BCH codes and distance multi- or fractional colorings in hypercubes asymptotically

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Abstract

The main result is a short and elementary proof for the author’s exact asymptotic results on distance chromatic parameters (both number and index) in hypercubes. Moreover, the results are extended to those on fractional distance chromatic parameters and on distance multi-colorings. Inspiration comes from radio frequencies allocation problem. The basic idea is the observation that binary primitive narrow-sense BCH codes or their shortenings have size asymptotically within a constant factor below the largest possible size, \( A(n, d) \), among all binary codes of the same length, \( n \), and the same minimum distance, \( d \), as \( n \to \infty \) while \( d \) is constant. Also a lower bound in terms of \( A(n, d) \) is obtained for \( B(n, d) \), the largest size among linear binary codes.

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1. Introduction

BCH codes are important error-correcting codes which are linear and cyclic, and named after Bose and Chaudhuri [4,5] and Hocquenghem [15], the discoverers of the binary BCH codes. The aim of this note is to present a simple and elementary proof that binary BCH codes are nearly optimal in a rather peripheral case when their minimum distance \( d \) is fixed and the length \( n \) tends to infinity, which makes the long codes useless for detection of transmission errors. As is shown in [31], those codes appear sufficient for determining the exact order of growth of classical parameters in vertex and edge distance colorings of growing hypercubes. This asymptotics is extended to distance multi-colorings and to distance fractional chromatic parameters. Relations with applications to the frequency assignment in cellular telecommunication are recalled. Results on coding that we refer to in what follows can be found in the monograph by MacWilliams and Sloane [23] or books by Hankerson et al. [13], van Lint [22], or by Roman [29]. We refer: to the author’s paper [31] for distance colorings; to the monograph by Scheinerman and Ullman [30] for fractional colorings and multi-colorings; and to [7] for a general overview of the frequency assignment.

We now recall some notation necessary for stating our main results. Given a graph \( G \) (\( G \) being loopless with multiple edges allowed), let \( \chi(G) \) and \( q(G) \) be, respectively, the chromatic number and the chromatic index of \( G \). Let \( L(G) \) stand for the line graph of \( G \). Then \( q(G) = \chi(L(G)) \). Given a positive integer \( d \), let \( G^d \) denote the \( d \)th power of \( G \),...
Theorem 2. Let \( G^d \) be the spanning supergraph of \( G \) with any two vertices being adjacent whenever their distance in \( G \) is between 1 and \( d \) inclusive. Hence \( G = G^1 \). Following [31], define the \( d^+ \)-distance chromatic parameters: number, \( \chi_d(G) \), and index, \( q_d(G) \), of \( G \) as follows.

\[
\chi_d(G) = \chi(G^d), \quad q_d(G) = \chi_d(L(G)) = \chi(L(G)^d).
\]  

The \( b \)-fold chromatic number of \( G \), denoted \( \chi^b(G) \), is defined to be the least cardinality \( a \) of a set of colors such that vertices can be colored with \( b \)-subsets of colors so that adjacent vertices are assigned disjoint subsets. Then \( q^b(G) := \chi^b(L(G)) \) is the \( b \)-fold chromatic index of \( G \). Thus, \( \chi^1 = \chi \) and \( q^1 = q \). Then the fractional chromatic parameter, say \( \pi_f \) with \( \pi = \chi \) or \( q \), namely number \( \chi_f(G) \) and index \( q_f(G) \), is defined to be

\[
\pi_f(G) = \lim_{b \to \infty} \frac{\pi^b(G)}{b} = \inf_b \frac{\pi^b(G)}{b},
\]  

where \( \pi = \chi \) or \( q \). If \( \pi \) therein is replaced by the corresponding \( d^+ \)-distance parameter \( \pi_d \) then the notions \( \pi^b_d(G) \) (or \( \pi_f, d_d(G) \)) are defined and they are called the \( d^+ \)-distance \( b \)-fold (or resp. \( d^+ \)-distance fractional) chromatic parameter: number, \( \chi^b_d(G) \) or \( \chi_f, d_d(G) \), if \( \pi = \chi \); and index, \( q^b_d(G) \) or \( q_f, d_d(G) \), if \( \pi = q \).

Remark 1.1. Each equality in (1) remains valid if on both sides of the equality the same superscript \( b \) or subscript \( f \) is added to the chromatic parameter therein, to \( \chi \) or \( q \) (or to both if applicable).

The symbol \( \Theta \) indicates the exact order of asymptotic growth. Recall that the floor \( \lfloor x \rfloor \) of a real number \( x \) is the integer part of \( x \). Moreover, the ceiling \( \lceil x \rceil = -\lfloor -x \rfloor \).

Theorem 1. Let \( Q_t \) be the \( t \)-cube. Then for any constants \( d, b \in \mathbb{N} \), \( d \) being a constant distance bound, if the dimension \( t \to \infty \) then all the three following distance chromatic parameters with the same \( \pi = \chi \) or \( q \) have the same order of growth, namely

\[
\begin{align*}
\chi_d(Q_t), & \quad \chi_f, d_d(Q_t), & \quad \chi^b_d(Q_t)/b = \Theta(t^{(d/2)}), \\
q_d(Q_t), & \quad q_f, d_d(Q_t), & \quad q^b_d(Q_t)/b = \Theta(t^{((d+1)/2)}).
\end{align*}
\]

Main part of the above theorem (concerning parameters \( \chi_d \) and \( q_d \) only, together with good bounds on them) was proved in the author’s paper [31]. A simplified thorough proof will be given in what follows. As a by-product, relatively good bounds on all of the above chromatic parameters have been obtained, see Theorems 15 and 16.

Recall the following notation from the theory of error-correcting codes. Let \( A(n, d) \) be the maximum size (the number of codewords) in any binary code (possibly nonlinear) of length \( n \) and minimum (Hamming) distance \( d \). Let \( B(n, d) \) be the corresponding maximum size among linear codes with the same parameters \( n \) and \( d \).

Moreover, let \( (2n)! \) stand for \( 2^n n! \). We are going to use the asymptotic inequality \( \lesssim \) in the following sense. Write \( a(n) \gtrsim c(n) \) whenever there exists an expression \( b(n) \) such that \( a(n) \geq b(n) \sim c(n) \) as \( n \to \infty \). Our main result on binary codes follows.

Theorem 2.

\[
B(n, d) \lesssim \frac{A(n, d)}{(2((d - 1)/2))!!}
\]

as \( n \to \infty \) and \( d \) is constant.

2. On coding and BCH codes

Some facts from coding theory are recalled in order to keep the paper self-contained. We restrict our attention to binary codes only. Thus, a code of length \( n \) is a subset of the linear space \( \{0, 1\}^n \) over Galois field GF(2), the code being linear if it is a linear subspace. The name \( (n, M, d) \) code stands for a code of length \( n \), size (or cardinality) \( M \), and minimum distance \( d \). A linear \( (n, M, d) \) code \( \mathcal{C} \) has size \( M = 2^k \) and is named an \( [n, k, d] \) code, where \( k = \log_2 M \) is the dimension of \( \mathcal{C} \).
Recall that (simple) shortening a binary code $\mathcal{C}$ (of length $\geq 2$) which includes codewords with distinct $i$th coordinates $x_i(=0, 1)$ consists in taking a cross-section of the code $\mathcal{C}$ at the $i$th coordinate, i.e., in taking all the codewords which have the same $x_i$ (just $x_i = 0$ if the code $\mathcal{C}$ is linear) and make up at least half of the code (which is exactly half if $\mathcal{C}$ is linear), and then deleting this $i$th coordinate from each chosen codeword. This shows

$$B(n+1, d) \leq 2B(n, d), \quad A(n+1, d) \leq 2A(n, d). \quad (3)$$

Simple shortening a (linear) $[n, k, d]$ code with $d < n$ results in a $[n−1, k−1, d]$ code which is linear and has length one smaller, halved size, and the minimum distance unchanged. Shortening to the length $l$ (from $n$) results after $n−l$ successive simple shortenings.

BCH codes are linear and cyclic (closed under cyclic permutations of their words). Each BCH code we deal with, say $\mathcal{C}$, is primitive, that is, of length $\bar{n} = 2^m − 1$ for an integer $m \geq 2$. Let $\beta$ be a primitive element of $\text{GF}(2^m)$, i.e., powers of $\beta$ range over all nonzero elements of $\text{GF}(2^m)$. We further assume that $\mathcal{C}$ is a narrow-sense BCH code, i.e., $\mathcal{C}$ is generated by a lowest degree polynomial $g(x) \in \text{GF}(2)[x]$ having $\beta, \beta^2, \ldots, \beta^{d−1}$ among its zeros, where $d$ is an integer from the interval $[2, \bar{n})$ and $d$ is called the designed distance of the BCH code; the trivial cases: $d = 1$ (in which $g(x) = 1$) and $d \geq \bar{n}$ being excluded unless $d = 3 = \bar{n}$. Note that if $\alpha$ is a root of $g(x)$ then so is $\alpha^2$ because $\alpha^2 g(x) = g(x^2)$. Hence replacing the even designed distance $d = 2\tau$ by $2\tau + 1$ gives the same BCH code $\mathcal{C}$. Therefore, we assume that $d = 2\tau + 1$. Furthermore, $\deg g(x)$, the degree of $g(x)$, is at most $m\tau$, because all roots of $g(x)$ are of the form $\beta^j$ with $j = 0, 1, \ldots, m−1$; $i = 1, 3, \ldots, 2\tau − 1$. Hence $k := \bar{n} − \deg g(x)$ is the dimension of $\mathcal{C}$ and therefore $|\mathcal{C}| = 2^k \geq 2^{n−m\tau}$. Thus, we get the following result.

**Proposition 3.** Let $\mathcal{C}$ be a primitive narrow-sense BCH code of length $2^m − 1 (= \bar{n})$ and with designed distance $d = 2\tau + 1$ (unless $d = 3 = \bar{n}$), $\tau \geq 1$, $m \geq 2$. Assume that $n \geq d$ and $2^{m−1} \leq n \leq 2^m − 1$ where $m = \lceil \log_2(n+1) \rceil$. Let $\mathcal{C}$ be a code of length $n$ such that either $\mathcal{C} = \mathcal{C}$ (and $n = \bar{n}$) or $\mathcal{C}$ is obtained by shortening $\mathcal{C}$ to the length $n$. Then $|\mathcal{C}| \geq 2^{n−m\tau} \geq 2^n/(2n)^\tau$.

3. The main results on codes

For auxiliary results which we recall below, the reader is referred to the monograph [23]. Finding the size $A(n, d)$ (or rather a code which realizes $A(n, d)$) is clearly among the most basic problems. Advantages of linearity motivate the study of $B(n, d)$. A method for determining the maximum linear code and its size $B(n, d)$ for $d \leq n \leq 2d$ is presented by Venturini [32]. Just in this case $A(n, d)$ is determined by the Plotkin bound and its relative sharpness (depending on the existence of Hadamard matrices) is shown by Levenshtein [19]. Nevertheless, even in this case it is an open problem to find a nontrivial lower bound for $B(n, d)$ in terms of $A(n, d)$.

Proof of Theorem 2 requires the following simple results.

**Theorem 4 (The Hamming bound).** For $d = 2\tau + 1, 2\tau + 2$,

$$A(n, d) \left(1 + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{\tau}\right) \leq 2^n.$$

Extending a code $\mathcal{C}$ of length $n$ consists of adding the $(n + 1)$th coordinate $x_{n+1}$ so that in the resulting code, $\mathcal{C}^+$, each word has an even weight, i.e., even number of 1’s. If the distance of two binary codewords is odd then so is their total of 1’s. Therefore, extending these two codewords increases their distance by one. Thus, $\mathcal{C}^+$ is an $(n + 1, M, d + 1)$ code if the minimum distance $d$ of $\mathcal{C}$ is odd. Hence $A(n + 1, 2\tau + 2) \geq A(n, 2\tau + 1)$. On the other hand, $A(n + 1, 2\tau + 2) \leq A(n, 2\tau + 1)$ because puncturing an $(n + 1, M, 2\tau + 2)$ code by deleting one coordinate can decrease the minimum distance to $2\tau + 1$. Furthermore, one can see that neither extending nor puncturing can spoil linearity if the source code is linear. Thus, we get the following known result.

**Theorem 5.** $A(n + 1, 2\tau + 2) = A(n, 2\tau + 1), B(n + 1, 2\tau + 2) = B(n, 2\tau + 1)$.

**Proposition 6.** Both $A(n, d)$ and $B(n, d)$ are decreasing in $d$, i.e., $A(n, d) \geq A(n, d + 1)$ and $B(n, d) \geq B(n, d + 1)$.
The proof follows for $1 \leq d < n$ and $M > 1$ by applying the known operation (which we call *weakening* a code and) which replaces a coordinate by zero in each codeword of an $(n, M, d + 1)$ code so that the resulting code is really weaker, is an $(n, M, d)$ code, that is, the coordinate is chosen so that the new minimum distance is $d$.

Even though "long BCH codes are bad"—in the sense that BCH codes are unable to keep both ratios $k/n$ and $d/n$ away from zero as $n$ is large, cf. Lin and Weldon [20] or Camion [6]—they are large enough as is shown in Proposition 3. The main result of Hocquenghem [15] as well as of Bose and Chaudhuri [4,5] is the following.

**Theorem 7 (The BCH bound).** If $d'$ is the minimum distance of a binary BCH code of designed distance $d$ then $d' \geq d$. Moreover, $d' \geq d + 1$ for even $d$ if the code is primitive narrow-sense.

We shall use the following abbreviations.

$$
\tau = \tau(d) = [(d - 1)/2],
$$

$$
r(d) = d \mod 2
$$

is the *parity indicator* (binary remainder) of $d$, its adjoint being

$$
r^*(d) = r(d + 1) = 1 - r(d)
$$

whence $r^*(d + 1) = r(d)$,

$$
\varepsilon(d) = 2 - r(d) = 1 + r^*(d) = 2^{r^*(d)} \quad (\in \{1, 2\}),
$$

$$
m = m(n, d) = \lceil \log_2(n + r(d)) \rceil = \begin{cases} 
\lceil \log_2(n + 1) \rceil & \text{for odd } d, \\
\lceil \log_2 n \rceil & \text{for even } d.
\end{cases}
$$

**Theorem 8 (The linear-code-coloring bound).** For $n \geq d \geq 2$ with the above abbreviations in use, the number of colors in a $(d - 1)^+$ coloring of $V(Q_n)$ is

$$
\mathcal{L}_{d-1}(Q_n) \leq 2^n/B(n, d) \leq \varepsilon(d)2^{m(n, d)\tau(d)} \quad (= \Theta(n^{-\tau(d)}))
$$

$$
\leq \varepsilon(d)(2n - 2r^*(d))^{\tau(d)}.
$$

**Proof.** Coloring of $V(Q_n)$ in which color classes are all translates of an $(n, B(n, d), d)$ linear code shows the first inequality in (9). The equality $m = m(n, d)$ is equivalent to conditions:

$$
\begin{align*}
2^{m-1} \leq n & \leq 2^m - 1 & \text{for odd } d, \\
2^{m-1} \leq n - 1 & \leq 2^m - 1 & \text{for even } d
\end{align*}
$$

which imply $n + r(d) \leq 2^m \leq 2n - 2r^*(d)$. This proves inequality (10) as well as asymptotics stated in (9). We pass on to the proof of the second inequality in (9).

1. Let odd $d = 2\tau + 1$. If $d'$ stands for the minimum distance of the code $C$ in Proposition 3 then $d' \geq d$ by the BCH bound and $B(n, d) \geq B(n, d')$ by Proposition 6. Hence the result follows from the definition of $B(\cdot, \cdot)$ and Proposition 3.

2. Let even $d = 2\tau + 2$. Then $B(n, 2\tau + 2) = B(n - 1, 2\tau + 1)$ by Theorem 5. Hence $2^n/B(n, d) = 2 \cdot 2^{n-1}/B(n - 1, 2\tau + 1) \leq 2 \cdot 2^{\lceil \log_2 n \rceil \tau}$ by part (1), which agrees with (9).

We shall need an improvement of the Hamming bound for even $d$. Let

$$
\bar{S}(n, d) = \begin{cases} 
\sum_{i=0}^{\tau} \binom{n}{i} & \text{for odd } d = 2\tau + 1, \\
\sum_{i=0}^{\tau} \binom{n - 1}{i} & \text{for even } d = 2\tau + 2.
\end{cases}
$$
Theorem 9 (The refined Hamming bound). For \( n \geq d \),

\[
2^n/A(n, d) \geq o(d)\tilde{S}(n, d).
\]

Proof. The refinement consists in the inequality for \( d = 2\tau + 2 \):

\[
2^n/A(n, 2\tau + 2) = 2^n/A(n - 1, 2\tau + 1) \quad \text{by Theorem (5)},
\]

\[
\geq 2 \sum_{i=0}^{\tau} \binom{n-1}{i} \quad \text{by the Hamming bound},
\]

which is as required and, moreover,

\[
\geq \sum_{i=0}^{\tau} \binom{n}{i} \quad \text{since } 2 \binom{n-1}{i} > \binom{n}{i} \quad \text{for } n > 2\tau \geq 2i \geq 0,
\]

which shows the refinement. \( \square \)

Proof of Theorem 2. Theorems 8 and 9 with abbreviations (4)–(8) in use give

\[
B(n, d) \geq \frac{2^n}{\bar{v}(d)2^{\tau\tau}} \geq A(n, d)\frac{\tilde{S}(n, d)}{2^{\tau}} \geq A(n, d)\frac{\tilde{S}(n, d)}{2^{\tau}(n - r^*(d))^\tau} \quad \text{by inequality (10).}
\]

This clearly implies Theorem 2 by taking the limit of the last quotient (of two polynomials in \( n \) of degree \( \tau \)). \( \square \)

In this proof (like in proof of Theorem 8) \( (n - r^*(d)) \in [2^{m-1}, 2^m - 1] \) whence \( 2^m \in [n + r(d), 2n - 2r^*(d)] \) and possibly \( 2^m = n + o(n) \) or \( 2^m = 2n + o(n) \) for some constant \( \alpha \in [1, 2] \). Therefore, inequality (11) implies the following improvements on Theorem 2.

Corollary 10. If \( n \rightarrow \infty \), \( d \) is kept constant, and \( \tau = \lfloor (d - 1)/2 \rfloor \) then

\[
B(n, d) \geq A(n, d)/2^{\tau\tau}!, \quad \text{as in Theorem 2,}
\]

\[
\geq A(n, d)/\tau! \quad \text{if } n = 2^m - o(2^m),
\]

\[
\geq A(n, d)/\alpha^\tau! \quad \text{if } n = 2^m/\alpha + o(2^m)
\]

for any real constant \( \alpha \in [1, 2] \), provided that \( (n - r^*(d)) \in [2^{m-1}, 2^m - 1] \).

For further use we are going to prove a refined simplification of the Johnson bound, an improvement of Hamming’s. The said simplification is a corollary which avoids the function \( A(n, d, w) \), a specification of \( A(n, d) \), with \( w \) being the weight of counted codewords. Let

\[
\tau = \lfloor (d - 1)/2 \rfloor,
\]

\[
J(n, d) = \frac{1}{\lfloor \tau \rfloor + 1} \binom{n}{\tau} \frac{r_n}{\tau + 1} \quad (= o(n^\tau)),
\]

where \( r_n = (n + 1) \mod (\tau + 1) \),

\[
\tilde{J}(n, d) = \begin{cases} J(n, d) & \text{for odd } d = 2\tau + 1, \\ J(n - 1, d) & \text{for even } d = 2\tau + 2. \end{cases}
\]

\( \tilde{S}(n, d) \), see the refined Hamming bound, can similarly be presented in terms of \( S(n, d) := \sum_{i=0}^{\tau} \binom{n}{i} \).
Theorem 11 (The (simplified) Johnson bound).

\[ 2^n / A(n, d) \geq S(n, d) + J(n, d). \]

This bound in \([23,29]\) is presented so that in \(J(n, d)\) our quotient \(r_n / (\tau + 1)\) with \(r_n = (n + 1) \mod (\tau + 1)\) is replaced by the difference \((n - \tau) / (\tau + 1) - [(n - \tau) / (\tau + 1)]\).

Theorem 12 (The refined Johnson bound). For \(n \geq d\),

\[ 2^n / A(n, d) \geq \varepsilon(d) (\tilde{S}(n, d) + \tilde{J}(n, d)), \quad \text{as } n \to \infty \text{ and } d \text{ is constant}. \]

Proof. The refinement consists in the inequality for \(d = 2 \tau + 2\) which, like in the refined Hamming bound, follows from that for odd \(d\), i.e., from the (simplified) Johnson bound stated above. In fact, the refinement consists in strict increase of the RHS of (12), i.e.,

\[ 2S(n - 1, 2 \tau + 2) - S(n, 2 \tau + 1) + 2J(n - 1, 2 \tau + 2) - J(n, 2 \tau + 1) > 0, \]

where the first difference has been proved to be positive but the latter one is surely positive if \(r_n = 0\) only. Otherwise \(r_n = 1 + r_{n-1} > 0\). To prove the refinement we subtract a part of \(J(n, 2 \tau + 1)\) with \(r_n \leftarrow r_{n-1}\) from \(2J(n - 1, 2 \tau + 2)\) which yields

\[ \frac{r_{n-1}}{(\tau + 1)\tau} \left( \frac{n - 1}{\tau - 1} \right) \left( \frac{2n - 2\tau}{n - 1} - \frac{n}{\tau + 1} \right) \geq 0 \]

because the last difference (within parentheses) is

\[ \geq \frac{2n - 2\tau - n}{\lfloor n / (\tau + 1) \rfloor} > 0. \]

Subtracting the remaining part with \(r_n \leftarrow 1\) from the last summand in the difference of \(S's\) yields

\[ 2 \left( \frac{n - 1}{\tau} - \frac{n}{\tau} - \frac{n}{\tau + 1} \right) \left( \frac{1}{(\tau + 1)\tau} \frac{n}{\tau + 1} \right) = \frac{1}{\tau} \left( \frac{n - 1}{\tau - 1} \right) \left( 2n - 2\tau - n - \frac{n}{(\tau + 1)\tau} \right) > 0 \]

since \(n \geq 2\tau + 2\). The asymptotic order in (13) is easily seen. \(\square\)

4. More on BCH codes

For more on the BCH bound, see \([22, 6.6]\). The exact value of \(d'\) in the BCH bound is known in special cases only. Additional results on the true minimum distance \(d'\) and exact size of the code \(C\) for \(n\) large enough follow.

Theorem 13 (Farr \([8]\)). The minimum distance \(d'\) of a binary primitive BCH code of length \(n = 2^m - 1\) and designed odd distance \(d = 2\tau + 1\) is \(d'' = d\) if

\[ \sum_{i=0}^{\tau+1} \binom{n}{i} > 2^{m\tau}, \]

which is the case for all \(m\) large enough if \(\tau\) is constant.
Theorem 14 (Mann [24]). If $C$ is a binary primitive narrow-sense BCH code of length $n = 2^m − 1$ and designed odd distance $d = 2τ + 1$, where $2τ − 2 < 2^{m/2}$, then

$$|C| = 2^n / 2^{mτ},$$

which is the case for all $m$ large enough if $τ = \text{const}$.

5. Distance multi-colorings and frequencies allocation

In what follows by a graph we mean a loopless multigraph. To color elements like vertices (or edges) of a graph means to assign a color to each element. A color class of a coloring is the set of unicolored elements. In proper (classical) coloring (the only one we deal with) unicolored elements are at distance larger than one. The distance between two edges in a graph $G$ is defined to be their vertex distance in the line graph $L(G)$. If no two unicolored elements are at distance $d$ or less, the coloring is called $d^+$-coloring, $d ∈ \mathbb{N}$. Then a color class is called either a $d^+$-matching if edges are colored or a $d^+$-independent set if vertices are colored. A $d^+$-matching number, $μ_d(G)$, and $d^+$-independence number, $χ_d(G)$, are the largest possible sizes of the color classes among corresponding colorings of $G$.

Given a function $ϕ(=ϕ_i)$ from elements (either vertices, $i = 0$, or edges, $i = 1$) to nonnegative integers, a $d^+$-$ϕ$-coloring of $G$ consists in assigning $ϕ(x)$ colors to each argument $x$ of $ϕ$ so that for any two elements at distance $d$ or less all assigned colors be distinct. The problem is to determine the corresponding $(d, ϕ)$-chromatic parameters: the chromatic number $χ_{d,ϕ}(G)$ and chromatic index $q_{d,ϕ}(G)$. These are $d^+$-distance chromatic parameters $χ_{d}(G)$ and $q_d(G)$ if all values of $ϕ$ are 1.

We are going to use the following lower bounds involving the number of vertices, $v(G)$, and that of edges denoted by $e(G)$ together with respective independence numbers

$$\frac{v(G)}{χ_d(G)} ≤ χ_d(G), \quad \frac{e(G)}{μ_d(G)} ≤ q_d(G). \quad (14)$$

On the other hand, constructing a coloring can give a good upper bound.

Both the new coloring problems can be reduced to classical ones at the expense of constructing some supergraphs of $G$. First, each nonzero value of $ϕ$ (at any $x$) is reduced to the number 1 (at each resulting copy of $x$) after passing on to a supergraph, say $Γ_{ϕ}^i(G)$ with $i = 0$ or 1. To get the supergraph for each successive element $x$ with $ϕ(x) > 1$ only, replace $x$ by $ϕ(x)$ copies of $x$; if $x$’s are edges ($i = 1$) then make each copy of any $x$ incident to both endvertices of the edge $x$, otherwise ($i = 0$) make the copies both mutually adjacent and adjacent to all neighbors of the vertex $x$ in the current graph. Thus, the original $ϕ$-coloring in $G$ is reduced to $\{0, 1\}$-coloring in the supergraph $Γ_{ϕ}^i(G)$ (or to ordinary coloring with single colors if $ϕ > 0$ on $G$).

Define $\tilde{G}$ to be the line graph $L(Γ_{ϕ}^i(G))$ of $Γ_{ϕ}^i(G)$ if $i = 1$ (i.e., if edges are to be colored; because coloring of edges is equivalent to the vertex coloring of the line graph), and let $\tilde{G} = Γ_{ϕ}^0(G)$ if $i = 0$. Finally, take a $d$th power $\tilde{G}^d$ of $\tilde{G}$ and remove from it all vertices $\tilde{y}$ which in the root graph $G$ are elements $y$ with $ϕ(y) = 0$. Let $\tilde{G}^d −$ be the resulting graph. Then the original $(d, ϕ)$-chromatic parameter of $G$ is equal to the classical chromatic number $χ(Γ_{ϕ}^d)$ of $Γ_{ϕ}^d$.

Finding the $(d, ϕ)$-chromatic number $χ_{d,ϕ}(G)$ can model optimal stationary assignment of radio frequencies (colors) from a limited band to the regions (cells = vertices) $x$ where edges of $G$ join neighboring cells (at unit distance apart) and $d$ is the upper bound on the distance within which interferences occur. A similar problem of so-called call chromatic number, denoted by $C_{ϕ}^d(G)$, with the constraint that all colors within any ball $B(x, R)$ of radius $R$ be different is quoted by Baldi [1]. Because a ball can contain distant vertices, the function $χ_{d,ϕ}$ suits better than $C_{ϕ}^d$ to applications in cellular telecommunication. By that way, recent models (cf. [10,9,25]) involving a metric in the set of radio frequencies (together with minimizing the cumulative interferences [25]) are still more realistic.

6. Proof of Theorem 1

Consider the following steps dealing with colorings in the hypercube.
6.1. Linear codes and colorings

Note that translates (or cosets) of a binary linear code of length \( t \) partition the vertex set \( V(Q_t) \) and that is why they can serve as color classes in distance colorings of vertices. A \( d^+ \)-coloring of edges is reduced to independent coloring of the subset, say \( E_j \), of edges parallel to the axis \( Ox_j \) for each \( j = 1, 2, \ldots, t \). Namely, a linear code of length \( t - 1 \), with minimum distance \( \geq d \), and of size \( B(t - 1, d) \) can be used for coloring all vertices of the half-cube \( x_j = 0 \) (the half-cube being isomorphic to \( Q_{t-1} \)). The color of each vertex is then moved to the incident edge in \( E_j \). The resulting edge coloring of \( E_j \) is a \( d^+ \)-coloring and so is the coloring of all edges of \( Q_t \).

6.2. Bounds on distance colorings

In order to use bounds (14), note that

\[
v(Q_t) = 2^t, \quad e(Q_t) = t 2^{t-1}, \quad A_d(Q_t) = A(t, d+1), \quad \mu_d(Q_t) = A(t-1, d).
\]

The last equality is due to the observation that a maximum \( d^+ \)-matching in the subset \( E_j \) is maximum in the cube \( Q_t \), see [31] for more details. Thus

\[
\frac{2^t}{A(t, d+1)} \leq A_d(Q_t) \leq \frac{2^t}{B(t, d+1)}, \tag{15}
\]

\[
\frac{t 2^{t-1}}{A(t-1, d)} \leq q_d(Q_t) \leq \frac{t 2^{t-1}}{B(t-1, d)}, \tag{16}
\]

with equalities if the corresponding linear code is of size \( B(\cdot, \cdot) = A(\cdot, \cdot) \). Now, using the refined Johnson bound (to cancel A’s) and the linear-code-coloring bound (for eliminating B’s) gives the following bounds which can be derived from [31, pp. 102–103].

**Theorem 15.**

\[
2 \left( \sum_{i=0}^{\tau} \binom{t-1}{i} + \frac{1}{\binom{t-1}{\tau + 1}} \binom{t}{\tau} \frac{t \mod (\tau + 1)}{\tau + 1} \right) \leq \chi_{2\tau+1}(Q_t) \leq 2^{1+\lceil \log_2 \tau \rceil} \leq 2 \cdot (2t - 2)^\tau
\]

for odd \( d = 2\tau + 1 \), otherwise for even \( d = 2\tau \),

\[
\sum_{i=0}^{\tau} \binom{t}{i} + \frac{1}{\binom{t-1}{\tau + 1}} \binom{t}{\tau} \frac{(t+1) \mod (\tau + 1)}{\tau + 1} \leq \chi_{2\tau}(Q_t) \leq 2^{\lceil \log_2 (t+1) \rceil} \leq (2t)^\tau,
\]

\[
t \cdot \left( \sum_{i=0}^{\tau} \binom{t-1}{i} + \frac{1}{\binom{t-1}{\tau + 1}} \binom{t}{\tau} \frac{t \mod (\tau + 1)}{\tau + 1} \right) \leq \chi_{2\tau+1}(Q_t) \leq t 2^{\lceil \log_2 \tau \rceil} \leq t (2t - 2)^\tau
\]

for odd \( d = 2\tau + 1 \), otherwise for even \( d = 2\tau + 2 \),

\[
2t \left( \sum_{i=0}^{\tau} \binom{t-2}{i} + \frac{1}{\binom{t-2}{\tau + 1}} \binom{t-2}{\tau} \frac{(t-1) \mod (\tau + 1)}{\tau + 1} \right) \leq \chi_{2\tau+2}(Q_t) \leq 2t \cdot 2^{\lceil \log_2 (t-1) \rceil} \leq 2t \cdot (2t - 4)^\tau;
\]
or equivalently,
\[ v(G) - z(G) \leq \chi_d(Q_d) - \chi(G) + \tilde{R}(t, d + 1) \]
\[ \leq \chi_d(Q_d) - \chi(G) + 2^{\lceil \log_2(t - 1 + (d - 1)/2) \rceil} \cdot \lfloor d/2 \rfloor \]
\[ \leq \chi(d + 1) \cdot (2t - 2(d - 1)/2)] \]

where, by (7), \( \varepsilon(d + 1) = 2 - (d \mod 2) \) whence \( \varepsilon(d + 1) = 1 + (d \mod 2) \).

This, due to (13), implies Theorem 1 for \( \chi_d \) and \( q_d \).

6.3. Bounds on multicolorings

The following result of basic importance is presented by Scheinerman and Ullman in their monograph [30, Sections 3.1 and 1.3].

Theorem 16. Let \( G \) be a loopless graph. The following chain of inequalities hold:

\[ \frac{v(G)}{z(G)} \leq \chi_f(G) \leq \chi_b(G) \leq \chi(G) \]

with equality for vertex-transitive \( G \), with equality for infinitely many \( b \)'s,

Due to formulas (1), using this result for \( G = Q_d^t \) or \( L(Q_t)^d \) (both graphs being vertex-transitive) completes the proof of Theorem 1.

7. Concluding remarks

Multicoloring was introduced for planar maps by Meyer [26] as a means for strengthening the five-color theorem in order to approach the four-color conjecture, still open at that time. Further approach was made by Hilton et al. [14] in terms of the fractional chromatic number defined by means of infimum as in formula (2) above.

Distance vertex colorings were initiated by Kramer and Kramer (cf. [18]) in the late sixties, for distance edge colorings, see [31]. Finding close formulas for distance chromatic parameters in seemingly easy cases can be a really hard problem. For instance, Berge’s elegant formula for the chromatic index of a cyclic multigraph (comprising a cycle \( C_n \) with arbitrarily multiplied edges) has not been extended yet, not even to the distance bound 2, cf. [11].

Applications in optical networks inspired the study of distance colorings of vertices in hypercubes, a few resulting papers [17, 27, 33], with bounds on chromatic number only, being published in the period 1997–2002. My related bounds of 1995 have been neither cited nor improved there yet.

In case of hypercubes it is quite interesting that a seemingly sparse subgraph of the \( d \)th power of the growing cube \( Q_j \), as \( t \to \infty \), can have the chromatic number of the same order as \( Q_d^t \) has. Let \( Q_d^{2^j} \) denote the spanning subgraph of \( Q_d^t \) such that two vertices are adjacent in \( Q_d^{2^j} \) whenever their distance in \( Q_d^t \) is exactly \( d, d \leq t \). Hence \( Q_d^{2^j} \) is bipartite if \( d \) is odd. Otherwise Linial et al. [21] prove that for any fixed even \( d \), \( \chi(Q_d^{2^j}) = \Theta(t^{d/2}) \), cf. Theorem 1. They note that the independence number \( \alpha(Q_d^{2^j}) = 2A(t - 1, 3) \) because \( Q_d^{2^j} \) splits into two isomorphic components both comprising vertices whose weights are of the same parity. On the other hand, \( \alpha(Q_d^t) = A(t, 3) \leq \alpha(Q_d^{2^j}) = 2A(t - 1, 3) \) due to (3), with equality if \( t \) is just below a power of 2, e.g., if \( 2^m - 3 \leq t < 2^m - 1 \) because, due to properties of the \( j - 1 \) times shortened binary Hamming code (see [2, 23 Chapter 17, pp. 533, 545, 566]), \( A(t, 3) = B(t, 3) = 2^{t-m} \) for \( t = 2^m - j \) with \( j = 1, 2, 3, 4 \). Hence \( \chi(Q_d^{2^j}) \leq \chi(Q_d^t) \) and equality does not prevail though holds for infinitely many values of \( t \).
Theorem 17. Parameters $q_d(Q_t)$ and $\chi_d(Q_t)$ grow exponentially as $t \to \infty$ and $d/t \geq \lambda$ for a positive constant $\lambda$.

The method of coloring the edges, presented in Section 6.1, gives the following.

Proposition 18. $q_d(Q_t) \leq t \cdot \chi_d^{-1}(Q_{t-1})$.

Equality therein holds for $d = 1, 2$.

Problem. What about strong inequality in this Proposition?

Theorem 16 and formulas (1), (15) and (16) imply that

$$\frac{t2^{t-1}}{A(t-1, d)} \leq q_{t,d}(Q_t) \leq q^b_{t,d}(Q_t) \leq q_d(Q_t) \leq \frac{t2^{t-1}}{B(t-1, d)},$$

where modifications $Q_t \leftarrow L(Q_t)$ together with $q \leftarrow \chi$ are possible, and

$$\frac{2^t}{A(t, d+1)} \leq \chi_d(Q^d_t) \leq \chi^b_d(Q^d_t) \leq \chi(Q^d_t) \leq \frac{2^t}{B(t, d+1)},$$

cf. (1) and Remark 1.1 to see possible formal modifications.

A few exact values are known. Most of them are implied by equalities of parameters $B$ and $A$ in either of the above inequalities. For instance, $B(n, d) = A(n, d) = 2^{n+1-d}$ if $d = 1, 2$. The equality $B = A$ for $d = 3$ is ensured by Hamming codes and by some of their shortenings for some $j$ in $n = 2^m - 1 - j$ ($j = 0, 1, 2, 3$, see [2, 23, Chapter 17, pp. 533, 545, 566]) whence for $d = 4$ and $n = 2^m - j$ due to Theorem 5. Quite a different example is the (nonlinear) punctured Preparata code $P(2r)^*(r)$ (with $r > 1$) which is [23, pp. 471–475] an $(n, 2^{n+1-4r}, 5)$ code whose size equals $A(n, 5)$ with $n = 4^r - 1$ and whose some translates as color classes can give equalities above with $\chi(Q^d_n) = (n + 1)^2/2$ and $\chi(Q_{n+1}) = (n + 1)^3/2$ for $n = 4^r - 1 \geq 15$ only. This suggests the open problem of replacing Theorem 8 by a general code-coloring bound.

Addition: Code-coloring bound has been improved considerably in [28] for any $n$ and $d = 3$ by using colorings with asymptotically perfect $(n, M, 3)$ codes as color classes. Those colorings stem from asymptotically perfect packings and coverings with unit spheres, see [16] for construction.

References

[8] E.H. Farr, see c:cross-ref[23, Chapter 9, Theorem 9].
[24] H.B. Mann, On the number of information symbols in Bose–Choudhuri codes, Inform. and Control 5 (1962) 153–162 (see cross-ref[23, Chapter 9, Cor. 8]).