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Color-bounded hypergraphs, I: General results[☆]

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ABSTRACT

The concept of *color-bounded hypergraph* is introduced here. It is a hypergraph (set system) with vertex set X and edge set $\mathcal{E} = \{E_1, \dots, E_m\}$, where each edge E_i is associated with two integers s_i and t_i such that $1 \le s_i \le t_i \le |E_i|$. A vertex coloring $\varphi : X \to \mathbb{N}$ is considered to be feasible if the number of colors occurring in E_i satisfies $s_i \le |\varphi(E_i)| \le t_i$, for all $i \le m$.

Color-bounded hypergraphs generalize the concept of 'mixed hypergraphs' introduced by Voloshin [V. Voloshin, The mixed hypergraphs, Computer Science Journal of Moldova 1 (1993) 45–52], and a recent model studied by Drgas-Burchardt and Łazuka [E. Drgas-Burchardt, E. Łazuka, On chromatic polynomials of hypergraphs, Applied Mathematics Letters 20 (12) (2007) 1250–1254] where only lower bounds s_i were considered.

We discuss the similarities and differences between our general model and the more particular earlier ones. An important issue is the chromatic spectrum – strongly related to the chromatic polynomial – which is the sequence whose kth element is the number of allowed colorings with precisely k colors (disregarding color permutations). Problems concerning algorithmic complexity are also considered.

1. Introduction

In this paper we initiate the study of a coloring concept concerning finite set systems. The following model will be considered, as announced in [3] under a somewhat different terminology: a *color-bounded hypergraph* is a four-tuple

$$\mathcal{H} = \{X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t}\}$$

where $(X, \mathcal{E})$ is a hypergraph (set system) in the usual sense, with a *finite* vertex set X and edge set $\mathcal{E}$,

$$X = \{x_1, \dots, x_n\}, \qquad \mathcal{E} = \{E_1, \dots, E_m\}$$

with $X \ne \emptyset$ and $|E_i| \ge 2$ for all edges; whereas $\boldsymbol{s} : \mathcal{E} \to \mathbb{N}$ and $\boldsymbol{t} : \mathcal{E} \to \mathbb{N}$ are positive integer-valued functions on $\mathcal{E}$, such that

$$1 \le \boldsymbol{s}(E_i) \le \boldsymbol{t}(E_i) \le |E_i| \quad \forall\, 1 \le i \le m.$$

To simplify the notation, we shall write

$$s_i := \boldsymbol{s}(E_i), \qquad t_i := \boldsymbol{t}(E_i), \qquad s := \max_{1\le i\le m} s_i, \qquad t := \max_{1\le i\le m} t_i.$$

A *vertex coloring* $\varphi : X \to \mathbb{N}$ is considered to be feasible if the number of colors occurring in E_i satisfies

$$s_i \le |\varphi(E_i)| \le t_i \quad \text{for all } 1 \le i \le m.$$

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Our model has been inspired, on the one hand, by the recent work of Drgas-Burchardt and Łazuka [8], who considered the case of arbitrarily specified lower bounds s_i but without upper bounds (which is equivalent to writing $t_i = |E_i|$ for all $i \le m$); and, on the other hand, by the area of *mixed hypergraphs*, introduced by Voloshin [23,24]. In the latter, the $\mathcal{C}$-edges and $\mathcal{D}$-edges can be characterized as $(s_i, t_i) = (1, |E_i| - 1)$ and $(s_i, t_i) = (2, |E_i|)$, respectively. The so-called 'bi-edges' are then those with $(s_i, t_i) = (2, |E_i| - 1)$; hence, these notions have a natural and unified description in our model. Moreover, the traditional concept of 'proper vertex coloring' in the usual hypergraph-theoretic sense can be described with $(s_i, t_i) = (2, |E_i|)$ for all edges.

It turns out that color-bounded hypergraphs provide not just a common generalization of the earlier coloring concepts, but in fact a much stronger model is obtained. This is demonstrated in the results of Section 4 on the possible numbers of colors in a feasible coloring, and of Section 5 on unique colorability; and partly of Section 6, too, concerning 2-regular hypergraphs. Striking differences between color-bounded and mixed hypergraphs are explored further in our paper [4] where hypergraphs of particular structures are studied. Nevertheless, some earlier results on mixed hypergraph coloring remain valid for color-bounded hypergraphs, too. These include some facts about the chromatic polynomial, which is considered in Section 3.

It was shown in [15] that mixed hypergraphs can represent list colorings of graphs (a survey on the latter can be found in [18]). The recent paper [12] explores many further types of graph coloring that can be expressed in terms of mixed hypergraphs. Due to its strength, our new model may lead to further such relations and a deeper understanding of the known ones. We are grateful to the referee for comments in this direction and for inviting our attention to the paper [12].

Some of the results presented here have been announced in [3].

1.1. Notation and terminology

Here we recall some standard definitions from the literature, and extend them to color-bounded hypergraphs.

Mixed hypergraphs. A *mixed hypergraph* $\mathcal{H}$ has vertex set X and two types of edges, the so-called $\mathcal{C}$-*edges* and $\mathcal{D}$-*edges*. A vertex coloring of $\mathcal{H}$ is feasible if every $\mathcal{C}$-edge has two vertices with a **c**ommon color, and every $\mathcal{D}$-edge has two vertices with **d**istinct colors. A *bi-edge* is a vertex subset, which is a $\mathcal{C}$-edge and $\mathcal{D}$-edge at the same time. As has been told, the edge types of mixed hypergraphs can be expressed in a simple unified way in the color-bounded model:

- $\mathcal{C}$-edge: $s_i = 1$ and $t_i = |E_i| - 1$,
- $\mathcal{D}$-edge: $s_i = 2$ and $t_i = |E_i|$,
- bi-edge: $s_i = 2$ and $t_i = |E_i| - 1$.

Concerning mixed hypergraphs, we refer to the monograph [25] as a comprehensive source of information, and to the survey [21] that discusses further recent results and includes selected open problems.

Feasible set, chromatic spectrum. Each feasible vertex coloring φ of a color-bounded hypergraph $\mathcal{H}$ induces a *color partition* $X_1 \cup \cdots \cup X_k = X$, where the partition classes are the (inclusionwise maximal) monochromatic subsets of X under φ. For $k = 1, 2, \ldots, n$ the number of color partitions of X into *precisely* k nonempty classes will be denoted by r_k. The *chromatic spectrum* of $\mathcal{H}$ is the n-tuple

$$(r_1, r_2, \ldots, r_n).$$

We consider two chromatic spectra $(p_1, p_2, \ldots, p_j)$ and $(r_1, r_2, \ldots, r_k)$ to be equal if one is a prefix of the other, and all the remaining entries of the other sequence are zeros. That is, assuming $j \le k$, we require $p_i = r_i$ for all $1 \le i \le j$ and $r_i = 0$ for all $j < i \le k$. Adopting this point of view, we usually write chromatic spectrum in the form omitting all zeros from the end. If two hypergraphs have equal chromatic spectra, they are said to be *chromatically equivalent*.

The *feasible set* of $\mathcal{H}$, denoted by $\Phi(\mathcal{H})$, is the set of possible numbers of colors in a coloring:

$$\Phi(\mathcal{H}) = \{k \mid r_k \neq 0\}.$$

The hypergraph $\mathcal{H}$ is *colorable* if $\Phi(\mathcal{H}) \neq \emptyset$; and otherwise it is called *uncolorable*. A small uncolorable color-bounded hypergraph is obtained on three vertices if we prescribe $s_i = 2$ on the vertex pairs and $t_i = 2$ on the 3-element set.

Upper and lower chromatic number, gaps. Assuming that $\mathcal{H}$ is colorable, the largest and smallest possible numbers of colors in a feasible coloring are termed the upper chromatic number and lower chromatic number of $\mathcal{H}$, respectively. In notation,

$$\chi(\mathcal{H}) = \min \Phi(\mathcal{H}), \qquad \overline{\chi}(\mathcal{H}) = \max \Phi(\mathcal{H}).$$

If $\mathcal{H}$ is uncolorable, these values are defined to be $\chi(\mathcal{H}) = \overline{\chi}(\mathcal{H}) = 0$.

A *gap* in the chromatic spectrum of $\mathcal{H}$, or a gap of $\Phi(\mathcal{H})$, is an integer $k \notin \Phi(\mathcal{H})$ such that $\min \Phi(\mathcal{H}) < k < \max \Phi(\mathcal{H})$. If $\Phi(\mathcal{H})$ has no gaps, then the spectrum or feasible set is termed *continuous* or *gap-free*; otherwise it is said to be *broken*.

More generally, a *gap of size* ℓ in $\Phi(\mathcal{H})$ means ℓ consecutive integers that are all missing from $\Phi(\mathcal{H})$, larger than $\chi(\mathcal{H})$ and smaller than $\overline{\chi}(\mathcal{H})$.

Stirling numbers. As usual, for $n \geq k > 0$ we denote by $S(n, k)$ the *Stirling number of the second kind*, which is the number of partitions of n elements into precisely k nonempty sets. We also write $[\lambda]_k := \lambda(\lambda-1)\cdots(\lambda-k+1)$ to denote a 'falling power'. Let us recall the following fundamental equation:

$$\lambda^n = \sum_{k=1}^{n} S(n, k) \cdot [\lambda]_k. \tag{1}$$

Other models. There are many other kinds of restrictions that look interesting and are worth studying. In the most general one, termed *pattern hypergraph*, each *edge* is associated with a collection of feasible color partitions on its set of vertices, and a vertex coloring of the entire hypergraph is feasible if and only if the induced color partition on each edge also is so. This concept was studied by Dvořák et al. [9], their main result characterizes those pattern types which admit gaps in the feasible set.

We note already at this early point that there is a natural way to assign values s_i, t_i to the edges E_i of a pattern hypergraph, too. Namely, for each edge we can take the smallest and largest numbers of nonempty classes, over the feasible color partitions of E_i. Similarly, the parameters χ, $\overline{\chi}$, and the set Φ have their obvious meaning for every colorable pattern hypergraph. Having these definitions at hand, the main results of Section 3 remain valid for this most general model, too.

From the many further variants of interest, let us just mention one studied in the papers [5,6]. Instead of – or, in addition to – bounds (s_i, t_i) on the cardinality of largest multicolored subset of E_i, it is also natural to consider bounds (a_i, b_i) on its largest *monochromatic* subset. Conditions of this type lead to another general model, provably different from the present one, but admit a common solution with color-bounded hypergraphs when restricted to certain particular classes. The class of mixed hypergraphs lies in the intersection of the models defined in terms of (s_i, t_i) and of (a_i, b_i).

Hypergraphs. For convenience, let us recall a few standard definitions, too. A hypergraph is *r-uniform* if $|E_i| = r$ for all of its edges, and *d-regular* if each vertex is incident with precisely d edges. The *dual* of $\mathcal{H} = (X, \mathcal{E})$ is obtained when we represent each edge $E_i \in \mathcal{E}$ by a new vertex x_i^* and each vertex $x_j \in X$ by a new edge E_j^*, while keeping the structure of incidences unchanged; i.e., $x_i^* \in E_j^*$ if and only if $x_j \in E_i$. A hypergraph is *linear* if any two of its edges have at most one vertex in common.

2. Preliminary results

In this preliminary section, we make some simple observations. First, connections of color-bounded hypergraphs with mixed hypergraphs are considered. After that, we determine the condition of colorability, and the lower and upper chromatic number of complete uniform hypergraphs with uniform color-bounds.

2.1. Simple reductions

There are particular situations where a color-bounded hypergraph can be reduced to one with smaller edges, or even to a *mixed* hypergraph. Below we mention such cases, collected in one assertion of several parts. Throughout, $\mathcal{C}$-edges and $\mathcal{D}$-edges are meant in the sense of mixed hypergraphs.

Remark 1. Let $\mathcal{H}$ be a color-bounded hypergraph. If no structural conditions are imposed, then the following reductions can be applied.

(a) If $s_i = t_i = |E_i|$, then one may replace E_i with all of its 2-element subsets as $\mathcal{D}$-edges, because every feasible coloring of E_i assigns mutually distinct colors to its vertices.

(b) If $s_i = 1$ and $t_i < |E_i| - 1$, then one may replace E_i with all of its $(t_i + 1)$-element subsets as $\mathcal{C}$-edges, because the original E_i is colored in a feasible way if and only if no $t_i + 1$ of its vertices are totally multicolored.

(c) More generally, if $t_i < |E_i|-1$, then one may modify the bounds (s_i, t_i) of E_i to $(s_i, |E_i|)$ and insert all the (t_i+1)-element subsets of E_i, with bounds $(1, t_i)$.

(d) In parts (b) and (c) it suffices to take all the $(t_i + 1)$-subsets incident with one arbitrarily chosen vertex of E_i, because in any coloring, each vertex of E_i can be completed to a largest polychromatic subset of the edge.

There is a consequence of these reductions that we find worth mentioning separately:

Corollary 1. *If $\mathcal{H}$ does not have any edges such that $|E_i| > 3$ and $s_i > 2$ hold simultaneously, then there exists a <u>mixed</u> hypergraph on the same vertex set, with precisely the same feasible colorings as $\mathcal{H}$.*

Proof. If $s_i = 1$, then reduction (b) replaces E_i with $\mathcal{C}$-edges of cardinality $t_i + 1$. The situation is similar if $s_i = 2$, except that E_i itself becomes a $\mathcal{D}$-edge under reduction (c) in this case. Finally, if $s_i \geq 3$, then $|E_i| \leq 3$ by assumption, and so the condition $s_i \leq t_i \leq |E_i|$ implies $s_i = t_i = |E_i| = 3$. Hence, the edge can be replaced with three 2-element $\mathcal{D}$-edges, applying reduction (a). □

It is important to note that the reductions listed in Remark 1 may lead out from a particular class of hypergraphs. For example, as shown in [4], already the 3-uniform color-bounded hypertrees (hypergraphs representable with subtrees of a tree) differ substantially from mixed hypertrees.

Let us note further that the edges with $2 < s_i < |E_i| = t_i$ usually *cannot be replaced* with any combinations of $\mathcal{D}$-edges. This is the reason for the fact (demonstrated in Section 4.1) that assuming a fixed number of vertices, some chromatic spectra are possible for color-bounded hypergraphs but cannot occur in mixed hypergraphs.

2.2. Complete uniform hypergraphs

Here we consider the following particular example.

Definition 1. Let $n \geq r \geq t \geq s \geq 1$ be integers. The *complete uniform color-bounded hypergraph* $\mathcal{K}_n(r; s, t)$ is the 4-tuple $(X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t})$ such that X consists of n vertices, and the edge set $\mathcal{E}$ contains all the r-element subsets of X with associated constant bounds $s_i = s$ and $t_i = t$.

Proposition 1. *The hypergraph $\mathcal{K}_n(r; s, t)$ is colorable if and only if*

(i) $t = r$, *or*
(ii) $s = 1$, *or*
(iii) $n \leq r - 1 + \left\lfloor \frac{r-1}{s-1} \right\rfloor (t - s + 1)$.

Proof. For $t = r$, the hypergraph is evidently colorable with n colors; and if $s = 1$, then all vertices can get the same color. This shows the sufficiency of the first two cases.

Suppose next that $\mathcal{K}_n(r; s, t)$ is colorable and that $t \neq r$, $s \neq 1$ hold. Assuming a coloring with at least $t + 1$ colors, this would yield some edges colored with more than t colors (since $t < r$). This is forbidden, hence there appear at most t color classes.

If there were $s - 1$ color classes whose union consists of at least r vertices, they would determine an edge with fewer than s colors. Therefore, even the $s - 1$ largest color classes can have at most $r - 1$ vertices in total. The remaining color classes are of size at most $\left\lfloor \frac{r-1}{s-1} \right\rfloor$ each, and their number is at most $t - (s - 1)$. Summing up, if $\mathcal{K}_n(r; s, t)$ is colorable, then

$$n \leq r - 1 + (t - s + 1) \left\lfloor \frac{r-1}{s-1} \right\rfloor$$

necessarily holds.

From now on, we assume that $t \neq r$, $s \neq 1$, and $n \leq r - 1 + (t - s + 1) \left\lfloor \frac{r-1}{s-1} \right\rfloor$. We need to prove that the hypergraph is colorable. Let us write $(r - 1)$ in the form $r - 1 = a(s - 1) + m$, where a and $m < s - 1$ are nonnegative integers. Applying that $a = \left\lfloor \frac{r-1}{s-1} \right\rfloor$ and $m = r - 1 - (s - 1) \left\lfloor \frac{r-1}{s-1} \right\rfloor$, we get the assumption rewritten in the form $n \leq at + m$. Hence, for $n' = at + m$ vertices there exists a vertex partition $X = X_1 \cup \cdots \cup X_t$ such that X_i has cardinality $a + 1$ for $1 \leq i \leq m$, and cardinality a for $m + 1 \leq i \leq t$. Moreover, a partition with t nonempty classes can be obtained for n vertices, too, by removing $n' - n$ vertices in any way so that at most $|X_i| - 1$ are deleted from each X_i.

It remains to observe that this vertex partition colors $\mathcal{K}_n(r; s, t)$ properly. Indeed, the union of the $s - 1$ largest classes has at most $a(s - 1) + m = r - 1$ elements, therefore every edge contains at least s colors. Moreover, there are just t colors, so that the upper bound is respected, too. This completes the proof. □

Remark 2. Along with the proof, we have also obtained that the upper chromatic number of a colorable color-bounded hypergraph $\mathcal{K}_n(r; s, t)$ is

- t, if $t \neq r$, and
- n, if $t = r$.

More generally, one may consider the complete (p, q)-uniform (s, t)-hypergraphs $\mathcal{K}_n(p, q; s, t)$, where all p-subsets of the underlying n-set have color-bounds (s, p) and all the q-subsets have bounds $(1, t)$. Here we assume $q \geq t \geq s$ and $p \geq s$. The case $q = t$ has been already included above, because it means no real upper bound and hence it is equivalent to $p = q = t$. On the other hand, if $q > t$, it is readily seen that every q-element subset contains at most t colors if and only if so does every $(t + 1)$-element subset. Consequently, $\overline{\chi}(\mathcal{K}_n(p, q; s, t)) = \overline{\chi}(\mathcal{K}_n(p, t + 1; s, t)) \leq t$ holds, and these two hypergraphs are chromatically equivalent. (In particular, both are colorable or both are uncolorable.) For this reason, the value of q is irrelevant and, along the lines of the proof above, we obtain:

Proposition 2. *The color-bounded hypergraph $\mathcal{K}_n(p, q; s, t)$ is colorable if and only if*

(i) $t = q$, *or*
(ii) $s = 1$, *or*
(iii) $n \leq p - 1 + \left\lfloor \frac{p-1}{s-1} \right\rfloor (t - s + 1)$. □

Putting $s = 2$ and $t = q - 1$, the following simple inequality (first observed in [19]) occurs as a particular case.

Corollary 2. *The complete mixed hypergraph on n vertices, with all p-element subsets as $\mathcal{D}$-edges and all q-element subsets as $\mathcal{C}$-edges is colorable if and only if $n \leq (p-1)(q-1)$.*

Already this rather restricted structure indicates that color-bounded hypergraphs are much more complex than mixed ones: while the latter would admit a one-line proof, the former needs some work.

We have already observed that the $s-1$ largest color classes can have at most $r-1$ vertices. As regards the lower chromatic number, this leads – in the same way as above – to:

Proposition 3. *For every $r \geq s \geq 2$ we have*

$$\chi(\mathcal{K}_n(r; s, r)) = s - 1 + \left\lceil \frac{n-r+1}{\left\lfloor \frac{r-1}{s-1} \right\rfloor} \right\rceil$$

and, if t is not smaller than this lower bound and $q > t$, or $t \geq s$ and $q = t$, then

$$\chi(\mathcal{K}_n(r, q; s, t)) = s - 1 + \left\lceil \frac{n-r+1}{\left\lfloor \frac{r-1}{s-1} \right\rfloor} \right\rceil. \quad \square$$

3. Chromatic polynomials

Most of the results proved in this section are valid in the general model of *pattern hypergraphs*, therefore we do not restrict ourselves to color-bounded ones here.

Let us extend the standard notion of chromatic polynomial $P(\mathcal{H}, \lambda)$ for this widest class of hypergraphs. By definition, the value of $P(\mathcal{H}, \lambda)$ for a positive integer $\lambda = k$ is the number of feasible colorings with at most k colors; that is, the number of mappings $\varphi : X \to \{1, \dots, k\}$ whose color partition $(\varphi^{-1}(1), \dots, \varphi^{-1}(k))$ induced on X is feasible for $\mathcal{H}$. (Here some of the sets $\varphi^{-1}(i)$ are allowed to be empty.)

Let us emphasize some substantial differences between r_k and the value of $P(\mathcal{H}, \lambda)$ at $\lambda = k$. The former does not count permutations of colors to be distinct, while the latter does; moreover, the former only takes into consideration the colorings with *precisely* k colors.

It is clear that if $\mathcal{H}$ is uncolorable, then its chromatic polynomial is $P(\mathcal{H}, \lambda) \equiv 0$, the identically zero function. For this reason, we assume throughout this section that $\mathcal{H}$ is *colorable*. (It does not mean that hypergraphs derived from $\mathcal{H}$, too, will be assumed to be colorable.) The chromatic polynomial will be written in the form

$$P(\mathcal{H}, \lambda) = \sum_{k \geq 0} a_k \lambda^k.$$

After recalling some known facts from the literature, we shall first observe that this is a legal notation, as the number of colorings with at most λ colors is indeed a polynomial of λ. For an easier formulation of some assertions, we shall use the notation

$$\mathcal{E}^{(r)} = \{E_i \in \mathcal{E} \mid |E_i| = r\}$$

and $m_r = |\mathcal{E}^{(r)}|$, for all $r \geq 2$. In particular, $\mathcal{E}^{(2)}$ is the graph formed by the 2-element edges of $\mathcal{H}$.

Known facts

1. If $s_i = 2$ and $t_i = |E_i|$ for all i (that is, hypergraph coloring in the usual sense), then
   (a) $P(\mathcal{H}, \lambda)$ is a polynomial of order $n = |X|$,
   (b) $a_n = 1$ and $a_0 = 0$,
   (c) $\sum_{k \geq 0} a_k = 0$, if there exists at least one edge,
   (d) $a_{n-1} = -m_2$, where $m_2 = |\mathcal{E}^{(2)}|$, [7]
   (e) $a_{n-2} = \binom{m_2}{2} - m_3 - t_2$, where t_2 is the number of triangles in the graph $\mathcal{E}^{(2)}$, provided that no $E_i \in \mathcal{E}^{(2)}$ is a subset of any $E_j \in \mathcal{E}^{(3)}$. This follows from a general equation of [7], although it is not stated there explicitly.
2. Facts (1a)–(1e) can be extended to hypergraphs such that

   $$s_i \geq 2 \quad \text{and} \quad t_i = |E_i| \quad \forall\, 1 \leq i \leq m,$$

   with some modifications in (1d) and (1e). In the present case, each edge E_i of $\mathcal{H}$ having $s_i = |E_i|$ is replaced with its 2-element subsets as edges, and $\mathcal{E}^{(r)}$ for $r \geq 2$ is meant to be the collection of r-element edges *after* this operation. Then a_{n-1} is determined as in (1d), and the formula of (1e) for a_{n-2} is proved to be valid under the further condition $s_i \leq |E_i| - 2$ for all $E_i \notin \mathcal{E}^{(2)}$, which also implies that $m_3 = 0$ holds [8].
3. If $\mathcal{H}$ is a mixed hypergraph, then $P(\mathcal{H}, \lambda) = \sum_{k=\chi}^{\overline{\chi}} r_k [\lambda]_k$, [24].

Turning now to color-bounded and, more generally, pattern hypergraphs, we begin with the following very general observation.

Proposition 4. *Let $\mathcal{H} = (X, \mathcal{E})$ be a hypergraph, and $\mathcal{P}$ a given set of partitions of X, whose members are the allowed color partitions for $\mathcal{E}$. For $k = 1, \dots, |X|$ denote by r_k the number of those partitions in $\mathcal{P}$ which have precisely k nonempty classes. Then the number $P(\mathcal{H}, \lambda)$ of allowed colorings of $\mathcal{H}$ with at most λ colors is a polynomial of λ, and it can be written as $P(\mathcal{H}, \lambda) = \sum_{k>0} r_k[\lambda]_k$.*

Proof. A unique ordering can be assigned to the classes of each partition in $\mathcal{P}$, according to their vertex of smallest subscript in the order $x_1, \dots, x_n$. Enumerating along this ordering, if $P \in \mathcal{P}$ has precisely k classes, then there are exactly $[\lambda]_k$ ways to assign distinct colors to its classes. □

Corollary 3. *For every $\mathcal{H}$, $P(\mathcal{H}, \lambda)$ is a polynomial of order $\overline{\chi} = \overline{\chi}(\mathcal{H})$ whose leading coefficient equals $r_{\overline{\chi}}$, that is the number of color partitions with the maximum number of colors.*

Corollary 4. *Two hypergraphs are chromatically equivalent if and only if they have the same chromatic polynomial.*

The following fact has been proved in [8] for the hypergraphs where $t_i = |E_i|$ holds for all edges.

Corollary 5. *Every chromatic polynomial can be written as the sum of chromatic polynomials of graphs, without any negative coefficients.*

Proof. It suffices to observe that $[\lambda]_k$ is the chromatic polynomial of the complete graph on k vertices, and that none of the $[\lambda]_k$ can have a negative coefficient in $P(\mathcal{H}, \lambda)$. □

Remark 3. By definition, all integers i with $1 \le i < \chi(\mathcal{H})$ are roots of $P(\mathcal{H}, \lambda)$.

Corollary 6. *If $\mathcal{H}$ is colorable and $\chi(\mathcal{H}) = \overline{\chi}(\mathcal{H})$, then the set of roots of $P(\mathcal{H}, \lambda)$ is*

$$\{i \in \mathbb{N} \cup \{0\} \mid i \le \chi(\mathcal{H}) - 1\}$$

and each root has multiplicity one; and vice versa, if the roots of $P(\mathcal{H}, \lambda)$ are $0, 1, \dots, k-1$ and each of them has multiplicity one, then $\mathcal{H}$ is colorable and $\chi(\mathcal{H}) = \overline{\chi}(\mathcal{H})$.

Proof. If $\chi(\mathcal{H}) = \overline{\chi}(\mathcal{H}) = k > 0$, then $P(\mathcal{H}, \lambda) = r_k \cdot \lambda(\lambda-1)\cdots(\lambda-k+1)$ by Proposition 4. Conversely, the given set of roots (without multiple roots) implies that $P(\mathcal{H}, \lambda)$ is of the form $r_k \cdot \lambda(\lambda-1)\cdots(\lambda-k+1)$, from which we obtain that $r_i = 0$ for every $1 \le i < k$ and also for $i > k$, so that k is the unique possible number of colors. □

For the explicit computation of $P(\mathcal{H}, \lambda)$, let us introduce the following notation:

- $\mathcal{H} + C_{i,j}$ — the hypergraph obtained from $\mathcal{H}$ by inserting the $\mathcal{C}$-edge $\{x_i, x_j\}$ with $s = t = 1$
- $\mathcal{H} + D_{i,j}$ — the hypergraph obtained from $\mathcal{H}$ by inserting the $\mathcal{D}$-edge $\{x_i, x_j\}$ with $s = t = 2$.

In order to compute $P(\mathcal{H}, \lambda)$, the following recursion may be useful.

Proposition 5. *If $\{x_i, x_j\}$ is neither a $\mathcal{C}$-edge nor a $\mathcal{D}$-edge, then*

$$P(\mathcal{H}, \lambda) = P(\mathcal{H} + C_{i,j}, \lambda) + P(\mathcal{H} + D_{i,j}, \lambda).$$

Proof. The first and second terms on the right-hand side count the numbers of those colorings of $\mathcal{H}$ with at most λ colors in which x_i and x_j get the same color or distinct colors, respectively. □

In the same way, one can also observe that

$$\overline{\chi}(\mathcal{H}) = \max(\overline{\chi}(\mathcal{H} + C_{i,j}), \overline{\chi}(\mathcal{H} + D_{i,j}))$$

for every $\mathcal{H}$. The analogous formula

$$\chi(\mathcal{H}) = \min(\chi(\mathcal{H} + C_{i,j}), \chi(\mathcal{H} + D_{i,j}))$$

is valid under the slight restriction that both hypergraphs $\mathcal{H} + C_{i,j}$ and $\mathcal{H} + D_{i,j}$ on the right-hand side are colorable; otherwise the uncolorable one has to be omitted and the lower chromatic number is equal to that of the colorable modified hypergraph.

Remark 4. This recursion leads to an analogue of the 'Splitting–Contraction Algorithm' developed by Voloshin for mixed hypergraphs [23].

Proposition 6. *If $\max_{1 \le i \le m} s_i \ge 2$, then the sum of the coefficients of $P(\mathcal{H}, \lambda)$ is equal to zero; and if $s_1 = \cdots = s_m = 1$, then the sum is equal to 1.*

Proof. The value $P(\mathcal{H}, 1)$ is equal to the sum of coefficients, and at the same time it counts the number of allowed colorings with just one color. The latter obviously is either 0 or 1. $\square$

Proposition 7. *The constant term in $P(\mathcal{H}, \lambda)$ is equal to zero.*

Proof. According to Proposition 4, each term in the formula for $P(\mathcal{H}, \lambda)$ is divisible by λ. $\square$

The following example shows that facts (1d) and (1e) do not remain valid if some edges have $t_i < |E_i|$.

Example 1. Let $\mathcal{H}$ have the only one edge, $E_1 = X$, with $s_1 = 1$ and $t_1 = n - 1$. Then

$$P(\mathcal{H}, \lambda) = \lambda^n - \prod_{i=0}^{n-1}(\lambda - i) = \lambda^{n-1}\sum_{i=0}^{n-1} i - \lambda^{n-2}\sum_{i=1}^{n-1}\sum_{j=0}^{i-1} ij + O(\lambda^{n-3})$$

$$= \binom{n}{2}\lambda^{n-1} - \left(\frac{1}{2}\binom{n}{2}^2 - \frac{1}{8}\binom{2n}{3}\right)\lambda^{n-2} + O(\lambda^{n-3})$$

because

$$\sum_{j<i\le n-1} ij = \sum_{i\le n-1} i\binom{i}{2} = \frac{1}{2}\sum_{i\le n-1} i^3 - \frac{1}{2}\sum_{i\le n-1} i^2$$

$$= \frac{1}{8}n^2(n-1)^2 - \frac{1}{12}n(n-1)(2n-1). \quad \square$$

The following result completely characterizes the chromatic polynomials under the assumption $P(1) = 0$.

Theorem 1. *Let $P(\lambda) = \sum_{k=0}^{\ell} a_k\lambda^k \not\equiv 0$ be a polynomial such that $P(1) = 0$, i.e. $\sum_{k=0}^{\ell} a_k = 0$. The following properties are equivalent.*

1. *$P(\lambda)$ is the chromatic polynomial of a color-bounded hypergraph.*
2. *$P(\lambda)$ is the chromatic polynomial of a mixed hypergraph.*
3. *$P(\lambda)$ is the chromatic polynomial of a pattern hypergraph.*
4. *$P(\lambda)$ satisfies all of the following conditions.*
   (i) *All coefficients a_k of $P(\lambda)$ are integers.*
   (ii) *The leading coefficient a_ℓ is positive.*
   (iii) *The constant term a_0 is zero.*
   (iv) *For every positive integer $j \le \ell$, the inequality*
   $$\sum_{k=j}^{\ell} a_k \cdot S(k, j) \ge 0$$
   is valid.

Proof. The condition $P(1) = 0$ means that if some hypergraph $\mathcal{H}$ has $P(\lambda)$ as its chromatic polynomial, then at least one edge $E_i \in \mathcal{H}$ has $s_i \ge 2$. In particular, if $\mathcal{H}$ is such a *mixed* hypergraph, then it contains at least one $\mathcal{D}$-edge. Due to the construction in [11], prescribing the numbers of k-colorings in an arbitrary way for $k = 2, \dots, \overline{\chi}$, there exists a mixed hypergraph with this given chromatic spectrum (provided that $r_1 = 0$, which is now the case). Since this already includes all possible variations, the equivalence of properties 1–3 follows.

In order to prove the equivalence of property 4 with 1 through 3, let us introduce the notation $r_j = \sum_{k=j}^{\ell} a_k \cdot S(k, j)$ for $j = 1, \dots, \ell$. On applying Eq. (1), we rewrite

$$P(\lambda) = \sum_{k=1}^{\ell} a_k\lambda^k = \sum_{k=1}^{\ell} a_k \sum_{j=1}^{k} S(k, j) \cdot [\lambda]_j = \sum_{j=1}^{\ell}[\lambda]_j \sum_{k=j}^{\ell} a_k \cdot S(k, j) = \sum_{j=1}^{\ell} r_j \cdot [\lambda]_j. \tag{2}$$

It is clear that $r_\ell = a_\ell$. Moreover, all the a_k are integers if and only if all the r_k are integers. Indeed, the *largest* subscript j for which r_j is not an integer would yield that a_j is not an integer either; and vice versa.

Suppose now that $P(\lambda)$ is equal to $P(\mathcal{H}, \lambda)$ for some mixed hypergraph $\mathcal{H}$. By Proposition 4, the meaning of r_j is the number of color partitions into precisely j nonempty vertex classes. Consequently, $r_j \ge 0$ must hold for all j, which implies (iv). Also, the condition $a_0 = 0$ follows by Proposition 7.

Conversely, suppose that the conditions 3(i)–3(iv) are valid for $P(\lambda)$. Applying Eq. (2) we obtain a sequence $(r_j)_{j>0}$ of nonnegative integers. We observe now that $r_1 = 0$ necessarily holds; this follows from the assumption $P(1) = 0$, because the right-hand side of (2) yields $P(1) = r_1$. Thus, due to [11], there exists a mixed hypergraph $\mathcal{H}$ whose chromatic spectrum is $(r_j)_{j>0}$. This completes the proof of the theorem. $\square$

Corollary 7. *For every color-bounded hypergraph $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t})$ there exists a mixed hypergraph with the same chromatic polynomial and with the same chromatic spectrum.*

Proof. For the case $\max_{1\le i\le m} s_i \ge 2$, this is just the equivalence of properties 1 and 2 above. In the other case, if $s_1 = \cdots = s_m = 1$, we replace each E_i with all its $(t_i + 1)$-element subsets as $\mathcal{C}$-edges, as described in reduction (b) of Remark 1. The color partitions of this 'mixed' $\mathcal{C}$-hypergraph are precisely those of $\mathcal{H}$. □

We shall see later that this property does not remain valid anymore when the hypergraph under consideration is assumed to be uniform. It is not valid either in the class of hypergraphs generated by the conditions (a_i, b_i) on the size of monochromatic subsets of edges (mentioned in the paragraph "Other models" of Section 1), as proved in [5].

Concerning pattern hypergraphs, 1-colorability does not imply that the chromatic spectrum is gap-free. This makes it possible to characterize the chromatic polynomials completely.

Proposition 8. *The polynomial $P(\lambda) = \sum_{i=0}^{\ell} a_k\lambda^k$ is the chromatic polynomial of some pattern hypergraph if and only if all of the following conditions hold.*

(i) *All coefficients a_k of $P(\lambda)$ are integers.*
(ii) *The leading coefficient a_ℓ is positive.*
(iii) *The constant term a_0 is zero.*
(iv) *For every positive integer $j \le \ell$, the inequality*

$$\sum_{k=j}^{\ell} a_k \cdot S(k, j) \ge 0$$

is valid.
(v) $\sum_{i=0}^{\ell} a_k = 0$ *or* 1.

Proof. The number of 1-colorings, that is the value $P(1) = \sum_{i=0}^{\ell} a_k$, is either 0 or 1, yielding the necessity of (v). For the case of $P(1) = 0$, the proof of Theorem 1 verifies the assertion for pattern hypergraphs as well. Moreover, the argument given there also implies that the conditions (i)–(iv) are necessary for the case $P(1) = 1$, too.

As regards $P(1) = 1$, the crucial point is that for the 1-colorable pattern hypergraphs (equivalently when $r_1 = 1$), the other entries of the chromatic spectrum can be arbitrarily prescribed nonnegative integers. To prove this, consider the nonnegative integers $r_2, \ldots, r_\ell$ and let a sufficiently large n be chosen such that $S(n, k) \ge r_k$ holds for each $k = 2, \ldots, \ell$. Create a hypergraph $\mathcal{H} = (X, \{X\})$ on n vertices. By the choice of n, we can prescribe for the hyperedge X exactly r_k feasible k-partitions for every $1 \le k \le \ell$, which results in a pattern hypergraph $\mathcal{H}$ with chromatic spectrum $(r_1 = 1, r_2, \ldots, r_\ell)$.

From this point the proof can be completed as in the case of Theorem 1. □

4. Gaps in the chromatic spectrum

In this section we study the feasible sets of color-bounded hypergraphs. The main results are the determination of largest gaps in hypergraphs with a given number of vertices, and the characterization of feasible sets of *uniform* color-bounded hypergraphs (Theorem 3).

4.1. Largest gap

It was proved in [10] that *any* finite set of integers greater than 1 is the feasible set of a mixed hypergraph, but the smallest number of vertices realizing a given set is not known. For gaps of size k, however, a construction on $2k+4$ vertices was given in [10]. This $2k + 4$ is the smallest possible order, which follows from another result of the same paper, although it was not stated explicitly until [5]. On the other hand, for color-bounded hypergraphs the minimum is much smaller, as shown by the following tight result.

Theorem 2. *If a color-bounded hypergraph has a gap of size $k \ge 1$ in its chromatic spectrum, then it has at least $k + 5$ vertices. Moreover, this bound is sharp; that is, for every positive integer k there exists a hypergraph $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t})$ with $|X| = k + 5$ vertices whose chromatic spectrum has a gap of size k.*

Before the proof, we make two simple observations.

Proposition 9. *Every* 1*-colorable color-bounded hypergraph has a gap-free chromatic spectrum.*

Proof. If all the vertices can be colored with the same color, every s_i equals 1. Then assuming an ℓ-coloring ($\ell \ge 2$) of $\mathcal{H}$, the union of any two color classes yields a proper $(\ell - 1)$-coloring, consequently no gaps can occur. □

Proposition 10. *Every n-colorable color-bounded hypergraph on n vertices has a gap-free chromatic spectrum.*

Proof. If the vertices can have mutually different colors, then for every edge E_i the equality $t_i = |E_i|$ holds, namely there is no upper restriction on the number of colors. In this case consider an arbitrary proper ℓ-coloring of $\mathcal{H}$. If $\ell < n$, there exists a color class with at least two vertices, and by splitting it into two parts we obtain a proper $(\ell + 1)$-coloring. □

Proof of Theorem 2. According to Propositions 9 and 10, there cannot occur any gaps of size k on fewer than $k+4$ vertices. Moreover, $n = k+4$ vertices would mean $\Phi(\mathcal{H}) = \{2, k+3\}$.

Assume for a contradiction that such a hypergraph exists. Note that $k + 4 \geq 5$ holds. Due to the 2-colorability, the inequality $s_i \leq 2$ is valid for every E_i. Considering a coloring φ with precisely $k+3 = n-1$ colors, there exist at least three single color classes, say $\{x_1\}, \{x_2\}, \{x_3\}$. The hypergraph has no $(k+2)$-coloring, hence the contraction of classes $\{x_1\}$ and $\{x_2\}$ results in an edge E_i colored with fewer than s_i colors. Taking into consideration that $s_i \leq 2$, this means E_i would become monochromatic by the contraction, thus $E_i = \{x_1, x_2\}$ with bounds $(s_i, t_i) = (2, 2)$. Similarly, there are $(2, 2)$-edges also for $\{x_2, x_3\}$ and $\{x_3, x_1\}$, hence every proper coloring assigns three different colors to the vertices x_1, x_2, x_3. This contradicts the supposed 2-colorability of $\mathcal{H}$, consequently there appears no gap of size k on fewer than $k+5$ vertices. □

To show that this bound is sharp, we present a hypergraph $\mathcal{H}_k = (X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t})$ with feasible set $\{3, k+4\}$ on $k+5$ vertices, for every positive integer k.

Example 2. Consider the hypergraph $\mathcal{H}_k$ with vertex set $X = \{x_1, x_2, y_1, y_2, \ldots, y_{k+3}\}$ and with edge set $\mathcal{E} = \{\{x_1, x_2, y_i, y_j\} \mid 1 \leq i < j \leq k+3\}$, where each of the edges has bounds $(s, t) = (3, 3)$.

If a proper coloring φ assigns different colors to x_1 and x_2, there appears exactly one more color on the vertices $y_1, y_2, \ldots, y_{k+3}$. It is clearly realizable with color classes $\{x_1\}$, $\{x_2\}$, and $\{y_1, y_2, \ldots, y_{k+3}\}$, hence $\mathcal{H}$ can be colored only with precisely three colors in this case.

On the other hand, if φ assigns the same color to x_1 and x_2, the $(3, 3)$-edges are properly colored if and only if any two distinct vertices y_i and y_j have colors different from each other and from $\varphi(x_1)$, too. Thus, in this case we obtain a proper $(k+4)$-coloring.

Since there are no other cases, the feasible set is $\{3, k+4\}$; that is, for every positive k the hypergraph $\mathcal{H}_k$ has a gap of size k on $k+5$ vertices. □

Remark 5. In pattern hypergraphs, the minimum order for a gap of size k is equal to $k+2$. This bound is attained by the hypergraph $\mathcal{H} = (X, \{X\})$ with $|X| = k+2$ and $r_1 = r_{k+2} = 1, r_2 = \cdots = r_{k+1} = 0$; i.e., where the vertex set is required to be either monochromatic or completely multicolored.

4.2. Feasible sets of uniform hypergraphs

As was shown in Section 3, the classes of mixed and color-bounded hypergraphs generate the same set of chromatic polynomials; moreover, the feasible sets are the same in any case. In hypergraphs with restricted structures, however, there appear substantial differences. We consider the following three types, the third one being the main issue of this subsection.

- Hypertrees

  These are hypergraphs that can be represented as subtrees of a tree, in the way that there exists a tree graph on the vertex set of the hypergraph, such that each hyperedge induces a subtree in it. The chromatic spectrum of mixed hypertrees is gap-free and their lower chromatic number is at most two [13]. Hence, their feasible sets determine precisely the intervals of the form $[1, \ldots, k]$ or $[2, \ldots, \ell]$, where $k \geq 1$ and $\ell \geq 2$.

  On the other hand, we have proven in the paper [4] that color-bounded hypertrees can have arbitrary large gaps in the chromatic spectrum, and any positive integer can occur as a lower chromatic number. Any set S of integers with $\min S \geq 3$ can be obtained as the feasible set of some color-bounded hypertree; if the lower chromatic number equals 1 or 2, however, then the chromatic spectrum is necessarily gap-free.
- Interval hypergraphs

  Mixed interval hypergraphs – hypertrees whose underlying tree is a path – have a gap-free chromatic spectrum with lower chromatic number 1 or 2 [10], whilst in the color-bounded case the spectrum still remains gap-free but the lower chromatic number can be any positive integer [4].
- r-uniform hypergraphs

  The 2-uniform mixed and color-bounded hypergraphs are practically the same: the $(1, 2)$-edges have no effect on coloring, and after their deletion we get a 2-uniform mixed hypergraph (i.e., a 'mixed graph').

  The larger cases, where $r \geq 3$, will be treated in this subsection and essential differences will be demonstrated in comparison with mixed hypergraphs.

We now consider the feasible sets belonging to r-uniform hypergraphs, for different values of r.

For the following observations the classes of possible feasible sets will be denoted by $\mathcal{F}_m$ and $\mathcal{F}_c$ regarding mixed and color-bounded hypergraphs, respectively. When we refer only to feasible sets of r-uniform hypergraphs, upper indices will be used: $\mathcal{F}_m^r$ and $\mathcal{F}_c^r$. Since mixed and color-bounded hypergraphs generate the same chromatic polynomials, the sets $\mathcal{F}_m$ and $\mathcal{F}_c$ are equal.

For r-uniform *mixed* hypergraphs, the feasible sets have been characterized in [2] as follows:

The set S of positive integers is a feasible set of an r-uniform mixed hypergraph ($r \geq 3$) if and only if

(i) $\min(S) \geq r$, or

(ii) $2 \le \min(S) \le r-1$ and S contains all integers between $\min(S)$ and $r-1$, or
(iii) $\min(S) = 1$ and $S = \{1, \dots, k\}$ for some natural number $k \ge r-1$.

Accordingly, the 3-uniform mixed hypergraphs generate all possible feasible sets from $\mathcal{F}_m$, except the set $\{1\}$. Increasing the value of r, for all integers $3 \le r_1 < r_2$ the inclusion $\mathcal{F}_m^{r_1} \supsetneqq \mathcal{F}_m^{r_2}$ holds. Thus, the classes $\mathcal{F}_m^r$ of possible feasible sets determine a strictly decreasing infinite set-sequence: $\mathcal{F}_m^3 \supsetneqq \mathcal{F}_m^4 \supsetneqq \cdots \supsetneqq \mathcal{F}_m^r \supsetneqq \cdots$. For every feasible set $S \in \mathcal{F}_m$, there are only finitely many values of r such that an r-uniform mixed hypergraph can have S as its feasible set, since in this case $r \le 1 + \max S$ necessarily holds. Consequently, there is no feasible set belonging to every element of the above nested sequence.

Contrary to this, we are going to prove that in the case of color-bounded r-uniform hypergraphs the classes $\mathcal{F}_c^r$ of possible feasible sets, for all $r \ge 3$, are the same.

Proposition 11. *For every color-bounded hypergraph $\mathcal{H}_1$ having edges only of sizes not larger than r, there exists a chromatically equivalent r-uniform color-bounded hypergraph $\mathcal{H}_2$; that is, $P(\mathcal{H}_1, \lambda) = P(\mathcal{H}_2, \lambda)$.*

Proof. Any given hypergraph $\mathcal{H}_1 = (X_1, \mathcal{E}_1, \boldsymbol{s}_1, \boldsymbol{t}_1)$ with edges not larger than r can be extended to $\mathcal{H}_2$ in the following way. For each vertex $x_i \in X_1$, we take additional $r-1$ copies, and join them with x_i in an r-element $(1, 1)$-edge to ensure that in each coloring they get the same color as x_i. Then every edge E_j of $\mathcal{H}_1$ can be extended to an r-element edge of $\mathcal{H}_2$ by adjoining $r - |E_j|$ 'copy vertices' of some $x_i \in E_j$ to it, whilst the color-bounds remain unchanged. Clearly, the feasible colorings of $\mathcal{H}_1$ and $\mathcal{H}_2$ are in one-to-one correspondence, thus the two hypergraphs have the same chromatic polynomial. □

Proposition 12. *For each integer $r \ge 3$, the r-uniform color-bounded hypergraphs generate all possible feasible sets from $\mathcal{F}_c$.*

Proof. For $r = 3$, already the mixed hypergraphs generate all the feasible sets from $\mathcal{F}_m$ except the set $\{1\}$. Obviously, the 3-uniform color-bounded hypergraphs determine all these feasible sets and also the set $\{1\}$. A trivial example for the latter has three vertices joined by a 3-element $(1, 1)$-edge. Due to Corollary 7, $\mathcal{F}_m = \mathcal{F}_c$ holds, hence we obtain that $\mathcal{F}_c = \mathcal{F}_c^3$. Applying Proposition 11, we obtain for any $r > 3$ that every 3-uniform color-bounded hypergraph has some r-uniform chromatic equivalent, therefore $\mathcal{F}_c^r \supseteq \mathcal{F}_c^3$. But $\mathcal{F}_c^3$ contains all the possible feasible sets of color-bounded hypergraphs, thus for every $r \ge 3$ the equality $\mathcal{F}_c^r = \mathcal{F}_c$ holds. □

The class of feasible sets occurring for mixed hypergraphs was characterized in the paper [10]. Combining that result and the above proposition we obtain:

Theorem 3. *For every integer $r \ge 3$, a set S of positive integers is the feasible set of an r-uniform color-bounded hypergraph if and only if*

(i) $\min S \ge 2$, *or*
(ii) $\min S = 1$ *and* $S = \{1, \dots, k\}$ *for some natural number* $k \ge 1$. □

Comparing the possible feasible sets of r-uniform mixed and color-bounded hypergraphs, we can conclude that for $r = 2$ they are the same (classical graphs), and for $r = 3$ there is only one feasible set – namely, $\{1\}$ – appearing in the color-bounded case and not belonging to any mixed hypergraphs. But increasing the value of r the difference becomes more and more substantial.

Now, we take some observations concerning the chromatic spectra of (r-uniform) color-bounded hypergraphs. It was proven for mixed hypergraphs in [11] that any vector $(r_1, r_2, \dots, r_k)$ with $r_1 = 0$ and $r_2, \dots, r_k \in \mathbb{N} \cup \{0\}$ can be obtained as the chromatic spectrum of some mixed hypergraph. In the construction of the proof there occur only $\mathcal{C}$-edges of size 3 and $\mathcal{D}$-edges of size 2. This mixed hypergraph can be considered color-bounded as well, and since it has edges of size not larger than 3, we can apply Proposition 11 to get, for each $r \ge 3$, an r-uniform color-bounded hypergraph with the above chromatic spectrum. As a consequence, we obtain:

Proposition 13. *For every finite sequence $r_2, r_3, \dots, r_k$ of nonnegative integers and for every $r \ge 3$ there exists some r-uniform color-bounded hypergraph whose chromatic spectrum is $(r_1 = 0, r_2, \dots, r_k)$.* □

This means that under the assumption $P(1) = 0$, the characterization of chromatic polynomials in Theorem 1 is valid for r-uniform color-bounded hypergraphs, too.

As was proven in Section 3, the possible chromatic polynomials – and also the chromatic spectra – are the same in the case of mixed and color-bounded hypergraphs. Considering a fixed integer $r \ge 3$, however, by the characterization from [2] and by our Theorem 3, there exist feasible sets and hence chromatic spectra, too, occurring for r-uniform color-bounded, but not occurring for r-uniform mixed hypergraphs. We are going to point out that, even when assuming a fixed common feasible set and r-uniform hypergraphs, the corresponding spectra can be different.

Let $r = 4$ and the feasible set $\{1, 2, 3\}$ be considered.

For 4-uniform mixed hypergraphs, this means that there occur only $\mathcal{C}$-edges of size 4, hence any partition of the vertex set into 1, 2, or 3 color classes induces a feasible coloring. Therefore $r_1 = 1$, $r_2 = S(n, 2)$, and $r_3 = S(n, 3)$ hold, where n denotes the number of vertices. In particular, $r_1 = 1$ and $r_2 = 15$ together imply that $n = 5$ and $r_3 = 25$.

On the other hand, in 4-uniform color-bounded hypergraphs the 1-colorability means that the lower bound s_i equals 1 for every edge E_i, and after the contraction of (1, 1)-edges, any 2-partition of the n vertices yields a proper coloring. Thus $r_1 = 1$ and $r_2 = S(n, 2)$. But if there exists an edge with bounds (1, 2), then not all of the 3-partitions are feasible, hence $0 < r_3 < S(n, 3)$ can hold. Analysis shows that if $r_1 = 1$ and $r_2 = 15$, then the first three entries in the chromatic spectrum belonging to the feasible set {1, 2, 3} form one of the following triples:

- (1, 15, 25) — five vertices and four or five 4-element edges, all of them with color-bounds (1, 3);
- (1, 15, 7) — five vertices with one (1, 2)-edge of size four;
- (1, 15, 1) — five vertices with two different (1, 2)-edges of size four.

As has been observed, only the first one can belong to 4-uniform mixed hypergraphs.

5. Uniquely colorable hypergraphs

A hypergraph $\mathcal{H}$ is called *uniquely colorable* if $\chi(\mathcal{H}) = \overline{\chi}(\mathcal{H}) = k$ for some $k \in \mathbb{N}$, and $r_k = 1$. General properties of uniquely colorable *mixed* hypergraphs have been studied in [22,1]. Moreover, uniquely $(n-1)$-colorable and uniquely $(n-2)$-colorable mixed hypergraphs are characterized in [16]. Though it is co-NP-complete to decide whether a mixed hypergraph (given together with one of its colorings) is uniquely colorable [22], it may be the case that uniquely $(n-c)$-colorable mixed hypergraphs admit a relatively simple structural description for any constant c. In sharp contrast to this, for color-bounded hypergraphs we prove:

Theorem 4. *It is co- NP-complete to decide whether a hypergraph $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t})$ on n vertices is uniquely $(n-1)$-colorable.*

Proof. We are going to apply the following result of [17] from the algorithmic theory of balanced incomplete block designs:

It is NP-complete to decide whether a Steiner Triple System, whose blocks are viewed as the $\mathcal{D}$-edges of a 3-uniform (mixed, or 'usual') hypergraph, is colorable with 14 colors.

Now let $X = \{x_1, \ldots, x_{n-2}\}$ be the vertex set of an 'input' Steiner Triple System $\mathcal{S} = STS(n-2) = (X, \mathcal{B})$ of order $n-2$, whose k-colorability (as a $\mathcal{D}$-hypergraph) should be decided for a given integer k. Due to the result quoted above, in our case $k = 14$ will be a suitable choice.

We construct a color-bounded hypergraph $\mathcal{H}$ on the vertex set $X \cup \{z_1, z_2\}$ – i.e., with two new vertices z_1, z_2 – in which the hyperedges and color-bounds are defined as follows:

- $B' = B \cup \{z_1, z_2\}$ with $\boldsymbol{s}(B') = 4$ and $\boldsymbol{t}(B') = 5$, for all blocks $B \in \mathcal{B}$;
- $W' = W \cup \{z_1, z_2\}$ with $\boldsymbol{s}(W') = 1$ and $\boldsymbol{t}(W') = k+2$, for all $(k+1)$-tuples $W \in \binom{X}{k+1}$;
- $e_{i,j} = \{x_i, z_j\}$ with $\boldsymbol{s}(e_{i,j}) = \boldsymbol{t}(e_{i,j}) = 2$, for all $1 \le i \le n-2$ and $j = 1, 2$.

We analyze the colorings φ of $\mathcal{H}$ by considering the following two cases.

Case 1: $\varphi(z_1) = \varphi(z_2)$

Due to the presence of the $\mathcal{D}$-edges $e_{i,j}$, the color-bound functions $\boldsymbol{s}, \boldsymbol{t}$ reduce to the conditions $|\varphi(W)| \le k+1$ for all W and $|\varphi(B)| = 3$ for all B. We may disregard the former, as it does not yield any real restriction. On the other hand, since $\mathcal{S}$ is a Steiner Triple System, the blocks $B \in \mathcal{B}$ cover all vertex pairs, and hence the latter equation means that any two vertices in X must get distinct colors. Thus, X is $(n-2)$-colored and $\mathcal{H}$ is $(n-1)$-colored.

This type of coloring is unique and it exists for any $STS(n-2)$ input $\mathcal{S}$; and it obviously colors the constructed $\mathcal{H}$ with a feasible $(n-1)$-coloring.

Case 2: $\varphi(z_1) \ne \varphi(z_2)$

Then the conditions reduce to $|\varphi(W)| \le k$ for all W and $|\varphi(B)| \ge 2$ for all B. Hence, such a coloring exists if and only if the input Steiner system $\mathcal{S}$ admits a coloring with at most k colors. By the theorem quoted above, this is NP-complete to decide.

Given any input $\mathcal{S}$, the color-bounded hypergraph $\mathcal{H}$ together with its $(n-1)$-coloring described in Case 1 can be constructed in polynomial time for any constant k. This $\mathcal{H}$ *is not* uniquely $(n-1)$-colorable if and only if $\mathcal{S}$ *is* k-colorable.

Finally, an n-element set has precisely $\binom{n}{2}$ partitions into $n-1$ nonempty sets, and it can be checked efficiently for each of those partitions whether it is a feasible coloring of $\mathcal{H}$. Moreover, the problem of finding another coloring of $\mathcal{H}$ has its obvious membership in NP. This completes the proof of the theorem. □

6. Regular hypergraphs and color-bounded edge-colorings of graphs

In the cases of classical and mixed hypergraphs, restricting the vertex degrees to at most 2 or prescribing that any two edges share at most one vertex – that is, *linear* hypergraphs – sometimes makes problems algorithmically easier to handle. For example, there can be given a well-characterized set of obstructions against the colorability of mixed hypergraphs with maximum degree two [20]. Efficient algorithms on this class have also been presented in [14].

In contrast to this, we are going to prove that every color-bounded hypergraph can be transformed to a chromatically equivalent 2-regular hypergraph. As a consequence, restricting the vertex degrees to at most 2, the time complexity of colorability problems does not change substantially. For comparison, let us mention that in [14] the mixed hypergraphs were shown to admit chromatically equivalent representations with mixed hypergraphs of maximum degree 3. It follows from known results on complexity that degree 3 cannot be reduced to degree 2, hence color-bounded hypergraphs yield a stronger model in this respect, too.

Proposition 14. *Every color-bounded hypergraph is chromatically equivalent to a 2-regular color-bounded linear hypergraph.*

Proof. Consider a hypergraph $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t})$. If there are vertices with degree 0 or 1, we can create some new edges containing them, with nonrestrictive bounds $s = 1$ and $t = |E|$. Thus, we can assume that each vertex of $\mathcal{H}$ has degree at least 2.

To construct a 2-regular hypergraph $\mathcal{H}^+$, for each vertex x_i of $\mathcal{H}$ we create $d(x_i)$ copies and let them form a (1,1)-edge. The edges of $\mathcal{H}$ can be transformed to edges of $\mathcal{H}^+$ in such a way that every vertex is replaced with one of its copies, and every 'copy vertex' occurs in exactly one edge of this type. The color-bounds remain unchanged, and the vertices of $\mathcal{H}$ do not belong to $\mathcal{H}^+$.

Obviously, the copies of a vertex x_i have the same color in every feasible coloring of $\mathcal{H}^+$, so they can be contracted and a feasible coloring of $\mathcal{H}$ is obtained; and vice versa. Thereby, $\mathcal{H}$ and $\mathcal{H}^+$ are chromatically equivalent. Moreover, $\mathcal{H}^+$ is a 2-regular hypergraph, where any two edges either are disjoint or have intersection of size 1, which completes the proof. □

It is important to note that this transformation generally does not preserve the special structural properties (e.g., hypertree, circular hypergraph). On the other hand, some properties can be ensured; for example, the 3-uniformity can be preserved by slightly modifying the construction.

According to the above proposition, it is enough to consider the 2-regular linear color-bounded hypergraphs regarding the general coloring properties. Let us observe that the dual of a 2-regular and linear hypergraph $\mathcal{H}$ is a *simple graph*. Coloring the vertices of $\mathcal{H}$ according to the color-bounds corresponds to an edge-coloring of the dual graph, where each vertex has the same color-bounds (s_i, t_i), as the corresponding edge in $\mathcal{H}$.

Definition 2. Consider a graph $G = (V, E)$ where each vertex x_i is associated with integer color-bounds: $1 \leq s_i \leq t_i \leq d(x_i)$. A *color-bounded edge-coloring* of G is a mapping from E to $\mathbb{N}$, such that for every vertex x_i, the incident edges are colored with at least s_i and at most t_i distinct colors. In this model it is convenient to assume that G has *no isolated vertices*.

Theorem 5. *Regarding the color-bounded edge-coloring of graphs:*

(i) *A set S of positive integers can be obtained as a feasible set if and only if* $\min S \geq 2$ *or* $S = [1, \dots, k]$ *for some* $k \geq 1$.
(ii) *The class of possible chromatic polynomials corresponds to the class of chromatic polynomials occurring for vertex colorings of color-bounded hypergraphs.*
(iii) *These properties remain valid in the restricted class of bipartite graphs, too.*

Proof. According to Proposition 14, for every color-bounded hypergraph $\mathcal{H}$, there exists a chromatically equivalent 2-regular, linear color-bounded hypergraph $\mathcal{H}^+$. The dual of $\mathcal{H}^+$ is a simple graph G. The vertex colorings of $\mathcal{H}^+$ are in one-to-one correspondence with the color-bounded edge-colorings of G, provided that each vertex of the latter has the same color-bounds as the corresponding edge in $\mathcal{H}^+$.

Conversely, every graph without isolated vertices has a dual hypergraph, and if there are assigned corresponding color-bounds, the feasible edge-colorings of the former and vertex colorings of the latter determine the same chromatic polynomial. This proves statement (ii).

Due to (ii), the possible chromatic spectra – and hence the feasible sets, too – are the same for the two structure classes. Taking into consideration the characterization of possible feasible sets of color-bounded hypergraphs, statement (i) follows.

To prove (iii), it suffices to observe that the dual graph of the constructed hypergraph $\mathcal{H}^+$ is bipartite: the two vertex classes correspond to the 'copy edges' and the original edges of $\mathcal{H}$, respectively. □

Remark 6. The validity of Theorem 5 can be extended to color-bounded edge-colorings of *multigraphs*, too. There can appear no additional feasible sets and chromatic polynomials, since also the dual of a multigraph is a hypergraph (though not necessarily linear).

7. Concluding remarks and open problems

In this paper we introduced a coloring concept of hypergraphs defined in terms of conditions on the polychromatic subsets of edges. Since the object is quite new, there are many interesting problems that remain open. Let us mention explicitly some of them in connection with the chromatic polynomial $P(\mathcal{H}, \lambda)$.

Problem 1. Characterize the polynomials $P(\lambda)$ with $P(1) = 1$ that are chromatic polynomials of some mixed/color-bounded hypergraphs.

Problem 2. Characterize the meaning of coefficients of the chromatic polynomials of mixed/color-bounded/pattern hypergraphs.

Problem 3. Characterize the meaning of roots of the chromatic polynomials of mixed/color-bounded/pattern hypergraphs.

Further properties of color-bounded hypergraphs are studied in the series of papers [4–6].

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