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Random planar graphs

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Abstract

We study various properties of the random planar graph R_n, drawn uniformly at random from the class $\mathscr{P}_n$ of all simple planar graphs on n labelled vertices. In particular, we show that the probability that R_n is connected is bounded away from 0 and from 1. We also show for example that each positive integer k, with high probability R_n has linearly many vertices of a given degree, in each embedding R_n has linearly many faces of a given size, and R_n has exponentially many automorphisms.
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1. Introduction

Let $\mathcal{P}_n$ denote the class of all simple planar graphs on the vertices $1, \ldots, n$, and let R_n denote a graph drawn uniformly at random from this class. The random planar graph R_n was introduced in [5]. We are interested here in the probability that R_n is connected, the number of vertices of a given degree, the number of faces of a given size in an embedding, the existence of given (planar!) subgraphs, and so on. It turns out that the rate of growth of $\ell(n) = |\mathcal{P}_n|$ is of central importance to our investigations.

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Let us sketch some relevant background, concerning the numbers of planar graphs and the numbers of edges in such graphs, before going on to give an outline of the paper.

First consider the number of unlabelled planar graphs. Let $u(n)$ denote the number of unlabelled simple planar graphs on n nodes, that is, the number of isomorphism classes of graphs in $\mathcal{P}_n$. It is shown by Denise et al. [5] that there is a constant γ_u, the *unlabelled planar graph growth constant*, such that $u(n)^{1/n} \to \gamma_u$ as $n \to \infty$; and, using work of Tutte [14] on counting triangulations, that $9.48 < \gamma_u < 75.9$. The upper bound was recently reduced to $\gamma_u \leqslant 32.2$ by Bonichon et al. [4].

Now let us return to considering labelled planar graphs. We shall see in Theorem 3.2 below that there is a constant γ_ℓ such that

$$(\ell(n)/n!)^{1/n} \to \gamma_\ell \quad \text{as } n \to \infty. \tag{1}$$

We call γ_ℓ the *labelled planar graph growth constant*. Since $\ell(n) \leqslant n!\, u(n)$ we have $\gamma_\ell \leqslant \gamma_u$, and so from the above we have $\gamma_\ell \leqslant 32.2$. We shall see that in fact $\gamma_\ell < \gamma_u$. In [2] Bender et al. give an asymptotic formula for the number of 2-connected graphs in $\mathcal{P}_n$, which shows that $\gamma_\ell \geqslant 26.1$. Thus

$$26.1 \leqslant \gamma_l < \gamma_u \leqslant 32.2. \tag{2}$$

A focus in investigating random planar graphs has been the number of edges in such graphs. It is shown in [5] that the expected number of edges $\mathbb{E}\,[|E(R_n)|]$ is at least $(3n-6)/2$. This is improved by Gerke and McDiarmid [6] who show $\mathbb{E}\,[|E(R_n)|] \geqslant \frac{13}{7}n + o(n)$; and (note that $\frac{13}{7} > 1.85$) that $\Pr\,[|E(R_n)| \leqslant 1.85n] = O(e^{-\delta n})$ for some $\delta > 0$.

In Osthus et al. [9] it is shown that with probability $1 - o(1)$ we have $|E(R_n)| \leqslant 2.56n$. This upper bound is improved in [4] to $2.54n$. Numerical computations for $n = 1000$ using the method suggested in [5] indicate that both upper and lower bounds are weak and that the correct value is close to $2.2n$. In [4] it is shown that the same upper bound of $2.54n$ applies also to the number of edges in unlabelled planar graphs (and to connected labelled or unlabelled planar graphs). Further it is shown that unlabelled planar graphs have at least $1.7n$ edges with probability $1 - o(1)$ (and the same result holds for the connected case).

As one last piece of background on what is known about R_n, note that it has recently been shown by Bodirsky et al. [3] how to generate R_n exactly in time polynomial in n. See [1] and the references therein for a wealth of results on unlabelled connected planar maps (embedded in the plane) with a given number of edges. From now on, we shall consider only the labelled case, except in Corollaries 4.7 and 4.8.

Outline of the paper: In the next section we will show that the random planar graph R_n is connected with probability at least $1/e$. In Section 3 we will use this result to establish result (1) above on the growth rate of $\ell(n)$. From that we will deduce in Section 4 that with high probability R_n has linearly many vertices of a given degree, and in each embedding there are linearly many faces of a given size. In addition, we show that if H is a fixed planar graph then R_n contains with high probability linearly many vertex disjoint copies of H, proving a conjecture of Taraz [13]. We deduce also the result mentioned above, that $\gamma_\ell < \gamma_u$; and we see that with high probability R_n has exponentially many automorphisms, and similarly for random unlabelled planar graphs.

In Section 5, we show that the probability that R_n is connected is bounded away from one. In fact, we show that R_n contains an isolated vertex with probability bounded away from zero. Similarly, we deduce that the probability that R_n contains a fixed planar graph H as a connected component is also bounded away from zero. Finally we show, assuming the 'labelled planar graph isolated vertices conjecture' (Conjecture 3.4), that as $n \to \infty$ the probability that R_n is connected tends to a certain explicit constant, and similarly for the probability that R_n has a component H.

2. Connectivity I

In this section we will obtain bounds on the probability that a random planar graph is connected and bounds on the expected number of components.

Theorem 2.1. *The random number $\kappa(R_n)$ of components of the random planar graph R_n is stochastically dominated by $1 + X$ where X has the Poisson distribution with mean 1. In particular* $\Pr[R_n \ \mathit{connected}] \geqslant 1/e$ *and* $\mathbb{E}[\kappa(R_n)] \leqslant 2$.

Indeed, we have the following more general result, which immediately implies Theorem 2.1.

Theorem 2.2. *Let $\mathcal{C}$ be any non-empty finite set of graphs, such that for each graph G in $\mathcal{C}$, if u and v are vertices in distinct components of G then the graph obtained from G by adding an edge joining u and v is also in $\mathcal{C}$. Let R denote a graph sampled uniformly at random from $\mathcal{C}$. Then the random number $\kappa(R)$ of components of R is stochastically dominated by $1 + X$ where X has the Poisson distribution with mean 1. In particular,* $\Pr[R \ \mathit{connected}] \geqslant 1/e$ *and* $\mathbb{E}[\kappa(R)] \leqslant 2$.

We believe that the lower bound $1/e$ on $\Pr[R_n$ connected$]$ from Theorem 2.1 is too low and that the true value is closer to 0.95. Nevertheless, the bound from Theorem 2.2 might well be tight. It would therefore be interesting is to exhibit a natural class $\mathcal{C}$ of graphs which shows that $1/e$ is the best possible value. As a digression we note that if we take $\mathcal{C}$ in Theorem 2.2 to be the class $\mathcal{F}_n$ of all forests on n labelled vertices, then the corresponding (exact) result is

$$\lim_{n\to\infty} \Pr[\text{random forest in } \mathcal{F}_n \text{ is connected}] = e^{-1/2}.$$

This follows from the result in [12] that $|\mathcal{F}_n| \sim e^{1/2} n^{n-2}$.

In order to prove Theorem 2.2 we need two lemmas.

Lemma 2.3. *Consider a Markov chain (X_t), with countable state space S, transition probabilities $p_{uv} = \Pr[X_{t+1} = v | X_t = u]$, and with steady-state probabilities π_v. Let the function $f : S \to \{0, 1, 2, \ldots, m\}$ be such that if $p_{uv} > 0$ then $f(v) \leqslant f(u) + 1$, so that f has 'at most unit increase'. Let $S_k = \{s \in S : f(s) = k\}$ and let $\sigma_k = \sum_{v \in S_k} \pi_v$ for each*

$k = 0, 1, \ldots, m$. *Let* $a_1, \ldots, a_m \geqslant 0$ *be such that if* $S_k \neq \emptyset$ *then*

$$\sum_{v \in \cup_{j<k} S_j} p_{uv} \geqslant a_k \quad \text{for each } u \in S_k$$

and if $S_k = \emptyset$ *then* $a_k = 0$. *Also, let* $b_0, \ldots, b_{m-1} \geqslant 0$ *be such that if* $S_k \neq \emptyset$ *then*

$$\sum_{v \in S_{k+1}} p_{uv} \leqslant b_k \quad \text{for each } u \in S_k$$

and if $S_k = \emptyset$ *then* $b_k = 0$. *Then*

$$a_{k+1}\sigma_{k+1} \leqslant b_k \sigma_k \quad \text{for each } k = 0, \ldots, m-1.$$

Proof. For each $k = 1, 2, \ldots, m$ let $d_k = \sum_{u \in S_k} \left(\pi_u \sum_{v \in \cup_{j<k} S_j} p_{uv}\right)$: then $d_k \geqslant a_k \sigma_k$. Also, for each $k = 0, 1, \ldots, m-1$ let $u_k = \sum_{u \in S_k} \left(\pi_u \sum_{v \in S_{k+1}} p_{uv}\right)$: then $u_k \leqslant b_k \sigma_k$. Let $k \in \{0, 1, \ldots, m-1\}$ and consider the set $A = \cup_{j \leqslant k} S_j$. Since $x_{uv} = \pi_u p_{uv}$ defines a circulation on S, the flow u_k out of A equals the flow into A, which is at least d_{k+1}; and so $u_k \geqslant d_{k+1}$. Thus

$$b_k \sigma_k \geqslant u_k \geqslant d_{k+1} \geqslant a_{k+1} \sigma_{k+1}$$

as required. $\square$

Lemma 2.4. *Let* $0 \leqslant r_k \leqslant s_k$ *for each* $k \geqslant 0$, *and suppose that* $x_0 = y_0 = 1$, $x_k = \prod_{j=0}^{k-1} r_j$ *and* $y_k = \prod_{j=0}^{k-1} s_j$ *for* $k \geqslant 1$ *and suppose that* $\sum_{j \geqslant 0} y_j$ *is finite. Let* $p_k = x_k / (\sum_{j \geqslant 0} x_j)$ *and* $q_k = y_k / (\sum_{j \geqslant 0} y_j)$ *for* $k \geqslant 0$. *Then the distribution* (p_k) *is stochastically dominated by the distribution* (q_k).

Proof. It suffices to consider the case where $r_k = s_k$ for each $k \geqslant 0$ other than some $\hat{k}$. [For we could then move from $r_0, r_1, r_2, \ldots$ to $s_0, r_1, r_2, \ldots$ to $s_0, s_1, r_2, \ldots$ and so on, and use an approximation argument if $s_k \neq r_k$ for infinitely many $k \geqslant 0$.] Let $k \geqslant 0$ be an integer. We must show that $\sum_{j \leqslant k} p_j \geqslant \sum_{j \leqslant k} q_j$. Let $c = \sum_{j=0}^{\hat{k}} x_j = \sum_{j=0}^{\hat{k}} y_j$.

Suppose first that $r_{\hat{k}} = 0$. Then $x_{\hat{k}+1} = x_{\hat{k}+2} = \cdots = 0$. If $0 \leqslant k \leqslant \hat{k}$ then

$$\sum_{j \leqslant k} p_j = \frac{\sum_{j \leqslant k} x_j}{c} \geqslant \frac{\sum_{j \leqslant k} x_j}{\sum_{j \geqslant 0} y_j} = \frac{\sum_{j \leqslant k} y_j}{\sum_{j \geqslant 0} y_j} = \sum_{j \leqslant k} q_j,$$

as required. If $k > \hat{k}$, then $\sum_{j \leqslant k} p_j = 1 \geqslant \sum_{j \leqslant k} q_j$.

Now suppose that $r_{\hat{k}} \neq 0$. Let $\gamma = s_{\hat{k}} / r_{\hat{k}}$ so that $\gamma \geqslant 1$, and let $d = \sum_{j=\hat{k}+1}^{\infty} x_j$. Note that $\sum_{j \geqslant 0} x_j = c + d$ and $\sum_{j \geqslant 0} y_j = c + \gamma d$. If $0 \leqslant k \leqslant \hat{k}$ then

$$\sum_{j \leqslant k} p_j = \frac{\sum_{j \leqslant k} x_j}{c+d} \geqslant \frac{\sum_{j \leqslant k} x_j}{c + \gamma d} = \frac{\sum_{j \leqslant k} y_j}{c + \gamma d} = \sum_{j \leqslant k} q_j,$$

as required. Suppose then that $k > \hat{k}$. As $\sum_{j=\hat{k}+1}^{k} x_j \leqslant d$ we obtain

$$\sum_{j \leqslant k} p_j = \frac{c + \sum_{j=\hat{k}+1}^{k} x_j}{c+d} \geqslant \frac{c + \gamma \sum_{j=\hat{k}+1}^{k} x_j}{c+\gamma d} = \frac{c + \sum_{j=\hat{k}+1}^{k} y_j}{c+\gamma d}$$
$$= \sum_{j \leqslant k} q_j. \qquad \square$$

Proof of Theorem 2.2. Form a graph $G(\mathcal{C})$, with a vertex v_G for each graph G in $\mathcal{C}$, and with vertices v_G and v_H adjacent if they differ in exactly one edge. Let V denote the union of all the vertex sets of all the graphs in $\mathcal{C}$ and let $n = |V|$.

Perform a random walk on $G(\mathcal{C})$ as follows. Suppose that we are currently at a vertex v_G. Pick a pair e of distinct vertices uniformly at random from V. If e forms an edge in the current graph G and if the graph H obtained from G by deleting e is also in $\mathcal{C}$ then move to v_H; if e is not an edge in the current graph G and if the graph H obtained from G by adding e is also in $\mathcal{C}$ then move to v_H; and if neither of the above conditions hold then stay still. Since the transition matrix is symmetric, the uniform distribution is a stationary distribution for this random walk.

We shall apply Lemma 2.3. Let $f(v_G) = \kappa(G) - 1$ for each vertex v_G in $G(\mathcal{C})$. Then f has at most unit increase. Let m be the maximum value of $f(G)$ over $G \in \mathcal{C}$. Observe that S_k denotes the set of graphs from $\mathcal{C}$ which have exactly $k+1$ components, and for each $k = 0, 1, \ldots, m$ we have $S_k \neq \emptyset$ and so $\sigma_k = |S_k|/|\mathcal{C}| \neq 0$. Observe also that the transition probabilities p_{uv} of the Markov chain are either zero or equal to some fixed constant p_0 (which is equal to $1/\binom{n}{2}$). Hence, the constants a_k and b_k of Lemma 2.3 can be set as follows. For $k = 1, 2, \ldots, m$ let

$$a_k := p_0 \cdot \min_{G \in \mathcal{C} : \kappa(G) = k+1} \#\{e \notin E(G) \mid G + e \in \mathcal{C} \ \wedge \ \kappa(G+e) = k\}$$

and for $k = 0, 1, \ldots, m-1$ let

$$b_k := p_0 \cdot \max_{G \in \mathcal{C} : \kappa(G) = k+1} \#\{e \in E(G) \mid G - e \in \mathcal{C} \ \wedge \ \kappa(G-e) = k+2\}.$$

(Here we take the maximum over an empty set to be 0.) As the number of possible edges between two disjoint sets X and Y is $|X||Y|$, and if $0 < |X| \leqslant |Y|$ then $|X||Y| > (|X|-1)(|Y|+1)$, it follows that we may take $a_k = p_0 \cdot [\binom{k}{2} + k(n-k)]$. Since the number of edges in a spanning forest of a graph with $k+1$ components is $n-k-1$, we may take $b_k = p_0 \cdot [n-k-1]$. Note that for these values we have $b_k/a_{k+1} \leqslant 1/(k+1)$ for each $k = 0, 1, \ldots, m-1$. (See also Proposition 7 and Corollary 8 of [5].)

Let $r_k = \sigma_{k+1}/\sigma_k$ for $k = 0, 1, \ldots, m-1$ and let $r_k = 0$ for $k \geqslant n$. Also, let $s_k = 1/(k+1)$ for each $k \geqslant 0$. Then by Lemma 2.3

$$r_k = \sigma_{k+1}/\sigma_k \leqslant b_k/a_{k+1} \leqslant s_k$$

for each $k = 0, 1, \ldots, m-1$. Now $x_k = \sigma_k/\sigma_0$ and hence $p_k = \sigma_k$ for all $k \geqslant 0$ in the notation of Lemma 2.4, so that $\kappa(R_n) - 1$ has the distribution (p_k). Also, $q_k = e^{-1}(1/k!)$ for each $k \geqslant 0$, so that the distribution (q_k) is the Poisson distribution with mean 1. Hence the theorem follows from Lemma 2.4. $\square$

Actually, basically the same proof can be used to bound the probability that a random planar graph conditioned on having only components of size at least i is connected. We obtain the following extension of Theorem 2.2.

Theorem 2.5. *Let $\mathcal{C}$ be any non-empty finite set of graphs, such that for each graph G in $\mathcal{C}$, if u and v are vertices in distinct components of G then the graph obtained from G by adding an edge joining u and v is also in $\mathcal{C}$. Let R denote a graph sampled uniformly at random from $\mathcal{C}$. Then for each positive integer i, the random number of components of R of order at least i is stochastically dominated by $1 + X$, where X has the Poisson distribution with mean $1/i$. In particular, if each component of each graph in $\mathcal{C}$ has order at least i then* $\Pr[R\ connected] \geqslant e^{-1/i}$ *and* $\mathbb{E}[\kappa(R)] \leqslant 1 + 1/i$.

Proof. The only difference to the proof of Theorem 2.2 is that we take $f(v_G)$ to be the number of components of order at least i, and the values of a_k and b_k have to be redefined. The minimum value in the definition of a_k is now obtained for the graph having k components with exactly i vertices and one component with $n - ki$ vertices (if such a graph is in $\mathcal{C}$), yielding $a_k = p_0 \cdot [\binom{k}{2} i^2 + ki(n - ki)]$. Similarly, the maximum for b_k is obtained for a graph having k components with exactly i vertices and a tree component with $n - ki - 1$ edges (if such a graph is in $\mathcal{C}$), yielding $b_k = p_0 \cdot [(n - ki - 1 - 2(i - 1))]$. Hence, $b_k/a_{k+1} \leqslant 1/(i(k + 1))$ for each $k = 0, 1, \ldots, m - 1$, from which the claim now follows in exactly the same way as in the proof of Theorem 2.2. $\square$

3. The number of planar graphs

The following lemma, which is a basic property of superadditive functions (see for example [8, Lemma 11.6]) will be very useful in our next theorem.

Lemma 3.1. *Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that $f(i + j) \geqslant f(i) \cdot f(j)$ for all $i, j \in \mathbb{N}$, and let $c = \sup_n f(n)^{1/n}$ (where c could be ∞). Then $f(n)^{1/n} \to c$ as $n \to \infty$.*

Theorem 3.2. *There exists a finite constant $\gamma_\ell > 0$ such that*

$$\left(\frac{\ell(n)}{n!}\right)^{1/n} \to \gamma_\ell \quad as\ n \to \infty.$$

Recall that we call γ_ℓ the *labelled planar graph growth constant*.

We shall deduce Theorem 3.2 from a more general result. Consider a non-empty class $\mathcal{C}$ of finite graphs, which is closed under isomorphism. Call $\mathcal{C}$ *small* if there is a constant d such that the number $f(n)$ of graphs in $\mathcal{C}$ on the vertices $1, \ldots, n$ satisfies $f(n) \leqslant d^n n!$ for all n sufficiently large; that is, $\mathcal{C}$ is small if $\sup_n (f(n)/n!)^{1/n} < \infty$. Call $\mathcal{C}$ *addable* if (a) a graph G is in $\mathcal{C}$ if and only if each component of G is in $\mathcal{C}$; and (b) for each graph G in $\mathcal{C}$, if u and v are vertices in distinct components of G then the graph obtained from G by adding an edge joining u and v is also in $\mathcal{C}$. For example, we saw earlier that the class of forests is small, and clearly it is addable.

Theorem 3.3. *Let $\mathcal{C}$ be a non-empty class of finite graphs which is small and addable. Then there is a finite constant $c > 0$, the growth constant for $\mathcal{C}$, such that*

$$\left(\frac{f(n)}{n!}\right)^{\frac{1}{n}} \to c \ \text{as } n \to \infty.$$

Proof. We let $g(n) := f(n)/(e^2 \cdot n!)$ for each $n \geqslant 1$. Let $c = \sup_n g(n)^{1/n}$. Then $c < \infty$ since $\mathcal{C}$ is small; and $c > 0$ since $\mathcal{C}$ is non-empty and so $g(n) > 0$ for some n. Denote by $f_c(n)$ the number of connected graphs on the vertices $1, \ldots, n$ which are in $\mathcal{C}$. Note that Theorem 2.2 (applied to the graphs in $\mathcal{C}$ with vertices $1, \ldots, n$) implies that $f_c(n) \geqslant f(n)/e$. Now $f(i+j) \geqslant \binom{i+j}{i} \cdot f_c(i) f_c(j)$ for all $i, j \in \mathbb{N}$: this is clear if $i \neq j$, and if $i = j$ we could add an edge between the two components so that we do not need to divide by two. We deduce that

$$g(i+j) = \frac{f(i+j)}{e^2(i+j)!} \geqslant \frac{1}{e^2 \cdot i! \cdot j!} \cdot f_c(i) f_c(j) \geqslant \frac{f(i)}{e^2 \cdot i!} \cdot \frac{f(j)}{e^2 \cdot j!} = g(i) \cdot g(j).$$

The theorem now follows from Lemma 3.1. [Observe that by the definition of c we also have $f(n) \leqslant e^2 c^n n!$ for each positive integer n.] $\square$

Proof of Theorem 3.2. Clearly, the class $\mathcal{P}$ of all labelled planar graphs is addable. As we saw in the introductory remarks, we have $\ell(n) \leqslant n! \cdot u(n) \leqslant n! \cdot c^n$ for an appropriate constant c and all sufficiently large n. Hence, $\mathcal{P}$ is also small and the theorem thus follows from Theorem 3.3. $\square$

As we will see shortly, the expected number I_n of isolated vertices in R_n plays an important role. Clearly,

$$I_n \ = \ n \cdot \Pr[v_n \text{ is isolated in } R_n] \ = \ n \cdot \frac{\ell(n-1)}{\ell(n)}.$$

From [5] we know that $\ell(n) \geqslant (6n - 15)\ell(n-1)$ for $n \geqslant 5$, so that

$$I_n \leqslant \frac{n}{6n - 15} \qquad \text{for } n \geqslant 5.$$

In Theorem 5.1 below we show that

$$I_n \geqslant \alpha e^{-1} + o(1)$$

for an appropriate constant $0 < \alpha < 1$. That is, I_n is contained in the interval $[\alpha e^{-1} - \varepsilon, 1/6 + \varepsilon]$ for all $\varepsilon > 0$ and n sufficiently large. This prompts:

Conjecture 3.4 (*The labelled planar graph isolated vertices conjecture*). *I_n tends to a limit as $n \to \infty$.*

Note that, if this conjecture is true, then in fact the limit must be γ_ℓ^{-1}, where γ_ℓ is the labelled planar graph growth constant, since

$$\frac{1}{n}\sum_{j=2}^{n}\log I_j = \frac{1}{n}\log\frac{n!}{\ell(n)} \to -\log\gamma_\ell.$$

Thus an equivalent conjecture is that

$$I_n \to \gamma_\ell^{-1} \quad \text{as } n \to \infty.$$

4. Degrees, faces, subgraphs

We turn now to questions such as, how many vertices of degree k, or how many triangular faces does the random planar graph R_n typically have (in some embedding)? We start by proving a more general result.

Let H be a graph on the vertex set $\{1,\ldots,h\}$, and let G be a graph on the vertex set $\{1,\ldots,n\}$ where $n > h$. Let $W \subset V(G)$ with $|W| = h$, and let the 'root' r_W denote the least element in W. We say that H *appears* at W in G if (a) the increasing bijection from $\{1,\ldots,h\}$ to W gives an isomorphism between H and the induced subgraph $G[W]$ of G; and (b) there is exactly one edge in G between W and the rest of G, and this edge is incident with the root r_W. We let $f_H(G)$ be the number of appearances of H in G, that is the number of sets $W \subseteq V(G)$ such that H appears at W in G.

Theorem 4.1. *Let $\mathcal{C}$ be a non-empty class of finite graphs which is addable and small, with growth constant c. Let $\tilde{R}_n = \tilde{R}_n(\mathcal{C})$ be uniformly distributed over the graphs in $\mathcal{C}$ with vertex set $\{1,\ldots,n\}$. Let H be a connected graph on the vertex set $\{1,\ldots,h\}$ in $\mathcal{C}$, and let $\alpha = 1/(9e^2c^h(h+2)h!)$. Then there exists n_0 such that*

$$\Pr[f_H(\tilde{R}_n) \leqslant \alpha n] < e^{-\alpha n} \quad \textit{for all } n \geqslant n_0. \tag{3}$$

Before we prove this result, let us note that in particular it implies that there is a constant $\alpha > 0$ such that the probability that R_n fails to have a subgraph K_4 is $O(e^{-\alpha n})$. This gives rise to a fast expected-time colouring algorithm for planar graphs as follows. First we check if there is a subgraph K_4, in linear time, see [10]. If there is one we apply the quadratic time algorithm to four-colour planar graphs, which follows from the proof of the four-colour theorem, see [11], to colour the graph optimally. In the remaining cases, which happen with probability at most $O(e^{-\alpha n})$, we colour the graph optimally in subexponential time $O(c^{\sqrt{n}})$ by using the $\sqrt{n}$-separator theorem. It follows that we can colour a random planar graph optimally in quadratic expected time. This observation is due to Anusch Taraz and Michael Krivelevich.

Proof of Theorem 4.1. We shall often write x instead of $\lfloor x \rfloor$ or $\lceil x \rceil$ to avoid cluttering up formulae. This should cause the reader no problems. Let $\beta = e^2c^h(h+2)h!$. Observe that the bound on α implies that $\alpha\beta < 1$ and we may therefore write $(\alpha\beta)^\alpha = 1 - 3\varepsilon$, where

$0 < \varepsilon < \frac{1}{3}$. Note that

$$\frac{1-\varepsilon}{(1-3\varepsilon)(1+\varepsilon)^2} > 1. \tag{4}$$

Let $f(n)$ denote the number of graphs in $\mathcal{C}$ on n vertices. By Theorem 3.3 there is a positive integer n_0 such that for each $n \geqslant n_0$ we have

$$(1-\varepsilon)^n \cdot n!\, c^n \leqslant f(n) \leqslant (1+\varepsilon)^n \cdot n!\, c^n. \tag{5}$$

Let $\delta = \alpha h$ and note that $\delta \leqslant 1$.

Assume that Eq. (3) does not hold for some $n \geqslant n_0$. We intend to show that then

$$f((1+\delta)n) > (1+\varepsilon)^{(1+\delta)n} \cdot [(1+\delta)n]! \cdot c^{(1+\delta)n},$$

contradicting (5). In order to see this, we construct graphs G' in $\mathcal{C}$ on vertex set $\{1, \ldots, (1+\delta)n\}$ as follows. First we choose a subset of δn special vertices ($\binom{(1+\delta)n}{\delta n}$ choices) and a graph $G \in \mathcal{C}$ on the remaining n vertices that satisfies $f_H(G) \leqslant \alpha n$. By assumption there are at least

$$e^{-\alpha n} \cdot f(n) \geqslant e^{-\alpha n}(1-\varepsilon)^n c^n n!$$

such graphs G. Next we consider the δn special vertices. We partition them into αn (unordered) blocks of size h. On each block B we put a copy of H such that the increasing bijection from $\{1, \ldots, h\}$ to B is an isomorphism between H and this copy. Call the lowest numbered vertex in B the *root* r_B of the block. For each block B we choose a non-special vertex v_B and add the edge $r_B v_B$ from the root to this vertex: observe that H appears at B in G'. This completes the construction of $G' \in \mathcal{C}$. For each choice of special vertices, and each $G \in \mathcal{C}$ on the remaining n vertices, we construct

$$\binom{\delta n}{h \cdots h} \cdot \frac{1}{(\alpha n)!} \cdot n^{\alpha n} = \frac{(\delta n)! n^{\alpha n}}{(h!)^{\alpha n}(\alpha n)!} \geqslant \frac{(\delta n)!}{(h!\alpha)^{\alpha n}}$$

graphs G'.

How often is the same graph G' constructed? Call an oriented edge $e = uv$ *good* in G' if it is a cut-edge in G', the component $\tilde{G}$ of $G' - e$ containing u has h nodes, u is the least of these nodes, and the increasing map from $\{1, \ldots, h\}$ to $V(\tilde{G})$ is an isomorphism between H and $\tilde{G}$. Observe that each added oriented edge $r_B v_B$ is good. Indeed, there is exactly one good oriented edge for each appearance of H in G. We shall see that G' contains at most $(h+2)\alpha n$ good oriented edges. It will then follow that the number of times that G' can be constructed is at most $\binom{(h+2)\alpha n}{\alpha n} \leqslant ((h+2)e)^{\alpha n}$.

We may bound the number of good edges in G' as follows. (a) There are exactly αn added oriented edges $r_B v_B$. (b) There are at most αn good oriented edges $e = uv$ in $E(G)$ (that is, such that the unoriented edge is in G): for in this case the entire component of $G' - e$ containing u must be contained in G (if it contained any other vertex it would have more than h vertices), and so the number of them is at most $f_H(G)$. (c) There are at most $h\alpha n$ 'extra' good oriented edges. To see this, consider a block B, and let $\tilde{H}$ denote the connected graph formed from the induced subgraph $G'[B]$ (which is isomorphic to H) together with the vertex v_B and the edge $r_B v_B$. Each 'extra' good oriented edge must be a cut edge in such a graph $\tilde{H}$ oriented away from v_B, and in each graph $\tilde{H}$ there are at most h cut-edges.

We may put the above results together to obtain

$$\begin{aligned} f((1+\delta)n) &\geqslant \binom{(1+\delta)n}{\delta n} \cdot e^{-\alpha n}(1-\varepsilon)^n c^n n! \cdot \frac{(\delta n)!}{(h!\alpha)^{\alpha n}} \cdot ((h+2)e)^{-\alpha n} \\ &= [(1+\delta)n]! \cdot c^{(1+\delta)n} \cdot (1-\varepsilon)^n \cdot \left(e^2 c^h (h+2) h! \alpha\right)^{-\alpha n} \\ &\overset{(5)}{\geqslant} f((1+\delta)n)\ (1+\varepsilon)^{-(1+\delta)n} \cdot (1-\varepsilon)^n \cdot (1-3\varepsilon)^{-n} \\ &\geqslant f((1+\delta)n) \left(\frac{1-\varepsilon}{(1-3\varepsilon)(1+\varepsilon)^2}\right)^n \\ &\overset{(4)}{>} f((1+\delta)n), \end{aligned}$$

yielding the desired contradiction. $\square$

We shall use the last result with $\mathcal{C}$ as the class $\mathcal{P}$ of all labelled planar graphs, which we have seen is addable and small. By choosing appropriate graphs H we are able to deduce that with high probability, the random planar graph R_n has (a) linearly many vertices of each given degree, (b) for all embeddings linearly many faces of each given size, and (c) exponentially many automorphisms. After that, we consider appearances of a given *plane* graph in R_n.

Comment: We might be interested in a random connected graph in $\mathcal{P}_n$, or perhaps a random connected or 2-vertex-connected graph in $\mathcal{P}_n$, and so on. For suitable definitions of 'addable' and 'appears' there are corresponding versions of the last theorem: we do not pursue such results here.

Theorem 4.2. *For a graph G let $d_k(G)$ denote the number of vertices with degree equal to k. There exists a constant $d > 0$ such that, for each positive integer k, if we set $\alpha_k = d/(\gamma_\ell^k(k+2)!)$, graph constant), then for all sufficiently large n*

$$\Pr[d_k(R_n) < \alpha_k n] \leqslant e^{-\alpha_k n}. \tag{6}$$

Proof. Consider the case when H is a star on $k+1$ vertices. More precisely, let H be the graph on vertices $\{1, \ldots, k+1\}$ in which vertex $k+1$ is connected to all other vertices. Since the appearances of H define distinct vertices of degree k, we have $d_k(G) \geqslant f_H(G)$, and so the result follows from Theorem 4.1. $\square$

Observe that if $h \geqslant 2$ and H is the star on vertices $\{1, \ldots, h\}$ in which (in contrast to the above proof) vertex 1 is connected to each other vertex, and if G is the star on vertices $\{1, \ldots, h+1\}$ in which vertex 1 is connected to each other vertex, then H appears h times in G, each time centred at vertex 1, and so the appearances of H are not vertex disjoint. However, if the graph H is 2-edge-connected then the appearances of H in any graph G have to be vertex disjoint.

Theorem 4.3. *For a planar graph G let $f_k(G)$ denote the number of faces of size k, minimized over all embeddings of G. There exists a constant $d' > 0$ such that, for each integer*

$k \geqslant 3$, *if we set* $\beta_k = d'/(\gamma_\ell^k (k+1)!)$, *then for all sufficiently large n*

$$\Pr[f_k(R_n) < \beta_k n] \leqslant e^{-\beta_k n}. \tag{7}$$

Proof. We again apply Theorem 4.1. This time we choose a k-cycle on vertices $\{1, \ldots, k\}$ as the graph H. Since H is 2-edge-connected, the appearances of H have to be vertex disjoint. Also, in each embedding of the graph R_n on the sphere, each appearance of H corresponds to a k-face of its component of the graph. Hence the number of k-faces in any plane embedding of R_n is at least the number of appearances of H less twice the number $\kappa(R_n)$ of components. But by Theorem 2.1, $\Pr[\kappa(R_n) \geqslant j] \leqslant 1/j!$, and the theorem follows. □

Let us briefly consider large vertex degrees and face-sizes. It is not difficult to check that Theorem 4.1 remains true if the order h of the graph H depends on n, as long as it does not grow too quickly. We obtain:

Proposition 4.4. *With probability* $1-o(1)$ *a random planar graph* R_n *has maximum degree* $\Omega(\log n/\log\log n)$, *and has the property that every embedding contains a face of size* $\Omega(\log n/\log\log n)$.

We may easily obtain an upper bound of about $\log_2 n$ on the maximum size of a face, to pair with the second half of the above result.

Proposition 4.5. *Let* $\omega(n) \to \infty$ *as* $n \to \infty$, *and let* $\delta(n)$ *be the probability that* R_n *has an embedding with a face of size at least* $\log_2 n + \omega(n)$. *Then* $\liminf_{n\to\infty} \delta(n) = 0$, *and if the labelled planar graph isolated vertices conjecture, Conjecture 3.4, holds then* $\delta(n) \to 0$ *as* $n \to \infty$.

Proof. If f denotes $\log_2 n + \omega(n)$, then

$$\ell(n+1) \geqslant \delta(n)\ell(n) \cdot 2^f = \delta(n)\ell(n)\, n\, \omega(n),$$

so that

$$\delta(n)\omega(n) \leqslant \frac{\ell(n+1)}{n\ell(n)},$$

and the result follows. □

Now we consider graph automorphisms. We let $aut(G)$ denote the number of automorphisms of a graph G.

Theorem 4.6. *There are constants* $\alpha, \beta, \gamma > 0$ *such that*

$$\Pr[2^{\alpha n} \leqslant aut(R_n) \leqslant 2^{\beta n}] = 1 - o(2^{-\gamma n}).$$

Proof. Let H be the graph on the vertex set $\{1, 2, 3\}$ with the two edges $\{1, 2\}$ and $\{1, 3\}$. Then the number of automorphisms $aut(G)$ is at least $2^{f_H(G)}$. Thus by Theorem 4.1, there are constants $\alpha, \gamma > 0$ such that

$$\Pr[aut(R_n) < 2^{\alpha n}] = o(2^{-\gamma n}).$$

Now consider the upper bound on $aut(R_n)$. Let $\beta > 0$ satisfy $2^{\beta-\gamma} > \gamma_u/\gamma_l$. The isomorphism class of a graph G in $\mathcal{P}_n$ (that is, the set of graphs in $\mathcal{P}_n$ isomorphic to G) has size $n!/aut(G)$. Thus if $aut(G) \geqslant 2^{\beta n}$ then the isomorphism class of G in $\mathcal{P}_n$ has size at most $n!/2^{\beta n}$. Hence

$$u(n) \geqslant \ell(n) \Pr[aut(R_n) \geqslant 2^{\beta n}] \cdot 2^{\beta n}/n!$$

and so

$$\Pr[aut(R_n) \geqslant 2^{\beta n}] \leqslant \frac{u(n)}{\ell(n)/n!} 2^{-\beta n} = \left(\frac{\gamma_u}{\gamma_l} 2^{-\beta} + o(1)\right)^n = o(2^{-\gamma n}). \qquad \square$$

From the lower bound on γ_ℓ and the upper bound on γ_u in (2), we see that we may choose $\beta = 0.31$.

Corollary 4.7. *The labelled planar graph growth constant γ_l and the unlabelled planar graph growth constant γ_u satisfy $\gamma_l < \gamma_u$.*

Proof. Again we use the observation that the isomorphism class of a graph G in $\mathcal{P}_n$ has size $n!/aut(G)$. Thus by the last theorem, the number of such graphs which are in isomorphism classes of size $> 2^{-\alpha n} n!$ is at most $2^{-\gamma n} \ell(n)$, which is at most $\frac{1}{2}\ell(n)$ for n sufficiently large. But then

$$u(n) \geqslant \tfrac{1}{2}\ell(n)/(2^{-\alpha n} n!),$$

that is

$$\ell(n)/n! \leqslant 2^{1-\alpha n} u(n);$$

and it follows that $\gamma_l \leqslant 2^{-\alpha}\gamma_u$. $\square$

Corollary 4.8. *Let U_n denote a graph sampled uniformly at random from the unlabelled simple planar graphs on n vertices. There is a constant $\delta > 0$ such that*

$$\Pr[aut(U_n) \leqslant 2^{\delta n}] = o(2^{-\delta n}).$$

Proof. The last result showed that $\gamma_l/\gamma_u < 1$. Let $\delta > 0$ satisfy $2^{-2\delta} > \gamma_l/\gamma_u$. Observe as before that if $G \in \mathcal{P}_n$ satisfies $aut(G) \leqslant 2^{\delta n}$ then the isomorphism class of G in $\mathcal{P}_n$ has size at least $2^{-\delta n} n!$. Hence

$$\ell(n) \geqslant u(n) \Pr[aut(U_n) \leqslant 2^{\delta n}] \cdot 2^{-\delta n} n!$$

and so

$$\Pr[aut(U_n) \leqslant 2^{\delta n}] \leqslant \frac{\ell(n)/n!}{u(n)} 2^{\delta n} = \left(\frac{\gamma_l}{\gamma_u} 2^{\delta} + o(1)\right)^n = o(2^{-\delta n}). \qquad \square$$

We now turn to copies in R_n of a *plane* graph H, that is of a graph H embedded in the plane. What does it mean for H to 'appear' in R_n? Let H and G be two plane graphs. Let us say that H *appears* in G at the vertex set $W \subseteq V(G)$, if (a) the underlying graph of

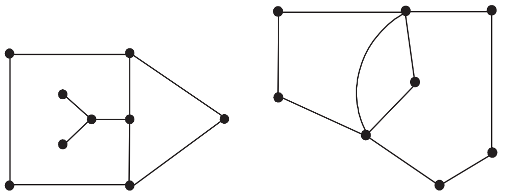

Fig. 1. Plane graphs H which cannot appear in *all* embeddings of R_n.

H appears at W in the underlying graph of G, (b) there is a continuous deformation of the plane taking H to the induced plane subgraph $G[W]$ of G, and (c) if no vertex of $V(G) \setminus W$ is contained in an interior face of $G[W]$.

We start with an illustrative example. Consider the two graphs H in Fig. 1. Then, obviously, for each planar graph G there does exist an embedding of G in which H does not appear. (Just start from an arbitrary embedding and then flip the interior vertices/vertex into the other face.) It is thus not possible to show that, for a given plane graph H, the random planar graph R_n contains linearly many copies of H in *each* embedding. On the other hand we know from Theorem 4.1 that R_n contains with high probability linearly many copies of each fixed planar graph.

By arguing as in the proof of Theorem 4.3 we may obtain

Proposition 4.9. *Let H be a connected plane graph. Let $f_H(G)$ denote the function which counts for a planar graph G the maximum over all embeddings of G of the maximum number of pairwise vertex disjoint appearances of H. Then there exists a constant $\alpha > 0$ such that*

$$\Pr[f_H(R_n) < \alpha n] \leqslant e^{-\alpha n} \quad \textit{for all sufficiently large } n.$$

If H is 3-connected then the claim remains true if $f_H(G)$ is defined by minimizing *over all embeddings of G.*

To close this section, we return to considering vertex degrees. We show that the expected number of vertices of degree 0, 1 or 2 in a random planar graph R_n can be easily related to other quantities of interest. Let $\bar{d}(n)$ denote the average degree of R_n: that is,

$$\bar{d}(n) = 2 \left(\sum_{G \in \mathcal{P}_n} |E(G)| \right) / \ (n\, \ell(n)),$$

where as before $\mathcal{P}_n$ denotes the collection of all simple planar graphs on n labelled vertices. From the results noted earlier on the number of edges in R_n we may see that $3.7 \leqslant \bar{d}(n) \leqslant 5.08$ for n sufficiently large.

Theorem 4.10. *Let X_i denote the number of vertices with degree i in a random planar graph R_n. Then*

$$\mathbb{E}[X_0] = \frac{n \cdot \ell(n-1)}{\ell(n)} = I_n,$$

$$\mathbb{E}[X_1] = (n-1)\cdot\mathbb{E}[X_0] = (n-1)I_n,$$
$$\mathbb{E}[X_2] = \mathbb{E}[X_1]\cdot\bar{d}(n-1),$$

where $\bar{d}(n-1)$ *is the average degree of* R_{n-1}.

Proof. Clearly,

$$\mathbb{E}[X_0] = n\cdot\Pr[v_n \text{ has degree 0 in } R_n] = n\cdot\frac{\ell(n-1)}{\ell(n)}$$

and similarly,

$$\mathbb{E}[X_1] = n\cdot\Pr[v_n \text{ has degree 1 in } R_n] = n\cdot\frac{(n-1)\ell(n-1)}{\ell(n)}.$$

To show the third claim we proceed as follows. Let $\mathcal{P}_n$ denote, as above, the collection of all simple labelled planar graphs on n vertices. Furthermore, we denote for a planar graph $G=(V,E)$ by $add(G)$ the number of edges $e\notin E$ such that $G+e$ is still planar.

Consider now the digraph with vertex set $\mathcal{P}_n$ and directed edges (G,H) if and only if $H=G+e$ for some edge $e\notin E(G)$. As the sum of all indegrees is equal to the sum of all outdegrees, we deduce

$$\sum_{G\in\mathcal{P}_n}|E(G)| = \sum_{G\in\mathcal{P}_n} add(G).$$

Observe also that all graphs in $\mathcal{P}_n$ in which vertex v_n has degree exactly two can be obtained from a graph G' on the remaining vertices by joining v_n to both vertices of an edge in G' or to both vertices of an edge e' that can be added to G' without destroying planarity. Hence,

$$\begin{aligned}\ell(n)\cdot\Pr[v_n \text{ has degree 2 in } R_n] &= \sum_{G'\in\mathcal{P}_{n-1}}\big(|E(G')| + add(G')\big)\\ &= 2\sum_{G'\in\mathcal{P}_{n-1}}|E(G')|\\ &= (n-1)\ell(n-1)\bar{d}(n-1).\end{aligned}$$

Hence,

$$\begin{aligned}\mathbb{E}[X_2] &= n\cdot\Pr[v_n \text{ has degree 2 in } R_n]\\ &= \frac{n(n-1)\ell(n-1)}{\ell(n)}\cdot\bar{d}(n-1)\\ &= \mathbb{E}[X_1]\cdot\bar{d}(n-1). \qquad\square\end{aligned}$$

5. Connectivity II

In this final section, we first give a lower bound for the probability that R_n has an isolated vertex, and more generally that R_n has a component isomorphic to a given planar graph H. After that, we see that if we assume the truth of Conjecture 3.4, the labelled planar graph isolated vertices conjecture, then we can determine the limiting values of these probabilities, and the limiting value of the probability that R_n is connected.

Theorem 5.1. *Let $\alpha > 0$ be such that R_n has at least αn vertices of degree 1 with probability $1 - o(1)$ as $n \to \infty$ (such an α exists by Theorem 4.2). Then*

$$\Pr[R_n \textit{ has an isolated vertex}] \geqslant \alpha e^{-1} + o(1).$$

Proof. Let us denote by $\tilde{\ell}(n)$ the number of labelled planar graphs on n vertices which contain at least αn vertices of degree 1, and by $\tilde{\ell}_c(n)$ the number of connected labelled planar graphs on n vertices which contain at least αn vertices of degree 1. Finally, let $\ell_s(n)$ denote the number of labelled planar graphs on n vertices which contain an isolated vertex.

By Theorem 2.1 we know that $\ell_c(n) \geqslant e^{-1} \cdot \ell(n)$. By our choice of α we know that $\tilde{\ell}(n) \geqslant (1 - \varepsilon(n)) \cdot \ell(n)$, where $\varepsilon(n) = o(1)$. Combining these inequalities we get

$$\tilde{\ell}_c(n) \geqslant (e^{-1} - \varepsilon(n)) \cdot \ell(n).$$

Clearly, every graph with at least αn vertices of degree 1 can be used to construct αn graphs with an isolated vertex by simply removing one of the edges incident to a vertex of degree one. In addition, one easily observes that if we start from *connected* graphs only, then every graph is generated at most $n-1$ times. (Note that this corresponds to the number of ways to reattach the isolated vertex.) Hence, we get

$$\ell_s(n) \geqslant \frac{\tilde{\ell}_c(n) \cdot \alpha n}{n-1} \geqslant (1 - e\,\varepsilon(n)) \cdot \alpha \cdot e^{-1} \cdot \ell(n)$$

from which the claim of the theorem follows immediately. $\square$

Actually, the proof of Theorem 5.1 easily generalizes to (finite) components different from isolated vertices. Recall that $aut(H)$ denotes the size of the automorphism group of a graph H.

Theorem 5.2. *Let H be a (fixed) planar graph on k vertices. Then*

$$\Pr[R_n \textit{ contains a component isomorphic to } H] \geqslant (1 + o(1)) \cdot e^{-1}\alpha^k / aut(H)$$

where $\alpha > 0$ is a constant as in Theorem 5.1.

Proof. We proceed as in the proof of Theorem 5.1. Every graph with at least αn vertices of degree 1 can be used to construct $\binom{\alpha n}{k} \cdot \frac{k!}{aut(H)}$ graphs with a component isomorphic to H. Again the fact that we start with connected graphs only implies that every graph is generated at most $(n-k)^k$ times. Hence, we get that the number of graphs with a component isomorphic to H is at least

$$\frac{\tilde{\ell}_c(n) \cdot \binom{\alpha n}{k} \cdot \frac{k!}{aut(H)}}{(n-k)^k}$$

from which the claim follows similarly as above. $\square$

Next we see that if we assume the truth of Conjecture 3.4, the labelled planar graph isolated vertices conjecture, then the probability that R_n is connected tends to a limit and we can determine this limit, and similarly for the probabilities appearing in the above two theorems. We start with some notation and two lemmas. As usual, let $\mathbb{E}[(X)_k] = \mathbb{E}[X(X-1)\cdots(X-k+1)]$ denote the kth factorial moment.

Lemma 5.3. *Assume that Conjecture 3.4 holds. Let $H_1, \ldots, H_m$ denote a fixed collection of pairwise non-isomorphic connected planar graphs. Furthermore, let $n_i := |V(H_i)|$; let $a_i := aut(H_i)$, the number of automorphisms of H_i; and let $X_n^{(i)}$ denote a random variable which counts the number of components isomorphic to H_i in the random planar graph R_n on n vertices. Then*

$$\mathbb{E}[(X_n^{(1)})_{k_1}\cdots(X_n^{(m)})_{k_m}] \to \left(\frac{1}{a_1\cdot\gamma_\ell^{n_1}}\right)^{k_1}\cdots\left(\frac{1}{a_m\cdot\gamma_\ell^{n_m}}\right)^{k_m} \quad as\ n\to\infty$$

for every $k_1, \ldots, k_m \in \mathbb{N}_0$, where γ_ℓ is the labelled planar graph growth constant.

Proof. Consider some fixed constants $k_1, \ldots, k_m \in \mathbb{N}_0$, and let $K = \sum_{s=1}^m k_s n_s$. In the following we assume without loss of generality that $k_i \geqslant 1$ for all $1 \leqslant i \leqslant m$. We construct a planar graph on n vertices with at least k_i components that are isomorphic to H_i as follows. First we choose the vertices of the components, then we insert appropriate copies of H_i on the vertices of each component and choose a planar graph on the remaining vertices. This can be done in exactly

$$\left(\prod_{i=1}^m \prod_{j=1}^{k_i} \binom{n-\sum_{s=1}^{i-1}k_s n_s-(j-1)n_i}{n_i}\cdot\frac{n_i!}{a_i}\right)\cdot\ell(n-K)$$

ways. Now let us deduce how often a specific planar graph G is constructed. Obviously, this depends on the number of components of G that are isomorphic to some H_i. If G contains t_i components that are isomorphic to H_i, then G is constructed exactly

$$\prod_{i=1}^m (t_i\cdot(t_i-1)\cdots(t_i-k_i+1))$$

times. Hence, if we denote by $\ell(n; t_1, \ldots, t_m)$ the number of planar graphs on n vertices with exactly t_i components that are isomorphic to H_i, then the definition of the expectation implies

$$\begin{aligned}
&\mathbb{E}[(X_n^{(1)})_{k_1}\cdots(X_n^{(m)})_{k_m}] \\
&\quad= \sum_{t_1,\ldots,t_m\geqslant 1}\left(\prod_{i=1}^m t_i\cdot(t_i-1)\cdots(t_i-k_i+1)\right)\frac{\ell(n;t_1,\ldots,t_m)}{\ell(n)} \\
&\quad= \left(\prod_{i=1}^m\prod_{j=1}^{k_i}\binom{n-\sum_{s=1}^{i-1}k_s n_s-(j-1)n_i}{n_i}\frac{n_i!}{a_i}\right)\cdot\frac{\ell(n-K)}{\ell(n)}
\end{aligned}$$

$$= \left(\prod_{i=1}^{m} \frac{1}{a_i^{k_i}} \right) \cdot \left(\prod_{i=1}^{K} (n-i+1) \right) \cdot \left(\prod_{i=1}^{K} \frac{\ell(n-i)}{\ell(n-i+1)} \right)$$
$$= \left(\prod_{i=1}^{m} \frac{1}{a_i^{k_i}} \right) \cdot \left(\prod_{i=1}^{K} I_{n-i+1} \right).$$

But now Conjecture 3.4 (which is equivalent to $\lim_{n\to\infty} I_n = \gamma_\ell^{-1}$) completes the proof, since K is a constant. □

In order to apply Lemma 5.3 the following lemma (see for example [7]) turns out to be very useful.

Lemma 5.4. *Let $(X_n^{(1)}, \ldots, X_n^{(m)})$ be m-vectors of random variables, where $m \geqslant 1$ is fixed. If $\lambda_1, \ldots, \lambda_m$ are such that, as $n \to \infty$,*

$$\mathbb{E}[(X_n^{(1)})_{k_1} \cdots (X_n^{(m)})_{k_m}] \to \lambda_1^{k_1} \cdots \lambda_m^{k_m}$$

for every $k_1, \ldots, k_m \in \mathbb{N}_0$, then $(X_n^{(1)}, \ldots, X_n^{(m)}) \xrightarrow{d} (Z_1, \ldots, Z_d)$, where $Z_i \in Po(\lambda_i)$ are independent.

With these two lemmas at hand we can now determine the limiting value of the probability that R_n is connected.

Theorem 5.5. *Assume that Conjecture 3.4 holds. Enumerate the connected graphs in $\mathcal{P}_1$ then $\mathcal{P}_2$ then $\mathcal{P}_3$ and so on, as $H_1, H_2, H_3, \ldots$. Let $n_i := |V(H_i)|$, and let $a_i := aut(H_i)$, the number of automorphisms of H_i. j vertices, Then*

$$\Pr[R_n \text{ is connected}] \to e^{-\sum_{i=1}^{\infty} 1/(a_i \gamma_\ell^{n_i})} \quad \text{as } n \to \infty.$$

Proof. For each positive integer k let

$$p_k := e^{-\sum_{i:n_i \leqslant k} 1/(a_i \gamma_\ell^{n_i})}.$$

From Lemma 5.3 together with Lemma 5.4 it follows that

$$\Pr[R_n \text{ has no component of order } \leqslant k] = p_k + o(1)$$

as $n \to \infty$. If X has the Poisson distribution with mean $1/(k+1)$, then by Theorem 2.5 we have

$$\Pr[R_n \text{ has at most one component of order } \geqslant k+1] \geqslant \Pr[X=0] = e^{-1/(k+1)}.$$

Hence

$$\Pr[R_n \text{ is disconnected}] \leqslant 1 - e^{-1/(k+1)} + 1 - p_k + o(1).$$

Thus

$$p_k - (1 - e^{-1/(k+1)}) + o(1) \leqslant \Pr[R_n \text{ is connected}] \leqslant p_k + o(1),$$

so

$$|\Pr[R_n \text{ is connected}] - p_k| \leqslant 1 - e^{-1/(k+1)} + o(1),$$

and the theorem follows. □

Finally, we note that Lemmas 5.3 and 5.4 also determine the limiting values of the probabilities considered in Theorem 5.2 and thus in Theorem 5.1.

Theorem 5.6. *Assume that Conjecture 3.4 holds. Let H be a fixed planar graph and let $aut(H)$ denote the number of automorphisms of H. Then the number of components of R_n isomorphic to H is asymptotically Poisson distributed with parameter λ, where $\lambda^{-1} := aut(H)\gamma_\ell^{|V(H)|}$; and so*

$$\Pr[R_n \text{ has a component isomorphic to } H] \to 1 - e^{-1/(aut(H)\cdot\gamma_\ell^{|V(H)|})} \quad \text{as } n \to \infty.$$

As a special case we obtain the limit of the probabilities that R_n contains an isolated vertex.

Corollary 5.7. *Assume that Conjecture 3.4 holds. Then the number of isolated vertices in R_n is asymptotically Poisson distributed with parameter $1/\gamma_\ell$; and so*

$$\Pr[R_n \text{ contains an isolated vertex}] \to 1 - e^{-1/\gamma_\ell} \quad \text{as } n \to \infty.$$

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