A classification of cubic $s$-regular graphs of order $14p$

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**Abstract**

A graph is $s$-regular if its automorphism group acts regularly on the set of its $s$-arcs. In this paper, we classify the cubic $s$-regular cubic graphs of order $14p$ for each $s \geq 1$ and each prime $p$.

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**1. Introduction**

Throughout this paper, graphs are finite, simple, undirected and connected. For a graph $X$, let $V(X)$, $E(X)$ and $\text{Aut}(X)$ denote the vertex set, the edge set and the full automorphism group of $X$, respectively. The arc set $A(X)$ of a graph $X$ is defined to be the set $\{(u, v), (v, u) \mid (u, v) \in E(X)\}$. For a vertex $v \in V(X)$, we denote $N(v)$ to be the set of vertices adjacent to $v$.

Let $X$ and $Y$ be graphs. A mapping $\alpha : V(X) \to V(Y)$ is a graph homomorphism (or homomorphism) if $\alpha(u)$ and $\alpha(v)$ are adjacent in $Y$ whenever $u$ and $v$ are adjacent in $X$. A surjective homomorphism $p : \tilde{X} \to X$ is a covering projection if $p|_{N(u)} : N(u) \to N(v)$ is a bijection for any $u, v \in V(X)$ and $v \in p^{-1}(u)$. The graph $X$ is usually referred to as the base graph and $\tilde{X}$ as the covering graph. By $\text{fib}_u = p^{-1}(u)$ and $\text{fib}_x = p^{-1}(x)$ we denote the fibre over $u \in V(X)$ and $x \in A(X)$, respectively. The group $\text{CT}(p)$ of all automorphisms of $\tilde{X}$ which fix each of the fibres setwise is called the covering transformation group. An automorphism of $\tilde{X}$ is said to be fibre-preserving if it maps a fibre to a fibre. All of fibre-preserving automorphisms form a group called the fibre-preserving group.

An $s$-arc in a graph $X$ is an ordered $(s + 1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices of $X$ such that $v_{i-1}$ is adjacent to $v_i$ for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i < s$. A graph $X$ containing at least one $s$-arc is said to be $s$-arc-transitive if $\text{Aut}(X)$ is transitive on the set of $s$-arcs in $X$. In particular, 0-arc-transitive means vertex-transitive, and 1-arc-transitive means arc-transitive or symmetric. A subgroup of the automorphism group of a graph $X$ is said to be $s$-regular if it acts regularly on the set of $s$-arcs of $X$. In particular, if the subgroup is the full automorphism group $\text{Aut}(X)$ of $X$ then $X$ is said to be $s$-regular. Thus, if a graph $X$ is $s$-regular then $\text{Aut}(X)$ is transitive on the set of $s$-arcs and the only automorphism fixing an $s$-arc is the identity automorphism of $X$.

A covering projection $p : \tilde{X} \to X$ is said to be a regular covering projection if the covering transformation group $\text{CT}(p)$ acts regularly on each fibre. If $\text{CT}(p)$ is isomorphic to an elementary abelian $p$-group, then the regular covering projection is called $p$-elementary abelian. An automorphism $\alpha \in \text{Aut}(X)$ lifts along $p$ if there exists an automorphism $\tilde{\alpha} \in \text{Aut}(\tilde{X})$ such that $\alpha p = p\tilde{\alpha}$. In this case we also say that $p$ is $\alpha$-admissible. A subgroup $G \leq \text{Aut}(X)$ lifts along $p$ if each $\alpha \in G$ lifts. The set of all lifts $G$ forms a group $\text{lift}(G) \leq \text{lift}(\tilde{X})$, called the lift of $G$. A regular covering projection $p$ is arc-transitive if some arc-transitive subgroup of $\text{Aut}(X)$ lifts along $p$.

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Two regular covering projections \( p : \tilde{X} \to X \) and \( p' : \tilde{X}' \to X \) of a graph \( X \) are isomorphic if there exist an automorphism \( \alpha \in \text{Aut}(X) \) and an isomorphism \( \tilde{\alpha} : \tilde{X} \to \tilde{X}' \) such that \( \alpha p = p' \tilde{\alpha} \). In particular, if \( \alpha \) is the identity automorphism of \( X \), then we say that \( p \) and \( p' \) are equivalent.

Let \( X \) be a connected graph and \( A \) a finite group, called the voltage group. Assign to each arc \((u, v)\) of \( X \) a voltage \( \xi(u, v) \in A \) such that \( \xi(u, v) = \xi(v, u)^{-1} \). This function \( \xi \) is called an (ordinary) voltage assignment of \( X \). Let \( \text{Cov}(X, \xi) \) be the derived graph with vertex set \( V(X) \times A \) and adjacency relation defined by \((u, a) \sim (v, a^\xi(u, v))\) whenever \( u \sim v \) in \( X \). Then the first coordinate projection \( p_1 : \text{Cov}(X, \xi) \to X \) is regular. Given a spanning tree \( T \) of the graph \( X \), a voltage assignment \( \xi \) is called \( T \)-reduced if the voltages on the tree arcs are the identity.

Tutte [21,22] showed that every finite cubic symmetric graph is \( s \)-regular for some \( s \geq 1 \), and this \( s \) is at most five. Since cubic graphs must have an even number of vertices, we must cubic symmetric graphs. It means that every cubic symmetric graph has an order of the form \( 2mp \) for a positive integer \( m \) and a prime number \( p \). In order to know all cubic symmetric graphs, we need to classify the cubic \( s \)-regular graphs of order \( 2mp \) for a fixed positive integer \( m \) and each prime \( p \). Conder and Dobcsányi [4,5] classified the cubic \( s \)-regular graphs up to order 2048 with the help of the “Low index normal subgroups” routine in MAGMA system [2]. Cheng and Oxley [3] classified the cubic \( s \)-regular graphs of order \( 2p \) (in fact, they classified the symmetric graphs of order \( 2p \) with any valency). Feng et al. [8–13] classified the cubic \( s \)-regular graphs of order \( 2p^2 \), \( 4p \), \( 6p \), \( 8p \) and \( 10p \left( 1 \leq i \leq 2 \right) \). In this paper, the cubic \( s \)-regular graphs of order \( 14p \) are classified for each \( s \geq 1 \) and each prime \( p \).

The following proposition is known as Burnside’s \( p^a q^b \)-Theorem.

**Proposition 1.1** ([20, Theorem 8.5.3]). Let \( p \) and \( q \) be primes and let \( a \) and \( b \) be non-negative integers. Then every group of order \( p^aq^b \) is soluble.

Let \( X \) be a graph and let \( N \) be a subgroup of \( \text{Aut}(X) \). Denote by \( X \) the quotient graph corresponding to the orbits of \( N \), that is the graph having the orbits of \( N \) as vertices with two orbits adjacent in \( X \) whenever there is an edge between those orbits in \( X \).

**Proposition 1.2** ([18, Theorem 9]). Let \( \tilde{X} \) be a connected symmetric graph of prime valency and \( G \) an \( s \)-arc-transitive subgroup of \( \text{Aut}(\tilde{X}) \) for some \( s \geq 1 \). If a normal subgroup \( N \) of \( G \) has more than two orbits, then it is semiregular and \( G/N \) is an \( s \)-arc-transitive subgroup of \( \text{Aut}(\tilde{X}) \) where \( \tilde{X} \) is the quotient graph of \( X \) corresponding to the orbits of \( N \). Furthermore, \( X \) is a regular covering of \( \tilde{X} \) with the covering transformation group \( N \).

**Proposition 1.3** ([6, Proposition 3.4]). Every \( 5 \)-regular cubic graph having \( 1 \)-regular subgroup is a covering graph of the Biggs–Conway graph (of order 2352), and in particular, is bipartite.

**Proposition 1.4.** Let \( p \geq 5 \) be prime and \( m \) be a positive integer such that \( \gcd (p, m) = 1 \). Let \( X \) be a cubic symmetric graph of order \( 2m \). Assume that \( p_i (i = 1, 2) : \tilde{X}_i \to X \) are arc-transitive \( p \)-elementary abelian covering projections and each \( \text{CT}(p_i) \) is normal in \( \text{Aut}(\tilde{X}_i) \). If \( \tilde{X}_1 \) and \( \tilde{X}_2 \) are not-isomorphic as covering graphs, then \( \tilde{X}_1 \) and \( \tilde{X}_2 \) are also non-isomorphic as graphs.

**Proof.** Suppose that there exists a graph isomorphism \( \tilde{\alpha} : \tilde{X}_1 \to \tilde{X}_2 \). Then the mapping \( p_\alpha : \tilde{X}_1 \to X \) is a regular covering projection whose covering transformation group is isomorphic to \( \text{CT}(p_\alpha) \). By the normality of \( \text{CT}(p_1) \), there is a unique Sylow \( p \)-subgroup, say \( B \), of \( \text{Aut}(\tilde{X}_1) \), which is actually isomorphic to \( \text{CT}(p_1) \). Let \( \tilde{X}_1/B \) be the quotient graph of \( \tilde{X}_1 \) corresponding to the orbits of \( B \). Since \( p_1 \) and \( p_2 \) are regular covering projections with the covering transformation group \( \text{CT}(p_i) \), there exist graph automorphisms \( i : \tilde{X}_1/B \to X \) and \( j : \tilde{X}_1/B \to X \) such that \( iq_0 = p_1 \) and \( jq_0 = p_2 \alpha \) where \( q_0 : \tilde{X}_1 \to \tilde{X}_1/B \) is the quotient map (see [16]). Then \( ij = p_1 = p_2 \alpha \), and hence \( \tilde{X}_1 \) and \( \tilde{X}_2 \) are isomorphic as covering graphs too. \( \square \)

2. A classification of cubic \( s \)-regular graphs of order \( 14p \)

The Heawood graph \( F_{14} \) is the unique cubic symmetric graph of order 14, which is actually 4-regular (see [17, p.173] and [5]). The graph \( F_{14} \) is illustrated in Fig. 1. We choose \( \alpha = (1 2)(3 14)(4 13)(5 12)(6 11)(7 10)(8 9) \), \( \beta = (1 12)(2 7 14 11 6 13)(3 8 9 10 5 4) \) and \( \gamma = (5 13)(6 14)(7 9)(10 12) \) as automorphisms of \( F_{14} \). Then one can check by GAP [15] that each proper arc-transitive subgroup of \( \text{Aut}(F_{14}) \) is conjugate in \( \text{Aut}(F_{14}) \) to the 1-regular group \( H = \langle \alpha, \beta \rangle \). Furthermore, \( \text{Aut}(F_{14}) = \langle \alpha, \beta, \gamma \rangle \) and \( H \) is a maximal subgroup of \( \text{Aut}(F_{14}) \).

Let \( T \) be a spanning tree of the graph \( F_{14} \) consisting of the edges

\[
\{1, 6\}, \{1, 14\}, \{2, 11\}, \{3, 8\}, \{4, 13\}, \{5, 10\}, \{7, 12\}, \{8, 9\}, \{9, 14\}, \{10, 11\}, \{11, 12\}, \{12, 13\}, \{13, 14\}.
\]

We orient the cotree arcs by setting

\[
x_1 = (1, 2), \quad x_2 = (2, 3), \quad x_3 = (3, 4), \quad x_4 = (4, 5), \quad x_5 = (5, 6),
\]
\[
x_6 = (6, 7), \quad x_7 = (7, 8), \quad x_8 = (9, 10).
\]

Malnič et al. [19] determined all pairwise non-isomorphic semisymmetric \( p \)-elementary abelian covers of the Heawood graph. Even though the arc-transitive covers were not explicitly given in [19], they can be easily deduced. Instead of repeating
Each non-trivial arc-transitive $p$-elementary abelian covering projection of the Heawood graph $F_{14}$ is isomorphic to a derived covering projection associated with one of the pairwise non-isomorphic $T$-reduced voltage assignments given in Tables 1 and 2 where the element $a$ in Table 1 runs over non-zero elements of the cyclic group $\mathbb{Z}_7$.

Furthermore, if $p = 2$, there exist 3 non-isomorphic covering projections which lift arc-transitive subgroups of $\text{Aut}(F_{14})$; if $p = 3, 5$ such ones; if $p = 7, 20$ such ones; if $p = 1 \mod (6)$ and $p \neq 7, 7$ such ones; otherwise, 3 such ones.

**Corollary 2.2.** Let $p$ be a prime.

1. Let $\tilde{X}_1$ be a $p$-elementary abelian($p > 7$) cover of $F_{14}$ in Table 2 whose covering projection admits a lift of $H$ as a maximal one. Then $\tilde{X}_1$ is 1-regular.

2. Let $\tilde{X}_2$ be a $p$-elementary abelian cover of $F_{14}$ in Tables 1 and 2 whose covering projection admits a lift of $\text{Aut}(F_{14})$. Then $\tilde{X}_2$ is 4-regular.

**Proof.** (1) Let $p : \tilde{X}_1 \to F_{14}$ be an associated covering projection and $A = \text{Aut}(\tilde{X}_1)$. Suppose to the contrary that $\tilde{X}_1$ is $s$-regular for some $s \geq 2$. By Tutte [21,22], $s \leq 5$ and so $|A| = 14p^i \cdot 48$ where $p \geq 11$ and $i = 1, 2, 6, 7$. Thus, $K := \text{CT}(p)$ is a Sylow $p$-subgroup of $A$. Let $B$ be the 1-regular subgroup of $\text{Aut}(\tilde{X}_1)$ lifted by $H = \langle \alpha, \beta \rangle$. Then $|B| = 14 \cdot 3 \cdot p^i$. The normality of $K$ in $B$ implies that $B \leq N_A(K)$, where $N_A(K)$ is the normalizer of $K$ in $A$. Since $X$ is at most 5-regular, $|A : N_A(K)| \mid 16$. By Sylow’s theorem, the number of Sylow $p$-subgroups of $A$ is $np + 1$ and $np + 1 = |A : N_A(K)|$. Thus, $np + 1 \mid 16$. Since $p \geq 11$, we have $np + 1 = 1$, and hence $K$ is normal in $A$. By Proposition 1.2, $A/K$ is an s-regular subgroup of the automorphism group of the Heawood graph $F_{14}$. This is impossible because otherwise 4-regular group $\text{Aut}(F_{14})$ lifts.

(2) Notice that $\text{Aut}(\tilde{X}_2)$ contains a 1-regular subgroup and a 4-regular subgroup. If $\tilde{X}_2$ were 5-regular, then by Proposition 1.3, it would be a covering of the Biggs-Conway graph of order 2352. It implies that $2352 = 2^3 \cdot 3 \cdot 7^2 \mid 14p^i$ for $(p, i) = (2, 8), (3, 7), (3, 8), (7, 3), (7, 8)$, a contradiction. Hence $\tilde{X}_2$ is 4-regular. □

By [5], one can see that there are eight cubic symmetric graphs of order $14p$ for $p \leq 13$, which are the 3-regular graph $F_{28}$ of order 28, the 1-regular graph $F_{24}$ of order 42, the 1-regular graph $F_{98A}$ of order 98, the 2-regular graph $F_{98B}$ of order 98, the 1-regular graph $F_{182A}$ of order 182, the 1-regular graph $F_{182B}$ of order 182, the 2-regular graph $F_{182C}$ of order 182 and the 3-regular graph $F_{182D}$ of order 182 (for explicit constructions, see [1,11,14]).

Let $C_{F_{14p}}$ and $D_{F_{14p}}$ be the derived graphs from the voltage assignments in the rows 2 and 3 of Table 2, respectively, where $p > 7$ and $p = 1 \mod (6)$. By Corollary 2.2(1) and Proposition 1.4, $C_{F_{14p}}$ and $D_{F_{14p}}$ are non-isomorphic as graphs.

**Theorem 2.3.** Let $p$ be a prime and let $X$ be a cubic symmetric graph of order $14p$. Then, $X$ is 1-, 2- or 3-regular. Furthermore,

1. $X$ is 1-regular if and only if $X$ is isomorphic to one of the graphs $F_{42}$, $F_{98A}$, $C_{F_{14p}}$ and $D_{F_{14p}}$ where $p > 7$ and $p = 1 \mod (6)$.
2. $X$ is 2-regular if and only if $X$ is isomorphic to one of the graphs $F_{98B}$ and $F_{182C}$.
3. $X$ is 3-regular if and only if $X$ is isomorphic to one of the graphs $F_{28}$ and $F_{182D}$.

**Remark** In fact, the graphs $C_{F_{14p}}$ and $D_{F_{14p}}$ are Cayley graphs on dihedral groups (see [14]).
Table 1
Arc-transitive $p$-elementary abelian covers of $F_{14}$

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<td>$H$</td>
<td>$p = 2$</td>
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</tbody>
</table>

Proof. Let $X$ be a cubic symmetric graph of order $14p$. We have already proved the theorem for $p \leq 13$. From now on we assume that $p > 13$. Let $A \subseteq Aut(X)$ and $P$ a Sylow $p$-subgroup of $A$. If $P$ is normal in $A$, by Proposition 1.2 $X$ is a regular covering of the Heawood $F_{14}$ with the covering transformation group $\mathbb{Z}_p$, and the normality of $P$ implies that the fibre-preserving group is arc-transitive. By Proposition 2.1, $X$ is isomorphic to one of the graphs $CF_{14p}$ and $DF_{14p}$. Thus, it suffices to show that $P$ is normal in $A$. 

Note: The table and proof are from a research paper, and the full context is not provided in this snippet.
Let $N_A(P)$ be the normalizer of $P$ in $A$. By Sylow's theorem, the number of Sylow $p$-subgroups of $A$ is $np + 1 = |A : N_A(P)|$. Since $X$ is at most 5-regular, $|A|$ is a divisor of $48 \cdot 14p$. Thus, $np + 1$ is a divisor of $48 \cdot 14 = 672$. Suppose to the contrary that $P$ is not normal in $A$. Since $np + 1 \geq 18$ and $np + 1 \mid 672 = 2^5 \cdot 3 \cdot 7$, we have $(p, n) = (19, 5), (23, 1), (37, 3), (41, 1), (47, 1), (61, 11), (67, 5), (83, 1), (167, 1)$ or (223, 1). Furthermore, since there exist no cubic symmetric graphs of order $14p$ for $p = 23, 41, 47$ or 83 by the result in [4], we have $(p, n) = (19, 5), (37, 3), (61, 11), (67, 5), (167, 1)$ or (223, 1). Since $|A : N_A(P)| = np + 1$, one can see that in all these cases, the order of $A$ is divisible by $2^5 \cdot 3 \cdot 7p$, which implies that $X$ is at...
least 3-arc-transitive. Let $M$ be a minimal normal subgroup of $A$ and $X$ the quotient graph of $X$ corresponding to the orbits of $M$.

If $M$ is elementary abelian then $X$ is 3-arc-transitive with order $2p$. By the result in [5], one can see that there exist no such cubic symmetric graphs of order $2p$ for $p = 19$, 37, 61, 67, 167 or 223. Thus, one may assume that $M = T_1 \times T_2 \times \cdots \times T_t$, where $T_i (1 \leq i \leq t)$ are isomorphic non-abelian simple groups. By Proposition 1.1, $|T_i|$ has at least three prime factors. Notice that $|A|$ is a divisor of $2^\ell \cdot 3 \cdot 7p$ where $p = 19$, 37, 61, 67, 167 or 223. Then $t = 1$ and $M$ is a non-abelian simple group. Thus, $M$ has order $2^\ell \cdot 3 \cdot 7$, $2^\ell \cdot 7p$, $3 \cdot 7p$ or $2^\ell \cdot 3 \cdot 7p$ for some $1 \leq \ell \leq 5$. By inspecting the orders of all finite simple groups (see [7]), one can see that $M$ has order 168 and $M \cong \text{PSL}(2,7)$. Since $|M|$ has a divisor 3, $M$ is not semiregular and this is impossible by Proposition 1.2. □

Acknowledgments

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References