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The minimal prime spectrum of rings with annihilator conditions

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ABSTRACT

In this paper, we study rings with the annihilator condition (a.c.) and rings whose space of minimal prime ideals, $\mathrm{Min}(R)$, is compact. We begin by extending the definition of (a.c.) to noncommutative rings. We then show that several extensions over semiprime rings have (a.c.). Moreover, we investigate the annihilator condition under the formation of matrix rings and classical quotient rings. Finally, we prove that if R is a reduced ring then: the classical right quotient ring $Q(R)$ is strongly regular if and only if R has a Property (A) and $\mathrm{Min}(R)$ is compact, if and only if R has (a.c.) and $\mathrm{Min}(R)$ is compact. This extends several results about commutative rings with (a.c.) to the noncommutative setting.
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Throughout this paper, R denotes an associative ring with identity. For a nonempty subset $S \subseteq R$, $\ell_R(S)$ and $r_R(S)$ denote the left annihilator and the right annihilator of S in R, respectively. We use $Z_l(R)$ (respectively, $Z_r(R)$) for the set of left (respectively, right) zero-divisors of R. We also write $\mathrm{Spec}(R)$ (respectively, $\mathrm{Min}(R)$) for the space of all (minimal) prime ideals of R.

Henriksen and Jerison [1] originally defined the annihilator condition for commutative reduced rings, giving an example of a reduced ring not possessing the annihilator condition. In 1986, Lucas [2] extended it as follows: a commutative ring R has the *annihilator condition* (briefly, (a.c.)) if for each finitely generated ideal I of R, there exists an element $b \in R$ whose annihilator equals the annihilator of I. The class of commutative rings with (a.c.) is very large. For example, this class includes the polynomial rings over any reduced ring [3], Bezout rings (finitely generated ideals are principal), and every subdirect sum of totally ordered integral domains. We recommend [2] for further examples.

On the other hand, a commutative ring R has *Property (A)* if every finitely generated ideal of R consisting entirely of zero-divisors has a nonzero annihilator. This property was introduced by Huckaba and Keller [3], and has been called Condition (C) by Quentel. The class of commutative rings with Property (A) is also quite large. It includes Noetherian rings [4, p. 56], rings whose prime ideals are maximal, polynomial rings, and rings whose total quotient rings are von Neumann regular [5]. A number of authors have studied commutative rings with Property (A) (for example, see [2–6]), and their results provide useful machinery in the study of commutative rings with zero-divisors.

For any $a \in R$, define $\mathrm{supp}(a) = \{P \in \mathrm{Spec}(R) \mid a \notin P\}$. Shin [7, Lemma 3.1] proved that for any ring R, $\{\mathrm{supp}(a) \mid a \in R\}$ forms a basis of open sets on $\mathrm{Spec}(R)$. We view $\mathrm{Min}(R)$ as a subspace. This topology is called the *hull-kernel* topology (or, for commutative rings, the *Zariski topology*). The hull-kernel topologies on $\mathrm{Spec}(R)$ and $\mathrm{Min}(R)$ have been studied extensively when R is a commutative ring, as in [1,3,5,8–11, etc.]. It is well known that if a ring R has an identity, then $\mathrm{Spec}(R)$ is compact. However, in general $\mathrm{Min}(R)$ is not compact even if R is a commutative reduced ring with identity by [10, Proposition 3.4]. The compactness of minimal prime ideal spaces plays a special role in the case of rings of continuous functions [1]. In the case of a commutative reduced ring R, Henriksen and Jerison, [1, Theorem 3.4] proved that R has the annihilator condition

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and $\mathrm{Min}(R)$ is compact if and only if for each $a \in R$, $r_R(a) = r_R(r_R(b))$ for some $b \in R$. Quentel [11, Proposition 9] proved that R has a Property (A) and $\mathrm{Min}(R)$ is compact if and only if the total quotient ring $T(R)$ of R is von Neumann regular. Huckaba and Keller [3, Theorem B] proved that R has a Property (A) and $\mathrm{Min}(R)$ is compact if and only if R has the annihilator condition and $\mathrm{Min}(R)$ is compact, if and only if the total quotient ring $T(R)$ of R is von Neumann regular. Shin [7, Theorem 4.9] asserts that the results of Henriksen and Jerison also hold in the case that R is a noncommutative reduced ring. More results on hull-kernel topologies for prime ideal spaces over noncommutative rings can be found in [7,12–15].

Rings with (a.c.) are closely connected with those having Property (A). For example, a commutative reduced Noetherian ring has both (a.c.) and Property (A). Also, Property (A) and (a.c.) are equivalent conditions when R is a reduced ring with compact $\mathrm{Min}(R)$, or when R is a reduced coherent ring, by [3]. However, Lucas [2] showed that Property (A) and (a.c.) are not always equivalent.

Recently, Hong et al. [16] defined Property (A) for noncommutative rings. A ring R has *right Property (A)* if for every finitely generated two-sided ideal $I \subseteq Z_l(R)$ there exists a nonzero element $a \in R$ such that $Ia = 0$, and one similarly defines left Property (A). A ring R has *Property (A)* if R has both right and left Property (A). In [16], several extensions of rings with Property (A) were studied, including matrix rings, polynomial rings, and classical quotient rings. Furthermore, the authors characterized when the space of minimal prime ideals for rings with Property (A) is compact.

In this paper we study rings with the annihilator condition (a.c.) and rings where $\mathrm{Min}(R)$ is compact. We begin by extending the definition of (a.c.) to noncommutative rings. We then show that several extensions over semiprime rings have (a.c.). Moreover, we investigate the annihilator condition under the formation of matrix rings and classical quotient rings. Finally, we prove that if R is a reduced ring then: the classical right quotient ring $Q(R)$ is strongly regular if and only if R has a Property (A) and $\mathrm{Min}(R)$ is compact, if and only if R has (a.c.) and $\mathrm{Min}(R)$ is compact. This extends several results about commutative rings with (a.c.) to the noncommutative setting.

The usefulness and significance of the annihilator condition stems from the fact that the class of rings with (a.c.) is very large. Most notably, right Bezout rings (hence von Neumann regular rings), quasi-Baer rings (hence prime rings), reduced p.p.-rings, and semiprime rings satisfying the ascending chain condition on annihilators all have (a.c.). Therefore the annihilator condition can be a useful tool when characterizing common properties of these rings. Moreover, by studying the relationships between the strong regularity of the classical right quotient ring $Q(R)$, the compactness of $\mathrm{Min}(R)$, and R possessing the annihilator condition, one arrives at a better understanding of von Neumann regular rings, biregular rings, and coherent reduced rings. These results have applications in the study of maximal quotient rings of noncommutative rings, and questions about the von Neumann regularity of such extensions.

1. Definition and examples of rings with the annihilator condition

We begin with the following definition.

Definition 1.1. A ring R is said to have the *right annihilator condition*, or (*right* (a.c.)), if for any finitely generated two-sided ideal I of R, there exists $c \in R$ such that $r_R(I) = r_R(RcR)$. Rings with left (a.c.) are defined similarly, and we say R has (a.c.) if R has left and right (a.c.).

Remark 1. (1) Recall the basic facts that $r_R(aR) = r_R(RaR)$ and $r_R(aR + bR) = r_R(aR) \cap r_R(bR)$ for $a, b \in R$. We will use these facts throughout, without further mention.

(2) A ring R has right (a.c.) if and only if for a 2-generated ideal $I = RaR + RbR$ of R, $r_R(I) = r_R(RcR)$ for some $c \in R$, if and only if for a 2-generated right ideal $J = aR + bR$ of R, $r_R(J) = r_R(cR)$ for some $c \in R$.

(3) Suppose R is a reduced ring. Then R has right (a.c.) if and only if for any $a, b \in R$, $r_R(\{a, b\}) = r_R(c)$ for some $c \in R$.

(4) If R is semiprime, then the annihilator condition is left-right symmetric.

The following example shows that the annihilator condition is, in general, not left-right symmetric. Recall that for a ring R with a ring endomorphism σ and a σ-derivation δ, the *Ore extension* $R[x; \sigma, \delta]$ of R is the ring of polynomials in x over R, written with coefficients on the left, with the usual addition and multiplication subject to the rule $xa = \sigma(a)x + \delta(a)$ for any $a \in R$. If $\delta = 0$, then $R[x; \sigma]$ is called the *skew polynomial ring* over R.

Example 1.2. Let $\mathbb{Z}_2$ be the ring of integers modulo 2 and $\mathbb{Z}_2[x, y]$ the polynomial ring over $\mathbb{Z}_2$ in commutating indeterminates x and y. Consider the ring $R = \mathbb{Z}_2[x, y]/\langle x^2, y^2\rangle = \{a_1 + a_2x + a_3y + a_4xy \mid a_1, a_2, a_3, a_4 \in \mathbb{Z}_2\}$, where $\langle x^2, y^2\rangle$ is the ideal of $\mathbb{Z}_2[x, y]$ generated by x^2 and y^2. By [2, Example 3.13], R does not have (a.c.). Now let σ be the homomorphism of R defined by $\sigma(f(x, y)) = f(0, 0)$, and $S = R[z; \sigma]$ be the skew polynomial ring over R.

Claim 1. *S has left (a.c.).*

For $f(z) = \sum_{i=0}^{m} \alpha_i z^i$ and $g(z) = \sum_{j=0}^{n} \beta_j z^j \in S$, let $h(z) = f(z) + g(z)z^{m+1}$. We claim that $\ell_S(Sf(z) + Sg(z)) = \ell_S(Sh(z))$. One inclusion is trivial since $Sh(z)S \subseteq Sf(z)S + Sg(z)S$. For the other inclusion we have two cases. Case 1: If $Sh(z) \cap \mathbb{Z}_2[z] \neq 0$, then $0 = \ell_S(Sh(z)) \supseteq \ell_S(Sf(z) + Sg(z))$ hence $\ell_S(Sh(z)) = 0 = \ell_S(Sf(z) + Sg(z))$. Case 2: If $Sh(z) \cap \mathbb{Z}_2[z] = 0$, then $\sigma(\alpha_i) = \sigma(\beta_j) = 0$ for all i, j, since $zh(z) \in Sh(z) \cap \mathbb{Z}_2[z] = 0$. Let $k(z) \in \ell_S(Sh(z))$. Since $\sigma(\alpha_i) = \sigma(\beta_j) = 0$ for all i, j, $k(0)Sh(z) = 0$. This implies $k(0)S\alpha_i = 0$ and $k(0)S\beta_j = 0$ for all i, j, and so $k(0)Sf(z) = 0$ and $k(0)Sg(z) = 0$. Equivalently, $k(z) \in \ell_S(Sf(z) + Sg(z))$. Therefore S has left (a.c.).

Claim 2. *S does not have right (a.c.).*

For $x, y \in S$, suppose that $r_S(xS + yS) = r_S(k(z)S)$, where $k(z) \in S$. Set $k(0) = k_0$. If $r \in R$ then either r is a unit or $zr = 0$. Thus, $r_S(xS + yS) \cap R = r_R(xR + yR)$ and similarly $r_S(k(z)S) \cap R = r_R(k_0R)$. Thus $r_R(xR + yR) = r_R(k_0R)$. A quick computation, as in [2, Example 3.13], shows this is impossible. Therefore S does not have right (a.c.).

The class of rings with (a.c.) is very large. For example, right Bezout rings (hence von Neumann regular rings), quasi-Baer rings (hence prime rings), and reduced p.p.-rings all have (a.c.). Moreover, we here provide more examples of rings with (a.c.).

In [1, Example 3.3], the authors showed that there exists a commutative reduced ring which does not have (a.c.). Thus semiprime rings do not, in general, have right (a.c.) and therefore a subdirect product of prime rings does not have right (a.c.) in general. However, we have the following (using the definition of fully ordered in given in [17]):

Proposition 1.3. *Every subdirect product of fully ordered semiprime rings has (a.c.).*

Proof. Suppose R is a subdirect product of fully ordered semiprime rings R_i. Let $\alpha, \beta \in R$ with $\alpha = (a_i), \beta = (b_i) \in \prod_{i \in I} R_i$. For any $i \in I$ and positive integer k, define the set $N_k(R_i) = \{a \in R_i \mid a^k = 0\}$. By [17, Lemma, p.130], $N_k(R_i)$ is an ideal of R_i such that $(N_k(R_i))^k = 0$. Since R_i is semiprime, $N_k(R_i) = 0$ for any positive integer k. Thus R_i is a domain by [17, Theorem 6, p.130]. Consequently, R is a subdirect product of fully ordered domains R_i. We will show that $r_R(\alpha R + \beta R) = r_R((\alpha^2 + \beta^2)R)$. Let $\gamma = (c_i) \in r_R((\alpha^2 + \beta^2)R)$. Then $(\alpha^2 + \beta^2)R\gamma = 0$ and so $(a_i^2 + b_i^2)(r_i)(c_i) = 0$ for any $(r_i) \in \prod_{i \in I} R_i$. This implies $a_i^2 r_i c_i = -b_i^2 r_i c_i$, and so by [17, Theorem 1, p.105] $a_i^2 r_i c_i = 0 = b_i^2 r_i c_i$. Thus, independent of a_i and b_i, we have $a_i r_i c_i = 0 = b_i r_i c_i$. This implies $(\alpha R + \beta R)\gamma = 0$ and so $r_R(\alpha R + \beta R) \subseteq r_R((\alpha^2 + \beta^2)R)$. The reverse inclusion is clear. Therefore R has right (a.c.). By symmetry R also has left (a.c.). □

Theorem 1.4. *If R is a semiprime ring with finitely many minimal prime ideals, then R has (a.c.).*

Proof. We first suppose that $|\mathrm{Min}(R)| = n$. Fix $a, b \in R$ and let $I = RaR + RbR$. Then $r_R(I) = \cap\{P \in \mathrm{Min}(R) \mid I \not\subseteq P\}$ by [18, Lemma 11.40]. Also, by [18, Theorem 11.41], $P = r_R(U)$ for some uniform ideal U of R and it is also a maximal right annihilator. Moreover, by [18, Theorem 11.43], the uniform dimension $\mathrm{u.dim}({}_RR_R) = n$. So there exist uniform ideals $U_1, U_2, \ldots, U_n$ such that $U_1 \oplus U_1 \oplus \cdots \oplus U_n$ is essential in R. We claim that $r_R(U_i) \neq r_R(U_j)$ for any $i \neq j$. To see this, suppose $r_R(U_i) = r_R(U_j)$ for some $i \neq j$. Since $U_iU_j = 0$, $U_i \subseteq r_R(U_j) = r_R(U_i)$. So $U_i^2 = 0$ and hence $U_i = 0$, a contradiction. Now, after relabeling the U_i, we may assume $r_R(I) = \cap P = \cap_{j=1}^{k} r_R(U_j)$, where $1 \leq k \leq n$. For each j fix an element $0 \neq u_j \in U_j$. Then, for each j, we have the equality $r_R(U_j) = r_R(u_jR)$ because $r_R(U_j)$ is maximal annihilator in R. We can also choose u_j so that there exists $v_j \in U_j$ with $u_jv_j \neq 0$. Note that $r_R((u_1 + u_2 + \cdots + u_k)v_jR) = r_R(u_jv_jR) = r_R(U_j)$. Thus we have

$$r_R(I) = \bigcap_{j=1}^{k} r_R(u_jR) = r_R(u_1R \oplus u_2R \oplus \cdots \oplus u_kR) = r_R((u_1 + u_2 + \cdots + u_k)R).$$

Therefore R has right (a.c.). □

In a semiprime ring, we note that there are multiple characterizations for R having only finitely many minimal prime ideals. For example, using [18, Theorem 11.43] we obtain:

Corollary 1.5. *Let R be a semiprime ring with a.c.c. on annihilators. Then R has (a.c.).*

Recall that a ring R is called *biregular* if every principal ideal of R is generated by central idempotent of R. By [16, Proposition 1.11], biregular rings have Property (A). We also have the following result.

Proposition 1.6. *If R is a biregular ring, then R has (a.c.).*

Proof. Fix $a, b \in R$ and let $I = RaR + RbR$. By hypothesis, $RaR = eR$ for some central idempotent e of R. Note that $I = eR \oplus I(1 - e)$. Then $I(1 - e) = Rb(1 - e)R = fR$ for some central idempotent f of R. Thus $I = eR \oplus fR = (e + f)R$. Therefore R has right (a.c.). By symmetry, R has (a.c.). □

From Proposition 1.6, [13, Theorem 6], and [13, Remark 2, p.4101], we have the following:

Corollary 1.7. *If R is a reduced ring whose prime ideals are maximal, then R has (a.c.).*

Remark 2. Note that if a ring R is right Noetherian right self-injective (i.e., a QF-ring) then R has Property (A) by [16, p.616]. However, $Z_2[x, y]/\langle x^2, y^2\rangle$ is a QF-ring by [18, Example 3.15B], but does not have (a.c.).

Huckaba and Keller [3, Corollary 1] proved that the polynomial ring over any commutative ring has Property (A). They also proved that commutative reduced nontrivial graded rings have (a.c.) [3, Theorem 4]. Thus the polynomial rings over commutative reduced rings have (a.c.). However, the following example shows that the polynomial ring over a commutative ring does not necessarily have (a.c.), and also that the reducedness condition is not superfluous.

Example 1.8. Let $R = \mathbb{Z}_2[x, y]/\langle x^2, y^2\rangle$ be the ring in [2, Example 3.13]. Then R is a commutative non-reduced Noetherian ring which does not have (a.c.). Now we consider the polynomial ring $R[z]$ over R. We claim $r_{R[z]}(\{x, y\}) \neq r_{R[z]}(g(z))$ for any $g(z) \in R[z]$. Suppose $f(z) = \sum_{i=0}^m f_i z^i \in r_{R[z]}(\{x, y\}) = r_{R[z]}(x) \cap r_{R[z]}(y)$, where $f_i \in R$ for each i. We then have $f(z)x = 0$ and so $f_i x = 0$ for all i. Similarly, $f_i y = 0$ for all i. Thus $f_i \in r_R(\{x, y\}) = \{0, xy\}$ for all i. Hence

$$r_{R[z]}(\{x, y\}) = xyR[z].$$

Suppose by way of contradiction that $r_{R[z]}(\{x, y\}) = r_{R[z]}(g(z))$ for some $g(z) \in R[z]$. Put $g(z) = g_0 + g_1 z + \cdots + g_n z^n$, where $g_j \in R$ for each j. Notice that $g(z)xy = 0$, hence $g_j \in J(R)$ for each j, and in particular $g_j^2 = 0$. Then

$$g(z)^2 = g_0^2 + g_1^2 z^2 + \cdots + g_n^2 z^{2n} + 2(g_0 g_1 z + \cdots + g_{n-1} g_n z^{2n-1}) = 0$$

since the characteristic of R is 2. Thus $g(z) \in r_{R[z]}(g(z)) = r_{R[z]}(\{x, y\}) = xyR[z]$. Hence $g(z)x = 0$, and so $x \in r_{R[z]}(g(z))$, contradicting the fact that $x \notin r_{R[z]}(\{x, y\})$.

We now extend many of the known results for commutative polynomial rings to noncommutative polynomial extensions such as Ore extensions, skew monoid rings, and so forth. This provides many useful examples of rings with (a.c.).

Lemma 1.9 (*[19, Lemma 1.5]*). *Let R be a semiprime ring with an automorphism σ and a σ-derivation δ. If, for some fixed $a, b \in R$, we have $aR\sigma^n(b) = 0$ for all integers $n \geq 0$, then $aR(\sigma^{n_1}\delta^{m_1} \cdots \sigma^{n_t}\delta^{m_t})(b) = 0$ for all integers $m_i, n_j \geq 0$.*

We invite the reader to compare the following theorem with Examples 1.2 and 1.8.

Theorem 1.10. *If R is a semiprime ring with an automorphism σ and a σ-derivation δ, then $R[x; \sigma, \delta]$ has (a.c.).*

Proof. Let $S = R[x; \sigma, \delta]$ and $I = Sf(x)S + Sg(x)S$, where $f(x) = a_0 + a_1 x + \cdots + a_m x^m$, $g(x) = b_0 + b_1 x + \cdots + b_n x^n \in S$. Note that for any $r \in R$,

$$rx = x\sigma^{-1}(r) - \delta(\sigma^{-1}(r)).$$

Using this equation repeatedly, we can rewrite left polynomials as right polynomials, so take $f(x) = c_0 + xc_1 + \cdots + x^m c_m$ and $g(x) = d_0 + xd_1 + \cdots + x^n d_n$. Let $h(x) = f(x) + x^{m+1} g(x)$. We claim that $r_S(I) = r_S(h(x)S)$. Since $h(x) \in I$, $r_S(I) \subseteq r_S(h(x)S)$. Let $k(x) = e_0 + e_1 x + \cdots + e_k x^k \in r_S(h(x)S)$. Then $h(x)Sk(x) = 0$, or equivalently,

$$(c_0 + xc_1 + \cdots + x^m c_m + x^{m+1} d_0 + x^{m+2} d_1 + \cdots + x^{m+n+1} d_n)Rx^t(e_0 + e_1 x + \cdots + e_k x^k) = 0$$

for any integer $t \geq 0$. Then $d_n R\sigma^t(e_k) = 0$ for any integer $t \geq 0$. By Lemma 1.9, $d_n R(\sigma^{n_1}\delta^{m_1} \cdots \sigma^{n_t}\delta^{m_t})(e_k) = 0$ for all integers $m_u, n_v \geq 0$. This implies that

$$d_n r\sigma^t(e_{k-1}) + d_{n-1} r\sigma^t(e_k) = 0. \tag{$*$}$$

If we multiply by $R\sigma^t(e_k) = 0$ on the right side of Eq. $(*)$, then $d_{n-1}R\sigma^t(e_k)R\sigma^t(e_k) = 0$. Since R is semiprime, $d_{n-1}R\sigma^t(e_k) = 0$ and hence $d_{n-1}R\sigma^t(e_k) = 0$. Continuing this process, we have $d_j R\sigma^t(e_s) = 0$ and also $c_i R\sigma^t(e_s) = 0$ for any integer $t \geq 0$, $0 \leq i \leq m$, $0 \leq j \leq n$ and $0 \leq s \leq k$. Thus $f(x)Sk(x) = 0$ and $g(x)Sk(x) = 0$. Hence $k(x) \in r_S(I)$, and therefore S has right (a.c.).

Similarly, let $h'(x) = f(x) + g(x)x^m$, written as left polynomials. Then $h'(x) \in I$ and so $\ell_S(I) \subseteq \ell_S(Sh'(x))$. Let $k'(x) \in \ell_S(Sh'(x))$. We can write $k'(x)$ as a left polynomial, using the argument in the previous paragraph; so take $k'(x) = e'_0 + xe'_1 + \cdots + x^k e'_k$. Then $k'(x)Sh'(x) = 0$ and so we have $e'_s R\sigma^t(a_i) = 0$ and $e'_s R\sigma^t(b_j) = 0$ for all $t \geq 0$, again by the argument in the preceding paragraph. Thus $k'(x) \in \ell_S(Sf(x)) \cap \ell_S(Sg(x)) = \ell_S(I)$. Therefore S has left (a.c.). □

Theorem 1.10 shows that Property (A) and (a.c.) behave quite differently in noncommutative rings, as there exists a polynomial ring over a semiprime ring which does not have either left or right Property (A) by [16, Example 2.7].

Recall that a monoid G is called a *unique product monoid* (simply, *u.p.-monoid*) if for any two nonempty finite subsets $A, B \subseteq G$ there exists some element $c \in G$ which is uniquely presented in the form ab where $a \in A$ and $b \in B$. The class of u.p.-monoids is quite large and important (see [20,21] for details). For example, this class includes the right or left ordered monoids, submonoids of a free group, and torsion-free nilpotent groups.

Let R be a ring and G a u.p.-monoid. Assume that G acts on R by means of a homomorphism into the automorphism group of R. We denote by $\sigma_g(r)$ the image of $r \in R$ under $g \in G$. The skew monoid ring $R * G$ is a ring which as a left R-module is free with basis G and multiplication defined by the rule $gr = \sigma_g(r)g$.

Lemma 1.11. *If G is a u.p.-monoid with $|G| \geq 2$, then $|G| = \infty$.*

Proof. Assume $|G| < \infty$. Take $g \neq 1_G$, where $g \in G$ and 1_G is the identity of G. Eventually the sequence $g, g^2, \ldots$ repeats. Thus $g^m = g^n$ for some positive integers $m > n$, and so $g^{m-n} = 1_G$ by [20, Lemma 1, p.119]. Hence there exists a smallest positive integer k such that $g^k = 1_G$. We now take $A = B = G$. For $h \neq 1_G$, since $h1_G = 1_G h = h$, we see that h does not have a unique representation as a product in AB. Therefore 1_G must be the element which has a unique representation; but $1_G = 1_G 1_G = gg^{k-1}$, a contradiction. □

Lemma 1.12 *([19, Theorem 1.1]). Let R be a semiprime ring and G a u.p.-monoid. Then $(a_0g_0+a_1g_1+\cdots+a_mg_m)(R*G)(b_0h_0+b_1h_1+\cdots+b_nh_n)=0$ with $a_i, b_j \in R$, $g_i, h_j \in G$ if and only if $a_iR\sigma_{g_i}(\sigma_g(b_j))=0$ for any $g \in G$ and $0 \le i \le m$ and $0 \le j \le n$.*

Theorem 1.13. *If R is a semiprime ring and G is a u.p.-monoid with $|G| \ge 2$, then $R*G$ has (a.c.).*

Proof. Let $R*G=S$ and $I=SgS+ShS$, where $g=a_0g_0+a_1g_1+\cdots+a_mg_m$ and $h=b_0h_0+b_1h_1+\cdots+b_nh_n \in S$ with $a_i, b_j \in R, g_i, h_j \in G$. Then by Lemma 1.11, we can put

$$k=a_0k_0+a_1k_1+\cdots+a_mk_m+b_0k_{m+1}+\cdots+b_nk_{m+n+1} \in S,$$

where $k_i \ne k_j$ for $i \ne j$. We first claim that $\ell_S(I)=\ell_S(Sk)$. Let $t=c_0t_0+c_1t_1+\cdots+c_st_s \in \ell_S(I)$. Then $tSg=tSh=0$, and so $c_uR\sigma_{t_u}(\sigma_p(a_i))=0$ and $c_uR\sigma_{t_u}(\sigma_p(b_j))=0$ for any $p \in G$ and $0 \le u \le s$, $0 \le i \le m$, $0 \le j \le n$ by Lemma 1.12. Hence $t \in \ell_S(Sk)$. By the same method as above, the reverse inclusion also holds. Therefore S has left (a.c.). Since G acts on R by means of a homomorphism into the automorphism group of R, we have $rg=g\sigma_g^{-1}(r)$. So, by symmetry considerations, S has right (a.c.). □

We mention here some standard constructions using skew monoid rings. For example, the skew Laurent polynomial ring $R[x,x^{-1};\sigma]$ with an automorphism σ over R is the skew monoid ring $R*G$ with $G=\{\ldots,x^{-2},x^{-1},1,x,x^2,\ldots\}$ and $\sigma_{x^n}(r)=\sigma^n(r)$ for each $n \in \mathbb{Z}$ and any $r \in R$. We can also remove the skew conditions by setting $\sigma_g=\mathrm{id}$ for all $g \in G$. The (non-skew) monoid ring is denoted $R[G]$. One can similarly define (skew, Laurent) power-series rings.

Corollary 1.14. *If R is a semiprime ring with an automorphism σ and G is a u.p.-monoid with $|G| \ge 2$, then $R[G]$, $R[x,x^{-1};\sigma]$, $R[[x,x^{-1};\sigma]]$ and $R[[x;\sigma]]$ have (a.c.).*

Proof. We only need to prove that $R[[x;\sigma]]$ has (a.c.). Given $f(x)=\sum_{i=0}^{\infty}a_ix^i$ and $g(x)=\sum_{i=0}^{\infty}b_ix^i \in R[[x,;\sigma]]$ we set $h(x)=f(x^2)+g(x^2)x=\sum_{i=0}^{\infty}(a_i+b_ix)x^{2i}$. Then the result follows by [19, Remark 2] (which is the power-series version of Lemma 1.12) and the same method as in the proofs of Theorems 1.10 and 1.13. □

Remark 3. If σ is not an automorphism, we can still form the rings $R[x;\sigma]$ and $R[[x;\sigma]]$. In the case that R is a semiprime ring and σ is an epimorphism then we claim, leaving the proof to the interested reader, that the results above still hold for left (a.c.), by the same method of proof.

Proposition 1.15. *A direct product of rings $\prod_{i\in I}R_i$ has right (a.c.) if and only if each R_i has right (a.c.).*

Proof. Assume each R_i has right (a.c.). Let $\alpha=\langle a_i\rangle$, $\beta=\langle b_i\rangle \in S=\prod_{i\in I}R_i$. For each $i \in I$, $r_{R_i}(a_iR_i+b_iR_i)=r_{R_i}(c_iR_i)$ for some $c_i \in R_i$. Then we note that $r_S(\alpha S+\beta S)=r_S(\gamma S)$, where $\gamma=\langle c_k\rangle$.

Conversely, suppose $S=\prod_{i\in I}R_i$ has right (a.c.). Fix $i \in I$ and elements $a_i, b_i \in R_i$. Let $\alpha=\langle a_j\rangle$, where $a_j=0$ if $j \ne i$, and $\beta=\langle b_j\rangle$, where $b_j=0$ if $j \ne i$. Since S has right (a.c.), $r_S(\alpha S+\beta S)=r_S(\gamma S)$ for some $\gamma=\langle c_k\rangle \in S$. Then we note that $r_{R_i}(a_iR_i+b_iR_i)=r_{R_i}(c_iR_i)$. □

Corollary 1.16. *Let G be a finite abelian group with $|G|=n$ and K a field such that $ch(K)$ does not divide n and it contains a primitive n-th root of 1. If a K-algebra R has right (a.c.), then the group ring $R[G]$ has right (a.c.).*

Proof. Since we have $R[G] \cong R\otimes_K K[G] \cong R\otimes_K K^n \cong R^n$, the result follows from Proposition 1.15. □

We denote the $n\times n$ full matrix ring over a ring R by $\mathbb{M}_n(R)$.

Proposition 1.17. *Let R be a finite ring with $|R|=n$. Then $\mathbb{M}_m(R)$ has (a.c.) for any $m \ge n$.*

Proof. Let J be a finitely generated ideal of $S=\mathbb{M}_m(R)$. Then $J=\mathbb{M}_m(I)$ for some ideal I of R. Since $|R|=n$, $I=Ra_1R+\cdots+Ra_kR$ for $1 \le k \le n$. Note that $r_S(J)=r_S(AS)$, where $A=a_1e_{11}+a_2e_{12}+\cdots+a_ke_{1k}$ since $SAS=J$. Therefore S has right (a.c.). By symmetry, S has left (a.c.). □

2. Extensions of rings with (a.c.)

In this section, we study the extensions of rings with (a.c.). We first study several types of matrix rings over rings with (a.c.). Throughout, we let e_{ij} denote the matrix units of $\mathbb{M}_n(R)$.

Theorem 2.1. *If a ring R has right (a.c.), then $\mathbb{M}_n(R)$ has right (a.c.) for any integer $n \ge 2$.*

Proof. Suppose that R has right (a.c.) and put $S = \mathbb{M}_n(R)$. Fix matrices $A_1, A_2 \in S$ and suppose $X = SA_1S + SA_2S$. Write the (i,j)-th entry of A_k as a_{ij}^k. Let $J = \sum_{i,j,k} Ra_{ij}^k R$. Since R has right (a.c.), $r_R(J) = r_R(cR)$ for some nonzero $c \in R$. We compute

$$a_{ij}^k I_n = \sum_{l=1}^{n} a_{ij}^k e_{ll} = \sum_{l=1}^{n} e_{li}(a_{ij}^k e_{ij})e_{jl} \in X$$

for all i, j, k. Hence

$$r_S(X) = r_S(\{a_{ij}^k I_n S\}) = r_S(cI_nS).$$

Therefore $\mathbb{M}_n(R)$ has right (a.c.). □

The converse of Theorem 2.1 is not true in general by the next example. Moreover, this example demonstrates that rings with right (a.c.) are not Morita invariant.

Example 2.2. (1) Consider the ring $R = \mathbb{Z}_2[x,y]/\langle x^2, y^2\rangle$, which does not have (a.c.). However, by Proposition 1.17, $\mathbb{M}_n(R)$ has right (a.c.) for any $n \geq 16$ since $|R| = 16$. Actually, we can show that $\mathbb{M}_n(R)$ has right (a.c.) for any $n \geq 2$. Let I be a nonzero proper ideal of R. Then $I \subseteq \{bx + cy + dxy \mid b, c, d \in \mathbb{Z}_2\}$ because $(1 + bx + cy + dxy)^2 = 1$. Put $I = a_1R + \cdots + a_tR$ for some nonzero elements $a_1, \ldots, a_t \in R$. It suffices to show that I is an ideal of R generated by at most two elements, since then the proof of Proposition 1.17 applies. Now for any i, $a_i \in \{x, y, xy, x+y, x+xy, y+xy, x+y+xy\}$. It is easy to see then that $xyR \subseteq a_iR$, so we may assume $a_iR \in \{xR, yR, (x+y)R, xyR\}$. Thus, if I is not a principal ideal then $I = xR + yR$.

(2) We claim that (a.c.) does not pass to corner rings. By (1), $S = \mathbb{M}_2(R)$ has right (a.c.). Let $e = e_{11}$. Then note that $SeS = S$ and $eSe \cong R$. Therefore eSe does not have right (a.c.).

We write $\mathbb{UM}_n(R)$ to denote the $n \times n$ upper triangular matrix ring over a ring R.

Theorem 2.3. *Fix $n \geq 2$. A ring R has right (a.c.) if and only if $\mathbb{UM}_n(R)$ has right (a.c.).*

Proof. Suppose that R has right (a.c.). Let $U = \mathbb{UM}_n(R)$ and $A = (a_{ij}), B = (b_{ij}) \in U$. Let $X_j = \sum_{i=1}^{j}\left(\sum_{k=i}^{j} a_{ik}R + b_{ik}R\right)$. These are exactly the elements which may appear as linear combinations of elements of the jth column of some matrix in $AU + BU$. We leave it to the reader to check that the right annihilator of $AU + BU$ is the same as the right annihilator of XU where $X = \sum_{j=1}^{n} X_j e_{jj}$. Since R has right (a.c.), there exist $c_1, c_2, \ldots, c_n \in R$ such that $r_R(X_j) = r_R(c_jR)$. One computes that $M \in r_U(XU)$ if and only if $e_{jk}M$ annihilates X_je_{jj}, for each j. Thus, if we set $C = \sum_{i=1}^{n} c_ie_{ii}$ we have $r_U(X) = r_U(CU)$. Therefore U has right (a.c.).

Conversely, let $a, b \in R$, again set $U = \mathbb{UM}_n(R)$, and now suppose U has right (a.c.). By hypothesis, $r_U(ae_{kk}U + be_{kk}U) = r_U(CU)$ for some $C \in U$. If we set $c = C_{kk}$ then $r_U(ae_{kk}U + be_{kk}U) = r_U(Ce_{kk}U) = r_U(ce_{kk}U)$. Trivially, $r_R(aR + bR) = r_R(cR)$, so R has right (a.c.). □

Note that in the previous two results we can replace the word “right” with “left” and the theorems remain true due to symmetry considerations.

In the following, we consider subrings of $\mathbb{UM}_n(R)$. Let

$$\mathbb{U}_n(R) = \left\{ M \in \mathbb{UM}_n(R) \,\middle|\, M = \sum_{i=1}^{n} ae_{ii} + \sum_{1\leq i<j\leq n} a_{ij}e_{ij} \right\},$$

denote the set of upper-triangular matrices with constant main diagonal and let

$$\mathbb{V}_n(R) = \left\{ M \in \mathbb{U}_n(R) \,\middle|\, M = \sum_{1\leq i\leq j\leq n} a_{ij}e_{ij},\ \text{where } a_{ij} = a_{(i+1)(j+1)} \right\}$$

be the set of upper-triangular matrices whose diagonals are each constant. Following [22], let $RA = \{rA : r \in R\}$ for any $A \in \mathbb{M}_n(R)$, and for $n \geq 0$ let $V = \sum_{i=1}^{n-1} e_{i(i+1)}$, where the e_{ij}'s are the matrix units. Then note that for any integer $n \geq 1$,

$$\mathbb{V}_n(R) = RI_n + RV + \cdots + RV^{n-1}.$$

Define $\rho : \mathbb{V}_n(R) \to R[x]/\langle x^n\rangle$ by $\rho(a_0I_n + a_1V + \cdots + a_{n-1}V^{n-1}) = \sum_{i=0}^{n-1} a_ix^i + \langle x^n\rangle$. One checks that ρ is a ring isomorphism. We then have the following:

Example 2.4. The ring $R = \mathbb{Z}_2[y]/\langle y^2\rangle$ has (a.c.). However $R[x]/\langle x^2\rangle \cong \mathbb{V}_2(R) \cong \mathbb{U}_2(R)$ does not. Hence the analog of Theorem 2.3 does not hold for these subrings. In particular, the trivial extension $T(R,R) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \,\middle|\, a, b \in R \right\}$ need not have (a.c.).

By Theorem 2.3, a ring R has right (a.c.) if and only if $\mathbb{UM}_2(R)$ has right (a.c.). However, for a ring A with right (a.c.) and an (A, A)-bimodule B, the upper matrix ring $\begin{pmatrix} A & B \\ 0 & A \end{pmatrix}$ does not necessarily have right (a.c.) by the following example.

Example 2.5. Let $A = \mathbb{C}[x, y]$ and let $B = \prod_{i\in I} A/M_i$, where $\{M_i\}_{i\in I}$ is the set of maximal ideals of A. Note that B is an (A, A)-bimodule, and A is a domain, hence clearly has right (a.c.). Let $R = \begin{pmatrix} A & B \\ 0 & A \end{pmatrix}$.

Suppose by way of contradiction that R does have right (a.c.). Then for $\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}, \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix} \in R$,

$$r_R\left(\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} R + \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix} R\right) = r_R\left(\begin{pmatrix} a & (\bar{n}_i) \\ 0 & b \end{pmatrix} R\right)$$

for some $\begin{pmatrix} a & (\bar{n}_i) \\ 0 & b \end{pmatrix} \in R$. We now compute $r_R\left(\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} R + \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix} R\right)$. If

$$\begin{pmatrix} r & (\bar{m}_i) \\ 0 & s \end{pmatrix} \in r_R\left(\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} R + \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix} R\right),$$

then $r = s = 0$, and $0 = x(\bar{m}_i) = (x\bar{m}_i)$ and $0 = y(\bar{m}_i) = (y\bar{m}_i)$. Thus $xm_i, ym_i \in M_i$ for each $i \in I$. This implies $x, y \in M_i$ or $m_i \in M_i$. This exactly characterizes the right annihilator.

Now, consider

$$\begin{pmatrix} 0 & (0, \ldots, 0, 1 + \langle x, y\rangle, 0, \ldots) \\ 0 & 0 \end{pmatrix} \in r_R\left(\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} R + \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix} R\right) = r_R\left(\begin{pmatrix} a & (\bar{n}_i) \\ 0 & b \end{pmatrix} R\right).$$

We then conclude $a \in \langle x, y\rangle$. If $a = 0$, then $\begin{pmatrix} 0 & (\bar{q}_i) \\ 0 & 0 \end{pmatrix} \in r_R\left(\begin{pmatrix} a & (\bar{n}_i) \\ 0 & b \end{pmatrix} R\right)$ for any $(\bar{q}_i) \in B$, contradicting our earlier computation. Thus $a \neq 0$, and since $a \in \langle x, y\rangle$, we may write $a = a(x, y) = xf(x, y) + yg(x, y)$ for some elements $f(x, y), g(x, y) \in \mathbb{C}[x, y]$. We claim that there exists a maximal ideal M containing a, but $M \neq \langle x, y\rangle$. If $y|a$ then take $M = \langle x - 1, y\rangle$. If $y \nmid a$ then since $a(x, y) \neq 0$, there exists a nonzero $k \in \mathbb{C}$ such that $a(x, k)$ is not a constant polynomial. Thus $a(x, k) = c(x - c_1)\cdots(x - c_t)$ for some $c, c_1, \ldots, c_t \in \mathbb{C}$ since $\mathbb{C}$ is algebraically closed. Then $a(c_1, k) = 0$ and so $a = a(x, y) \in \langle x - c_1, y - k\rangle = M$, which proves our claim. Now $\begin{pmatrix} a & (\bar{n}_i) \\ 0 & b \end{pmatrix} R \begin{pmatrix} 0 & (0, \ldots, 0, 1 + M, 0, \ldots) \\ 0 & 0 \end{pmatrix} = 0$ and so $\begin{pmatrix} 0 & (0, \ldots, 0, 1 + M, 0, \ldots) \\ 0 & 0 \end{pmatrix} \in r_R\left(\begin{pmatrix} a & (\bar{n}_i) \\ 0 & b \end{pmatrix} R\right)$ again contradicting our previous computation. Therefore R does not have right (a.c.).

We may conjecture that the homomorphic image of a ring with right (a.c.) has right (a.c.), and that for an ideal I of a ring R, if R/I has right (a.c.) and I has right (a.c.) as a ring (possibly without 1), then R has right (a.c.). However, the following example erases these possibilities.

Example 2.6. Let $A = \mathbb{Z}_2[x, y]$ and $I = \langle x^2, y^2\rangle$. Then $R = A/I$ does not have (a.c.). Let $J = \bar{x}R$, where $\bar{x} \in R$. Then $R/J \cong \mathbb{Z}_2[y]/\langle y^2\rangle$, and we note that R/J has (a.c.). Moreover, since $J = \{a\bar{x} + b\bar{x}\bar{y} \mid a, b \in \mathbb{Z}_2\}$, $r_J(S) = J$ for any $S \subseteq J$. Hence J has (a.c.) as a non-unital ring.

Proposition 2.7. *If R has right (a.c.) and I is an ideal of R, then $R/\ell_R(I)$ has right (a.c.).*

Proof. For elements $\bar{a}, \bar{b} \in \bar{R} = R/\ell_R(I)$, let $\bar{J} = \bar{R}\bar{a}\bar{R} + \bar{R}\bar{b}\bar{R}$. Since R has right (a.c.), $r_R(aR + bR) = r_R(cR)$ for some $c \in R$. We now claim that $r_{\bar{R}}(\bar{J}) = r_{\bar{R}}(\bar{c}\bar{R})$. Let $\bar{d} \in r_{\bar{R}}(\bar{J})$. Then $aRdI = 0$ and $bRdI = 0$, and so $dI \subseteq r_R(aR + bR) = r_R(cR)$. Hence $\bar{d} \in r_{\bar{R}}(\bar{c}\bar{R})$, which means $r_{\bar{R}}(\bar{J}) \subseteq r_{\bar{R}}(\bar{c}\bar{R})$. The reverse inclusion is obtained similarly, by reversing the implications. Therefore $R/\ell_R(I)$ has right (a.c.). $\square$

Finally, we consider the question of whether the classical right quotient ring $Q(R)$ has right (a.c.) when R has right (a.c.). This result plays an important role when studying the compactness of the space of minimal prime ideals in Section 3.

Proposition 2.8. *Suppose R has its classical right quotient ring $Q(R)$. If R has right (a.c.), then $Q(R)$ has right (a.c.). Moreover, the converse holds if R is reduced.*

Proof. Let $Q = Q(R)$ and $I = Qa_1b^{-1}Q + Qa_2b^{-1}Q$ for $a_1b^{-1}, a_2b^{-1} \in Q$. By hypothesis, $r_R(a_1R + a_2R) = r_R(a_3R)$ for some $a_3 \in R$. Note that $r_Q(I) = r_Q(a_1Q + a_2Q)$ since $b^{-1}Q = Q$. We will show that $r_Q(a_1Q + a_2Q) = r_Q(a_3Q)$. Suppose $cd^{-1} \in r_Q(a_1Q + a_2Q)$. Then $a_1Qc = 0$ and $a_2Qc = 0$; hence for every $u^{-1} \in Q$, $a_1Ru^{-1}c = 0$ and $a_2Ru^{-1}c = 0$. Let $u^{-1}c = c_1u_1^{-1}$ for some $c_1u_1^{-1} \in Q$. Then $a_1Rc_1 = 0$ and $a_2Rc_1 = 0$, and so $a_3Rc_1 = 0$. It then follows that

$$0 = a_3Rc_1u_1^{-1} = a_3Ru^{-1}c \Rightarrow 0 = a_3Qcd^{-1},$$

hence we obtain $r_Q(a_1Q + a_2Q) \subseteq r_Q(a_3Q)$. The reverse inclusion is obtained by reversing all the implications above. Therefore Q has right (a.c.).

Moreover, a ring R is reduced if and only if Q is reduced by [23, Theorem 16]. We can then prove that the converse holds by the same method as above. $\square$

3. Compact spaces of minimal prime ideals

Recall that for any $a \in R$, we defined $\mathrm{supp}(a) = \{P \in \mathrm{Spec}(R) \mid a \notin P\}$. Shin [7, Lemma 3.1] proved that for any ring R, $\{\mathrm{supp}(a) \mid a \in R\}$ forms a basis (for open sets) on $\mathrm{Spec}(R)$. This topology is called the *hull-kernel* topology. We regard $\mathrm{Min}(R)$ as a subspace of $\mathrm{Spec}(R)$. Also we will adopt the notations: $s(a) = \mathrm{supp}(a) \cap \mathrm{Min}(R)$ for any $a \in R$ and $S(q) = \mathrm{supp}(q) \cap \mathrm{Min}(Q(R))$ for any $q \in Q(R)$.

Huckaba and Keller [3, Theorem B], citing Quentel [11, Proposition 9], proved that for a commutative reduced ring R, the total quotient ring $T(R)$ of R is von Neumann regular if and only if R has (a.c.) and $\mathrm{Min}(R)$ is compact, if and only if R has Property (A) and $\mathrm{Min}(R)$ is compact. In [16, Theorem 3.3] it is proven that the classical right quotient ring $Q(R)$ is biregular if and only if R has Property (A) and $\mathrm{Min}(R)$ is compact, when R has a right maximal quotient ring which is reduced and R has the two-sided classical quotient ring $Q(R)$.

In this section we will extend the result of Huckaba and Keller [3, Theorem B] and Quentel [11, Proposition 9] to noncommutative rings, which is also a significant extension of [16, Theorem 3.3] as mentioned above. For notational convenience we let $Z(R)$ (respectively, $Z_l(R)$ and $Z_r(R)$) denote the set of (left, right) zero-divisors. Note that in a reduced ring, these sets agree.

Lemma 3.1. *Let R be a reduced ring. Then we have the following:*

(1) *Let J be a finitely generated ideal of R. The inclusion $J \subseteq P$ holds for some $P \in \mathrm{Min}(R)$ if and only if $r_R(J) \neq 0$.*
(2) *A prime ideal P of R is minimal if and only if for all $a \in P$, $r_R(a) \not\subseteq P$.*
(3) $Z(R) = \bigcup_{P \in \mathrm{Min}(R)} P$.
(4) *If $\mathrm{Min}(R)$ is compact, then for each $a \in R$, there exists a finitely generated ideal $J \subseteq r_R(a)$ with $r_R(J + aR) = 0$.*

Proof. (1) By [16, Lemma 3.1]. (2) By [24, Corollary 1.4].

(3) If $a \in Z(R)$, then $r_R(a) \neq 0$. By (1), $a \in P$ for some $P \in \mathrm{Min}(R)$. Conversely, let $b \in \bigcup_{P \in \mathrm{Min}(R)} P$. Then $b \in P$ for some P. By (2), $bc = 0$ for some $c \in R \setminus P$, and therefore $b \in Z(R)$.

(4) We refer the reader to the proof of [9, Proposition 1.16]. One needs only minor modifications for our case. □

Lemma 3.2. *Let R be a reduced ring with its classical right quotient ring $Q(R)$. Then:*

(1) $\mathrm{Min}(R) = \{M \cap R \mid M \in \mathrm{Min}(Q(R))\}$.
(2) $\mathrm{Min}(R)$ *is compact if and only if* $\mathrm{Min}(Q(R))$ *is compact.*

Proof. (1) Let $P \in \mathrm{Min}(R)$. We first claim that $P = PQ(R) \cap R$. To prove this, note that if $a \in PQ(R) \cap R$, then $ab \in P$ for some non-zero-divisor $b \in R$. Since P is completely prime and $b \notin P$ by Lemma 3.1(3), $a \in P$. We next show that $PQ(R)$ is a two-sided ideal of $Q(R)$. For any $rs^{-1} \in Q(R)$ and $ab^{-1} \in PQ(R)$ with $a \in P$, $s^{-1}a = a_1s_1^{-1}$ and hence $sa_1 = as_1 \in P$. Since $s \notin P$, $a_1 \in P$ and so $rs^{-1}ab^{-1} = ra_1s_1^{-1}b^{-1} \in PQ(R)$. Obviously, $PQ(R) \neq Q(R)$. Moreover, $PQ(R)$ is prime in $Q(R)$, for if A and B are ideals of $Q(R)$ with $AB \subseteq PQ(R)$, then we have $(A \cap R)(B \cap R) \subseteq P$ and so, say, $A \cap R \subseteq P$. Hence $A = (A \cap R)Q(R) \subseteq PQ(R)$, and therefore $PQ(R)$ is a prime ideal of $Q(R)$.

Next, for a minimal prime M of $Q(R)$, we claim that $M \cap R$ is minimal prime in R. To see this, if P is minimal prime in R with $P \subseteq M \cap R$, then $PQ(R) = M$ since M is minimal. Thus $P = PQ(R) \cap R = M \cap R$. Therefore $\mathrm{Min}(R) \supseteq \{M \cap R \mid M \in \mathrm{Min}(Q(R))\}$. Also, given $P \in \mathrm{Min}(R)$ and a minimal prime ideal $M \subseteq PQ(R)$ of $Q(R)$, then

$$P = PQ(R) \cap R \supseteq M \cap R$$

and by minimality $P = M \cap R$. This yields the reverse inclusion.

(2) We only prove sufficiency because the other implication can be proved by the same method. Suppose that $\mathrm{Min}(R)$ is compact and $\mathrm{Min}(Q(R)) = \cup_{i \in I} S(a_ib_i^{-1})$. We claim that $\mathrm{Min}(R) = \cup_{i \in I} s(a_i)$. If we let $P \in \mathrm{Min}(R)$, then by (1), $P = M \cap R$ for some minimal prime ideal M of $Q(R)$. Since $M \in \cup_{i \in I} S(a_ib_i^{-1})$, $M \in S(a_ib_i^{-1})$ for some $i \in I$. Hence $a_ib_i^{-1} \notin M$ and so $a_i \notin M \cap R = P$. Thus $P \in s(a_i)$ and therefore $\mathrm{Min}(R) = \cup_{i \in I} s(a_i)$. By hypothesis, $\mathrm{Min}(R) = \cup_{i=1}^n s(a_i)$. Letting $N \in \mathrm{Min}(Q(R))$, then by (1), $N \cap R \in \mathrm{Min}(R)$. Thus $N \cap R \in s(a_j)$ for some $1 \le j \le n$ and so $a_jb_j^{-1} \notin N$. It follows that $N \in S(a_jb_j^{-1})$ for some $1 \le j \le n$. Therefore $\mathrm{Min}(Q(R)) = \cup_{i=1}^n S(a_ib_i^{-1})$. □

It is a well known fact that strongly regular rings are reduced and biregular. But the converse is not true in general. For an example take the first Weyl algebra over a field of characteristic zero. Also, by [13, Theorem 6], reduced biregular rings are reduced p.p.-ring (recall, a ring is right p.p. if every principal right ideal is projective). But the converse is not true. For example, take a polynomial ring over a division ring. However, we have the following:

Lemma 3.3. *Let R be a ring with its classical right quotient ring $Q(R)$. The following statements are equivalent:*

(1) $Q(R)$ *is a reduced biregular ring;*
(2) $Q(R)$ *is a reduced p.p.-ring;*
(3) $Q(R)$ *is a strongly regular ring.*

Proof. (1) $\Leftrightarrow$ (2): This follows from [13, Theorem 6]. (3) $\Rightarrow$ (1): Obvious. (1) $\Rightarrow$ (3): Suppose that $Q(R)$ is a reduced p.p.-ring. For any $ab^{-1} \in Q(R)$, we have $r_{Q(R)}(ab^{-1}) = ef^{-1}Q(R)$ for some central idempotent $ef^{-1} \in Q(R)$, and so $ab^{-1}Q(R) \cap ef^{-1}Q(R) = 0$. Now we claim that $a+e$ is a non-zero-divisor in R. Assume that $(a+e)c = 0$ for some $c \in R$. Then $ac = -ec$ and so $ab^{-1}bc = -ef^{-1}fc \in ab^{-1}Q(R) \cap ef^{-1}Q(R) = 0$. Thus $bc \in r_{Q(R)}(ab^{-1}) = ef^{-1}Q(R)$ whence $bc = ef^{-1}rs^{-1}$ for some $rs^{-1} \in Q(R)$. Recall that ef^{-1} is a central idempotent, hence we have

$$bc = ef^{-1}bc = bcef^{-1} \Rightarrow c = cef^{-1} \Rightarrow fc = fcef^{-1} = ef^{-1}fc = 0 \Rightarrow c = 0$$

because b and f are non-zero-divisors in R. Therefore $a+e$ is not a zero-divisor in R. Let $u = ab^{-1} \in Q(R)$. Then we note that $ab^{-1}e = 0$ and hence

$$uQ(R) = ab^{-1}Q(R) = ab^{-1}(a+e)Q(R) = ab^{-1}aQ(R) = ab^{-1}ab^{-1}Q(R) = u^2Q(R).$$

Consequently, $Q(R)$ is strongly regular. $\square$

We are ready to prove our main result of this section.

Theorem 3.4. *Let R be a reduced ring with its classical right quotient ring $Q(R)$. Then the following statements are equivalent:*

(1) *$Q(R)$ is a strongly regular ring;*
(2) *R has (a.c.) and* $\mathrm{Min}(R)$ *is compact;*
(3) *R has Property (A) and* $\mathrm{Min}(R)$ *is compact.*

Proof. (1) $\Rightarrow$ (2): If we suppose that $Q(R)$ is strongly regular, then $Q(R)$ has (a.c.) and so does R by Proposition 2.8. Moreover, by [25, Proposition 4.1], every prime ideal of $Q(R)$ is maximal and so $\mathrm{Min}(Q(R)) = \mathrm{Spec}(Q(R))$ is compact. Therefore, by Lemma 3.2, $\mathrm{Min}(R)$ is compact.

(2) $\Rightarrow$ (3): We partially adapt the method in the proof of [1, Lemma 3.8]. Let I be a finitely generated ideal of R such that $I \subseteq Z_l(R)$. Let M be the set of all non-zero-divisors in R. Then M is an m-system disjoint from I. Hence I is contained in a prime ideal P that is disjoint from M. We now claim that P is minimal. Let $a \in P$. By [7, Theorem 4.9], $r_R(a) = r_R(r_R(b))$ for some $b \in R$. Assume (by way of contradiction) that $a, b \in Q$ for some $Q \in \mathrm{Min}(R)$. By Lemma 3.1(2), $r_R(a) \nsubseteq Q$ and $r_R(b) \nsubseteq Q$ and also $r_R(a) = r_R(r_R(b)) \subseteq Q$, contradicting the fact that $r_R(a) \nsubseteq Q$. Assume now (again, by way of contradiction) that $a, b \notin Q$ for some $Q \in \mathrm{Min}(R)$. Then $b \in r_R(a) \subseteq Q$, which is also a contradiction. Consequently, $a \in Q$ if and only if $b \notin Q$ for any $Q \in \mathrm{Min}(R)$. Note that $r_R(a+b) = 0$. If not, then $a+b \in T$ for some $T \in \mathrm{Min}(R)$ by Lemma 3.1(3), which is absurd by the preceding argument. Thus $a+b$ is not a zero-divisor. This implies that $a+b \notin P$ and so $b \notin P$. Thus we have for any $a \in P$, $ab = 0$ for some $b \notin P$. Hence by Lemma 3.1(2), P is minimal, and therefore by Lemma 3.1(1), $r_R(I) \neq 0$.

(3) $\Rightarrow$ (1): By Lemma 3.3 and [25, Theorem 4.5], it suffices to prove that every prime ideal of $Q(R)$ is maximal. Assume that $P \subsetneq Q$ are prime ideals in $Q(R)$ with P minimal. Choose $a \in Q \setminus P$. Since $\mathrm{Min}(R)$ is compact, $\mathrm{Min}(Q(R))$ is compact by Lemma 3.2. Since $Q(R)$ is reduced, there exists a finitely generated ideal $J \subseteq r_{Q(R)}(a)$ with $r_{Q(R)}(J + aQ(R)) = 0$ by Lemma 3.1(4). However, we note that $r_{Q(R)}(a) \subseteq P \subsetneq Q$ since $aQ(R)r_{Q(R)}(a) = 0$, $a \notin P$, and P is prime. Thus $J + aQ(R) \subseteq Q$. If $J + aQ(R) \nsubseteq Z_l(Q(R))$, then $J + aQ(R)$ contains a non-zero-divisor and so $Q(R) = J + aQ(R) \subseteq Q$, which is a contradiction. On the other hand, if $J + aQ(r) \subseteq Z_l(Q(R))$ then since $Q(R)$ has Property (A), $r_{Q(R)}(J + aQ(R)) \neq 0$, which is also a contradiction. Therefore every prime ideal of $Q(R)$ is maximal. $\square$

In view of Theorem 3.4 we may raise several questions about the possible redundancy of our hypotheses, which we answer in the following remark.

Remark 4. (1) In Theorem 3.4, the condition "R is reduced" is not superfluous. For example, let R be the ring of all sequences of 2×2 matrices over a field F which are eventually diagonal. Then R is von Neumann regular and all primes are maximal by [12, p.1865]. Thus $R = Q(R)$ is von Neumann regular, R has (a.c.) and $\mathrm{Min}(R)$ is compact. But, $R = Q(R)$ is not strongly regular.

(2) There exists a commutative reduced ring R with $\mathrm{Min}(R)$ compact, but the total quotient ring $T(R)$ is not strongly regular [11]. Thus the statements about (a.c.) and Property (A) in Theorem 3.4 are not redundant.

(3) In Theorem 3.4, we cannot replace "$Q(R)$ is strongly regular" by "R is strongly regular". For example, let R be the ring of integers. Then R has (a.c.) and Property (A). Moreover, $\mathrm{Min}(R)$ is compact. But R is not strongly regular, though its classical quotient ring is strongly regular.

(4) By [25, Proposition 4.1], if a ring R is biregular then every prime ideal of R is maximal. Note that if every prime ideal is maximal then $\mathrm{Min}(R) = \mathrm{Spec}(R)$ is compact. Thus if R is biregular then $\mathrm{Min}(R)$ is compact. But there exists a von Neumann regular ring R in which $\mathrm{Min}(R)$ is not compact by [14, Example 8.28]. Actually, the example is a unit-regular ring with general comparability.

(5) Note that if R is a von Neumann regular ring with general comparability, then R is biregular if and only if every prime ideal is maximal [14, Corollary 8.24]. Then we may conjecture that if R is a von Neumann regular ring with general comparability and $\mathrm{Min}(R)$ is compact, then R (or $Q(R)$) is biregular. Unfortunately, this is not true by the following example. Let D be a division ring and V be a right vector space over D with countably infinite basis. Let $R = \mathrm{End}_D(V)$ be the endomorphism ring of V, thinking of V as a right R-module. Then $R = Q(R)$ is von Neumann regular and right self-injective.

Thus by [14, Corollary 9.15], R satisfies general comparability. Moreover, since R is a prime ring, R has (a.c.) and $\mathrm{Min}(R)$ is compact. But $R = Q(R)$ is not biregular since the zero ideal is not maximal.

(6) In Theorem 3.4, can we replace "R is a reduced ring" by "R is right Noetherian"? Note that if R is right Noetherian, then $\mathrm{Min}(R)$ is compact. Thus the question is equivalent to asking: if R is right Noetherian and has right (a.c.) (or right Property (A)), is $Q(R)$ biregular (or von Neumann regular)? The answer is no, due to the following example. Let $S = \mathbb{Z}_2[x, y]/\langle x^2, y^2\rangle$. By Example 2.2, $R = \mathbb{M}_2(S)$ has right (a.c.). Since S is commutative and Noetherian, S has Property (A). Then by [16, Theorem 2.1], R has Property (A). Obviously, $\mathrm{Min}(R)$ is compact. But $Q(R)$ is neither biregular nor von Neumann regular. Assume that $Q(R)$ is biregular. Then $Q(R)(xye_{11}+xye_{22})Q(R) = EQ(R)$ for some central idempotent $E \in Q(R)$. Since $xye_{11}+xye_{22}$ is central in R, it is also central in $Q(R)$. Then $(xye_{11} + xye_{22})Q(R) = EQ(R)$ and so $E = (xye_{11} + xye_{22})A$ for some $A \in Q(R)$. It follows that $E = E^2 = (xye_{11} + xye_{22})A(xye_{11} + xye_{22})A = (xye_{11} + xye_{22})^2A^2 = 0$, contradicting the fact $(xye_{11} + xye_{22})Q(R) \neq 0$. Hence $Q(R)$ is not biregular. Moreover, $(xye_{11} + xye_{22})B(xye_{11} + xye_{22}) \neq xye_{11} + xye_{22}$ for any $B \in Q(R)$, and therefore $Q(R)$ is also not von Neumann regular.

A ring R is called *right coherent* if every finitely generated right ideal of R is finitely presented. A right coherent ring has the property that for any $a \in R$, $r_R(a)$ is a finitely generated right ideal of R by [18, Corollary 4.60]. Huckaba and Keller [3, Theorem 3] proved that for a commutative reduced coherent ring R, R has Property (A) if and only if R has (a.c.), if and only if $T(R)$ is von Neumann regular. We prove that this fact is also true for noncommutative rings.

Proposition 3.5. *Let R be a reduced right coherent ring with its classical right quotient ring $Q(R)$. Then the following statements are equivalent:*

(1) *$Q(R)$ is strongly regular;*
(2) *R has (a.c.);*
(3) *R has Property (A).*

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) follow from Theorem 3.4. (2) $\Rightarrow$ (1): We first claim that $\mathrm{Spec}(Q(R)) = \mathrm{Min}(Q(R))$. Let $P \in \mathrm{Spec}(Q(R))$ and $ab^{-1} \in P$. Then $a \in P$ and $r_R(a)$ is a finitely generated ideal of R by hypothesis. Since R has (a.c.), $\ell_R(r_R(a)) = \ell_R(c)$ for some $c \in R$. Thus we have $r_R(a) = r_R(\ell_R(c))$. By the same argument as in the proof that (2) implies (3) of Theorem 3.4, $P \in \mathrm{Min}(Q(R))$. Thus $\mathrm{Min}(Q(R))$ is compact and so is $\mathrm{Min}(R)$ by Lemma 3.2. Therefore $Q(R)$ is strongly regular by Theorem 3.4.

(3) $\Rightarrow$ (1): Again, it suffices to prove that every prime ideal of $Q(R)$ is maximal. Assume that $P \subsetneq Q$ are prime ideals $Q(R)$ with P minimal. Choose $ab^{-1} \in Q \setminus P$. Note that $r_{Q(R)}(ab^{-1}) = r_{Q(R)}(a) = r_R(a)Q(R)$. Now $r_R(a) = \sum_{i=1}^n x_iR$ for some $x_i \in R$ by hypothesis. Since $Q(R)$ is reduced, $r_{Q(R)}(a)$ is an ideal of $Q(R)$ and hence $r_{Q(R)}(a) = \sum_{i=1}^n Q(R)x_iQ(R)$. Let $\sum_{i=1}^n Q(R)x_iQ(R) = J$. Then $J = r_{Q(R)}(a) = r_{Q(R)}(ab^{-1}) \subseteq P$ by Lemma 3.1(2), and so $J + Q(R)ab^{-1}Q(R) \subseteq Q$. We note that $J + Q(R)ab^{-1}Q(R) \subseteq Z_l(Q(R))$. Since R has Property (A), $r_{Q(R)}(J + Q(R)ab^{-1}Q(R)) \neq 0$ and so by Lemma 3.1(1), $J + Q(R)ab^{-1}Q(R) \subseteq T$ for some $T \in \mathrm{Min}(Q(R))$. This contradicts the fact that $J + Q(R)ab^{-1}Q(R)$ is not contained in any minimal prime ideal since $J = r_{Q(R)}(ab^{-1})$. Hence every prime ideal of $Q(R)$ is maximal. $\square$

Remark 5. In Proposition 3.5, the conditions "reduced" and "right coherent" are not superfluous. First, we note that right Noetherian rings are right coherent by [18, Example 4.46(a)]. Thus we can use the example in Remark 4(6) to show that the hypothesis that R is reduced is needed. Next, let $R = \mathbb{Q}[x_1, x_2, \ldots]$ be the polynomial ring over the field of rational numbers $\mathbb{Q}$ with commuting indeterminates $x_1, x_2, \ldots$ with relations given by $x_ix_j = 0$ for all $i \neq j$. By [26, Ex. 13.17], R is reduced, $Q(R)$ is not strongly regular and $\mathrm{Min}(R)$ is not compact. Obviously R has Property (A). Since $r_R(x_1) = \sum_{i\neq 1} x_iR$, R is not coherent. Now we claim that R has (a.c.). Actually, for any $\alpha, \beta \in R$, $r_R(\alpha, \beta) = r_R(\alpha^2 + \beta^2)$ because the indeterminates appearing in α and β coincide with those appearing in $\alpha^2 + \beta^2$.

We finally raise the following questions.

Question 1. If R has right (a.c.), then does $R[x]$ (or $R[[x]]$) have right (a.c.)?

We note that if R is a commutative reduced left coherent ring then $\mathrm{Min}(R)$ is compact by [8, Corollary 4.2.16].

Question 2. Is $\mathrm{Min}(R)$ compact if R is a reduced left coherent ring?

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