite group with \( Z(G) = 1 \) and \( M \) is a non-abelian finite simple group satisfying \( N(G) = N(M) \), then \( G \cong M \). Here the set \( N(G) \) is defined by \( N(G) = \{ n \in N | G \text{ has a conjugacy class } C \text{ such that } |C| = n \} \).

It is shown in [4] that Conjecture 3 is related to another graph associated to a finite group \( G \), the so called Gruenberg–Kegel, or the prime graph of \( G \). The Gruenberg–Kegel graph of a finite group \( G \), denoted by \( GK(G) \), has the set of all primes dividing the order of \( G \) as its vertex set and two distinct primes \( p \) and \( q \) are joined by an edge if and only if the

* Fax: +98 21 66412178.
E-mail address: darafsheh@ut.ac.ir.

0166-218X/$ - see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.dam.2008.06.010
group $G$ has an element of order $pq$. The set of all the prime divisors of $G$ is denoted by $\pi(G)$ and the connected components of $GK(G)$ are denoted by $\pi_1, \pi_2, \ldots, \pi_{|\pi(G)|}$, where $t(G)$ denotes the number of connected components of $G$. In [15,17] it is proved that for any finite simple group $G$ we have $t(G) \leq 6$. A list of all the finite simple groups with disconnected Gruenberg–Kegel graph is available in various papers, for example one can refer to Table 1 in [5]. In [4] it is proved that Conjecture 3 is valid for simple groups $G$ with $t(G) \geq 3$.

Our main results in this paper are:

**Theorem A.** Let $P$ be a finite non-abelian simple group. If $G$ is a finite group with $\Gamma_G \cong \Gamma_P$. Then $|G| = |P|$.

**Theorem B.** Let $P$ be a simple group for which the Thompson’s conjecture holds. Then if $G$ is a finite group such that $\Gamma_G \cong \Gamma_P$, we have $G \cong P$.

Since Thompson’s conjecture has been verified for all the simple groups $P$ with $t(P) \geq 3$ in [4], so Conjecture 2 is valid for all the simple groups listed in Table II of [4]. Consequently Thompson’s conjecture is valid for a variety of simple groups $P$ with $t(P) = 2$. For example one can refer to [7] and the references quoted in that paper. In particular, since Thompson’s conjecture holds for all sporadic groups, Conjecture 2 also holds for these groups as well.

Since Conjecture 2 has been verified for a special class of finite non-abelian simple groups $P$, Conjecture 2 is valid for this special class as well. Our notation for graphs is standard and one can consult [2] for the graph concepts that we use here. Our notation for the name finite non-abelian simple groups is obtained from [6].

2. Preliminary results

In this section we state some results which are needed to prove our main theorems. For a non-abelian group $G$, the non-commuting graph of $G$ is denoted by $\Gamma_G$. The vertex set $V(\Gamma_G)$ of $\Gamma_G$ is $G - Z(G)$ and two vertices $x$ and $y$ are joined by an edge if and only if $xy \neq yx$.

Now if $H$ is a group and the graphs $\Gamma_G$ and $\Gamma_H$ are isomorphic then this means that there is a one-to-one correspondence $\varphi : G - Z(G) \rightarrow H - Z(H)$ preserving edges, i.e. if $x, y \in G - Z(G)$ and $xy \neq yx$ then $\varphi(x)\varphi(y) \neq \varphi(y)\varphi(x)$. Equivalently if we consider the complimentary graphs of $\Gamma_G$ and $\Gamma_H$, then we have the condition that $x, y \in G - Z(G)$ and $xy = yx$ implies $\varphi(x)\varphi(y) = \varphi(y)\varphi(x)$. The graph isomorphism $\Gamma_G \cong \Gamma_H$ implies also $|G - Z(G)| = |H - Z(H)|$. Furthermore if we assume only $G$ is non-abelian, then $|G - Z(G)| \neq 0$ implying $|H - Z(H)| \neq 0$ and consequently $H$ must be non-abelian. Since $|Z(H)| \leq |H - Z(H)| = |G - Z(G)|$, finiteness of $G$ implies that $Z(H)$ and consequently $H$ is a finite group. Therefore if in the isomorphism $\Gamma_G \cong \Gamma_H$ we assume $G$ is a non-abelian finite group, then the same conditions will be satisfied by $H$.

The degree of a vertex $v$ in a graph $\Gamma$ is defined to be the number of edges adjacent to $v$ and it is denoted by $d(v)$. It is easy to see that the degree of a vertex $g$ in the non-commuting graph $\Gamma_G$ of $G$ is equal to $d(G) = |G| - |C_G(g)|$ which can be written as $d(G) = |C_G(g)| (|G : C_G(g)| - 1)$. Therefore both $|C_G(g)|$ and $Z(G)$ are divisors of $d(G)$. In the following $P$ is a finite non-abelian group and $G$ is a group such that $\Gamma_P \cong \Gamma_G$ is a graph isomorphism.

**Lemma 1.** Let $\Gamma_P \cong \Gamma_G$. Then for any non-empty subset $S$ of $P$ which intersects $Z(P)$ in empty set, $|Z(G)|$ divides $|C_P(S)| - |Z(P)|$.

**Proof.** By definition from $\Gamma_P \cong \Gamma_G$ it follows that there is a one-to-one correspondence $\varphi : P - Z(P) \rightarrow G - Z(G)$ preserving edges of the graphs. Now any subset $S$ of $P$ with $Z(P) \cap S = \emptyset$ if the bijection $\varphi$ induces the one-to-one correspondence $\varphi : C_P(S) - Z(P) \rightarrow C_G(\varphi(S)) - Z(G)$. Therefore $|C_P(S)| - |Z(P)| = |C_G(\varphi(S))| - |Z(G)|$, from which it follows that $|C_P(S)| - |Z(P)| = |Z(G)| (|C_G(\varphi(S))| - |Z(G)|)$. Consequently $|Z(G)| = |C_P(S)| - |Z(P)|$ and the Lemma follows. ■

We remark that if $S = \{x\}, x \notin Z(P)$, then $|Z(G)| = |C_P(x)| - |Z(P)|$ which is mentioned in Lemma 3.1 of [1]. Obviously from $\Gamma_P \cong \Gamma_G$ it is evident that $|P| - |Z(P)| = |G| - |Z(G)|$ from which it follows that $|Z(G)| = |P| - |Z(P)|$ and consequently $Z(G)$ divides $|C_P(x)| - |Z(P)|$. Hence $|Z(G)| = |P| - |C_P(x)|$ for all $x \in P - Z(P)$. Hence $|Z(G)| = |C_P(x)| (|P : C_P(x)| - 1)$. If $P$ is a centerless group, i.e. $|Z(P)| = 1$, then $|P| = |C_P(x)| - 1$ implies that $|Z(G)|$ and $|C_P(x)|$ are coprime and from $|Z(G)| = |C_P(x)| (|P : C_P(x)| - 1)$ we obtain $|Z(G)| = |x^P| - 1$, where $x^P$ is the conjugacy class in $P$ containing $x$.

**Lemma 2.** Let $P$ and $G$ have trivial centers and $\varphi : \Gamma_P \rightarrow \Gamma_G$ be a graph isomorphism. Then for any $x \in P$ we have $\varphi(C_P(x)^*) = C_G(\varphi(x))^*$, where $A^* = A - \{1\}$.

**Proof.** By definition $\varphi : P - \{1\} \rightarrow G - \{1\}$ is a one-to-one correspondence preserving edges. For any $y \in \varphi(C_P(x)^*)$ we have $y = \varphi(t), t \in C_P(x)^*$, hence $tx = xt$ implying $\varphi(t)\varphi(x) = \varphi(x)\varphi(t)$ or $\varphi(y)\varphi(x) = \varphi(x)\varphi(y)$. Therefore $y \in C_G(\varphi(x))^*$ proving $\varphi(C_P(x)^*) \subseteq C_G(\varphi(x))^*$.

Conversely if $y \in C_G(\varphi(x))^*$, then $y \in G$ and $\varphi(y)\varphi(x) = \varphi(x)\varphi(y)$. Since $\varphi$ is onto there is $g \in G$ such that $\varphi(g) = y$. Hence $\varphi(g)\varphi(x) = \varphi(x)\varphi(g)$ implying $gx = xg$ or $g \in C_P(x)^*$, giving $y \in \varphi(C_P(x)^*)$. Therefore $C_G(\varphi(x))^* \subseteq \varphi(C_P(x)^*)$ and the Lemma is proved. ■

Before stating the next lemma we define the following concept. For a finite group $G$ we define $N(G) = \{n \in \mathbb{N} |$ there is a conjugacy class of $G$ with size $n\}$.

**Lemma 3.** If $\Gamma_P \cong \Gamma_G$, and $P$ and $G$ are centerless, then $N(P) = N(G)$. 
Proof. As in the proof of Lemma 2 let \( \varphi : P \to \{1\} \to G \to \{1\} \) be a one-to-one correspondence of the vertices of the graphs \( \Gamma_P \) and \( \Gamma_G \) with the extension \( \varphi(1) = 1 \). In this manner we also have \( |P| = |G| \). For \( x \in P \) let \( x^p \) denote the conjugacy class of \( x \) in \( P \). Using Lemma 2 we do the following calculation: 
\[
|x^p| = |P : \psi(x)| = |P : \varphi(x)| = |G : \psi(\varphi(x))| = |G : \varphi(\varphi(x))| = |G : C_G(\varphi(x))| = |G : \varphi(x)|^G.
\]
Since \( \varphi \) is a bijection we observe that there is a one-to-one correspondence between the conjugacy class sizes of \( P \) and \( G \), hence \( N(P) = N(G) \). 

3. Main results

Given a finite non-abelian simple group \( P \), we are concerned with a group \( G \) satisfying \( \Gamma_P \cong \Gamma_G \). As a matter of fact we want to prove that \( |P| = |G| \) and if possible \( P \cong G \). In [1] the following results related to above questions are obtained. For the alternating group \( A_n \), \( n \geq 5 \), the projective special linear groups \( \text{PSL}_2(2^n) \) and \( \text{PSL}_2(q) \), \( q \) odd, it is proved that if \( P \) is any of these groups and if \( G \) is a group satisfying \( \Gamma_P \cong \Gamma_G \), then \( |P| = |G| \). Furthermore it is also proved that if \( P \) is any of the simple groups \( \text{PSL}_2(2^n) \) or the Suzuki simple groups \( \text{Suz}(2^{2n+1}) \), \( n > 1 \), then \( \Gamma_P \cong \Gamma_G \) implies \( P \cong G \). As far as the author is aware no results of this kind have been published so far. Therefore first of all we show that if \( P \) is a non-abelian finite simple group and if the group \( G \) satisfies \( \Gamma_P \cong \Gamma_G \), then \( |P| = |G| \). Our notation for the names of the finite simple groups is the same as used in [6]. In what follows for the proof of Theorem 1 we need to prove some common property of the simple groups of Lie type. For the terminology and basic properties of these groups one can refer to [3].

**Lemma 4.** Let \( P \) be a simple group of Lie type defined over a field of \( q \) elements where \( q \) is a power of the prime \( p \). If \( S \) is a Sylow \( p \)-subgroup of \( P \), then the centralizer of \( S \) in \( P \) is non-trivial and is contained in \( S \).

**Proof.** Let \( B \) the Borel subgroup containing \( S \), so that \( B = N_P(S) \) and \( B = SH \), where \( H \) is a Cartan subgroup; \( C_P(S) \subset N_P(S) = B \). Also \( H \) is abelian of order prime to \( p \), so that \( C_P(S) = Z(S) \ Y \), where \( Y \) is a subgroup of \( H \). By ([3], p. 20) there is \( w \) in the Weyl group \( W \) which transforms every positive root to a negative root. Therefore the corresponding element \( n_w \) of \( P \) which is denoted again by \( w \) transforms \( S \) to its opposite, i.e. \( S_w = w^{-1} S w \) is generated by the root subgroups \( X_e \), where \( X_e \) is a generator of \( S \). By ([3], p. 68) we have \( P = (S, S_w) \). By ([3], p. 102) \( w^{-1} h(\chi) w = h(\chi^t) \) where \( \chi(s) = \chi(w^{-1}(s)) \) for all \( h \in H \) and for all \( s \in Y \). Therefore \( \chi(z) = \chi(1) \) for all \( z \in S \) and \( |S_w| = |S| \). Hence \( \sum s \in S \ Y = h^{-1} w h = h^{-1} \). From \( C_P(S) = Z(S) Y \) we see that each element of \( Y \) centralizes \( S \) and from \( w^{-1} y w = y^{-1} \) we obtain for all \( s \in S \):
\[
y^{-1} s w = y^{-1} w^{-1} s w = w^{-1} s w = w^{-1} s w y^{-1} = y^{-1} s w y^{-1}
\]
which implies that \( y^{-1} \) centralizes \( S_w \), hence each element of \( Y \) centralizes \( S_w \). Since \( Y \) centralizes both \( S \) and \( S_w \) this forces \( Y \) to be in the center of \( P \). But, \( P \) being a simple group, its center is trivial, and this forces \( Y \) to be trivial as well. This shows that \( C_P(S) = Z(S) \).

**Theorem 1.** Let \( P \) be a finite non-abelian simple group. If \( G \) is a group such that \( \Gamma_P \cong \Gamma_G \), then \( |P| = |G| \).

**Proof.** According to the classification of finite simple groups we have one of the following possibilities for \( P \): an alternating group \( A_n \), \( n \geq 5 \); a group of Lie type; or one of the 26 sporadic groups. Therefore we deal with the above possibilities of \( P \) separately. The case of \( P \cong A_n \), \( n \geq 5 \) has been dealt with in [1]. Therefore we deal with the other cases. But first note that \( Z(P) = 1 \) and from \( \Gamma_P \cong \Gamma_G \) it follows that \( |P| = |G| = |Z(G)| \), hence if we prove \( Z(G) = 1 \), then it follows that \( |P| = |G| \). We continue with the following cases:

If \( P \) is isomorphic to one of the 26 sporadic simple groups, by the remark made after Lemma 1, \( |Z(G)| \) is a divisor of \( |P| - 1 \) and \( |C_P(x)| - 1 \) for any \( x \in P \). One can observe (cf. [6]) the following common property of sporadic groups: If \( P \) is such a group, let \( p \) be the largest prime divisor of \( |P| \); then (i) \( p - 1 \) divides \( |P| \); (ii) \( p - 1 \) divides \( |P| \); there exist \( x \in P \) of order \( p \) such that \( C_P(x) = \langle x \rangle \). Now since \( |Z(G)| \) is a divisor of \( |P| - 1 \) we deduce \( |Z(G)| \) is a divisor of \( |P| - 1 \). From \( |Z(G)| \) is a divisor of \( |P| \), \( |Z(G)| \) is a divisor of \( |P| \), hence \( |Z(G)| \) is a divisor of \( |P| \), \( |P| \) proving \( |Z(G)| = 1 \).

In the following, we will consider all the simple groups of Lie type. Let \( P \) be a simple group of Lie type defined over a field with \( q \) elements where \( q \) is a power of the prime \( p \). By Lemma 4 if \( S \) is a Sylow \( p \)-subgroup of \( P \), then \( C_P(S) \) is non-trivial and is contained in \( S \). In particular if \( |C_P(S)| = q^t \), then \( t > 0 \) and \( q^t - 1 \) | \( |P| \). By Lemma 1 we have \( |Z(G)| \) is a divisor of \( |P| \), \( |C_P(S)| = q^t - 1 \), hence \( |Z(G)| \) is a divisor of \( |P| \). But \( |Z(G)| \) and \( |P| \) are relatively prime from which it follows that \( |Z(G)| = 1 \) and the Theorem is proved.

**Theorem 2.** Let \( P \) be a finite non-abelian simple group for which the Thompson’s conjecture holds. Then if \( G \) is a finite group such that \( \Gamma_P \cong \Gamma_G \), we have \( G \cong P \).

**Proof.** From \( \Gamma_P \cong \Gamma_G \) and using Theorem 1 we obtain \( Z(G) = 1 \). Now since \( G \) and \( P \) are centerless, by Lemma 3 it follows that \( N(G) = N(P) \). Since Thompson’s conjecture holds for \( P \) we deduce \( G \cong P \) and the Theorem is proved.

**Corollary 1.** Conjecture 2 holds for any of the following simple groups.

(a) A sporadic simple group,
(b) Any simple group with at least 3 prime graph components,
(c) A few of simple groups with 2 prime graph components such as: \( A_{p-1}(q)^2 \), \( A_{p-1}(q)^2 \), \( D_p(3) \), \( p \geq 5 \) a prime number not of the form \( 2^n + 1 \) \( L_{p+1}(2) \), \( E_6(q) \), \( E_7(q) \), \( D_n(q) \), \( n = 2^m + 4 \).
**Theorem 2.** Let $P$ be a simple alternating or a simple sporadic group or one of the following simple groups of Lie type: $A_n(q)$, $B_n(q)$, $C_n(q)$, $2D_n(q)$, $n \geq 4$, $F_4(q)$, $G_2(q)$. Let $A = \text{Aut}(P)$. If $G$ is a group such that $|\Gamma_A| \leq |\Gamma_C|$, then $|A| = |G|$.

**Proof.** By [4,5,7,8,10–14] Thompson’s conjecture holds for the above groups, hence the result follows by Theorem 2. ■

In [1], Theorem 3.16, p. 482, it is also proved that for $n > 2$, the graph isomorphism $\Gamma_{S_n} \cong \Gamma_{C_n}$ implies $|S_n| = |C_n|$, where $S_n$ denotes the symmetric group of degree $n$. We will show that if $P$ is a non-abelian simple group and $A = \text{Aut}(P)$ then $\Gamma_A \cong \Gamma_C$ will imply $|A| = |G|$ for certain classes of simple groups $P$. Of course if $n \neq 2, 6$, then $\text{Aut}(A_n) \cong S_n$, hence we obtain a generalization of the above result of [1].

**Theorem 3.** Let $P$ be a simple alternating or a simple sporadic group or one of the following simple groups of Lie type: $A_n(q)$, $B_n(q)$, $C_n(q)$, $2D_n(q)$, $n \geq 4$, $F_4(q)$, $G_2(q)$. Let $A = \text{Aut}(P)$. If $G$ is a group such that $|\Gamma_A| \leq |\Gamma_C|$, then $|A| = |G|$.

**Proof.** Since $Z(A)$ is trivial it is enough to show that $Z(G)$ is also trivial. If $n \geq 6$, $n \neq 6$, $n \geq 5$, the result follows by [1]. For $P \cong A_6 \cong S_6$, we have $A = \text{Aut}(P) \cong P \Gamma L_2(9)$. Where $P \Gamma L_2(9) = \Gamma Z/2(9)$ and $\Gamma L_2(9)$ is the group of all the semi-linear transformation of the vector space of dimension 2 over the Galois field with $q$ element. Using [6] we observe that $A = P \Gamma L_2(9)$ has elements $x$ and $y$ of orders 2 and 5, respectively such that $|C_A(x)| = 24$ and $|C_A(y)| = 5$. By Lemma 1 we must have $|Z(G)| | C_A(x)| = 1 = 23$ and $|Z(G)| | C_A(y)| = 1 = 4$, implying $Z(G) = 1$.

If $P$ is a sporadic group, then there exists $x \in P$ such that $C_A(x) = \langle x \rangle$ (Theorem 3.4 of [9]; more precisely:

(a) If $P$ is not $M^*, L^*$, let $p$ be the prime greatest divisor of $|P|$, then there exists $x \in P$ of order $p$ such that $C_A(x) = \langle x \rangle$ and $p - 1$ divides $|P|$. Moreover, by the proof of Theorem 3.4 of [9], $C_A(x) = C_A(x)$ and $p - 1$ divides $|P|$, therefore $p - 1$ divides $|A|$. Now Lemma 1 implies $Z(G) = 1$.

(b) If $P$ is the proof of Theorem 3.4 of [9] gives $x \in P$ of order 15 such that $C_A(x) = C_A(x) = \langle x \rangle$; since $15 - 1 = 14$ divides $|J_2|$ we are done.

(c) If $P$ is a sporadic group, the proof of Theorem 3.4 of [9] gives $x \in P$ of order 14 such that $C_A(x) = C_A(x) = \langle x \rangle$. But by our observation after the proof of Lemma 1, we must have $|Z(G)| | C_A(x)| = 1 = 13$ and $|Z(G)| | |A| - 1$. But $|A| = 2(M^*L = 2^3.3^3.3.7.11$ and $|A| - 1$ is prime to 13, hence $|Z(G)| = 1$ and we are done.

Finally, assume $P$ is a simple group of Lie type. By Theorem 3.1 page 107 of [9], if $P \neq A_3(2)$, $D_4(q)$, $q \leq 5$, there exists a torus $T$ of $A$ such that $C_A(T \cap P) = T$. But $T$ is abelian and $C_A(T \cap P) = T$, hence $C_A(T) = T$: moreover $|T|$ is given in Table III, page 108–109 of [9]. More precisely:

If $P = A_n(q)$, then $|T| = q^{n-1}$.

If $P = B_n(q)$, $C_n(q)$, $2D_n(q)$, $n \geq 4$, then $|T| = q^n + 1$.

If $P = F_4(q)$, then $|T| = q^6 + 1$.

If $P = G_2(q)$, then $|T| = q^2 + q + 1$.

In all the above cases clearly $|T| - 1$ divides $|P|$. Hence in every case $|T| - 1$ divides $|A|$. Therefore $|Z(G)| | |A|$ as well as $|Z(G)| | |A| - 1$. Hence $|Z(G)| = 1$ and the Theorem is proved.

**Corollary.** Let $P$ be a sporadic group, except $M^*L$ and $J_2$, or one of the following simple groups of Lie type: $A_{p-1}(q)$, $2D_p(3)$, $p \geq 5$ a prime number not of the form $2^m + 1$; $A_{p-1}(q)$, $p$ an odd prime number; $C_n(q)$, $n = 2^m \geq 2$, $q$ a power of 2; $2D_n(q)$, $n = 2^m \geq 4$; $F_4(q)$, $q$ odd; $G_2(q)$, $q \equiv \epsilon (\text{mod} 3)$, $\epsilon = \pm 1$, $q > 2$. Let $A = \text{Aut}(P)$. If $G$ is a centerless group such that $\Gamma_A \cong \Gamma_C$, then $A \cong G$.

**Proof.** Clearly $A$ is a centerless group. Since $G$ is assumed to be centerless we have $Z(G) = 1$, and from $\Gamma_A \cong \Gamma_C$ by Lemma 3 we obtain $N(A) = N(G)$. Since Thompson’s conjecture is true for all automorphism groups of the simple groups mentioned in the corollary, by Theorem 2 the result follows. ■

**Acknowledgements.**

This research was carried out at the mathematics and statistics department of the University of North Carolina at Charlotte during the year 2006–2007 while the author had a visiting position. The author would like to express his deep gratitude to the referee for his invaluable comments and suggestions which improved the paper. Also He would like to thank professor G. M. Seitz for his comments on the proof of Lemma 4.

**References.**