On the Duality Principle for Linear Dynamical Systems Over Commutative Rings

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ABSTRACT

The main result in this paper characterizes those commutative rings $R$ having the property that every linear dynamical system over $R$ verifies the duality principle [i.e., the system $\Sigma$ is observable (reachable) if and only if the dual system $\Sigma'$ is reachable (observable)]. This characterization is given in terms of the finitely generated faithful ideals of $R$, and it generalizes a result due to Ching and Wyman for the noetherian case. In case $R$ satisfies the additional condition of being a reduced ring, we prove that the duality principle holds in $R$ if and only if the height of every finitely generated ideal of $R$ is zero.

INTRODUCTION

The purpose of this paper is to study the duality principle, “reachability is dual to observability,” for linear dynamical systems over commutative rings. Namely, the duality principle holds in the commutative ring $R$ if for every linear dynamical system $\Sigma$ over $R$, $\Sigma$ is observable (reachable) if and only if the dual system $\Sigma'$ of $\Sigma$ is reachable (observable).

The classical case of this principle, i.e. when $R$ is a field, can be found in R. E. Kalman [4]. This case has been generalized by W. S. Ching and B. F. Wyman in [2]. They have proved that the duality principle holds for noetherian total quotient rings.

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The following is still open:

**Duality problem.** What kinds of commutative rings do satisfy the duality principle?

This paper is organized as follows. Section 1 contains some definitions and notation. Section 2 is devoted to proving the main result in this paper, specifically, that the duality principle holds in a commutative ring $R$ if and only if every finitely generated faithful ideal of $R$ contains a unit. Finally, we use this characterization to prove that for reduced rings the duality principle holds if and only if the height of every finitely generated ideal is zero.

1. PRELIMINARIES

This section is devoted to stating some definitions and notation.

Throughout this paper $R$ will denote a commutative ring with unit.

**Definition 1.1.** A constant discrete-time linear dynamical system $\Sigma$ over the ring $R$ consists of three $R$-modules $X$, $U$, and $Y$, together with three $R$-module maps

$$
F : X \rightarrow X, \\
G : U \rightarrow X, \\
H : X \rightarrow Y.
$$

Throughout this paper, the modules $X$, $U$, and $Y$ will be always assumed to be finitely generated free $R$-modules: $X \cong R^n$, $U \cong R^m$, $Y \cong R^p$. The integer $n$ will be called the rank of the $R$-system $\Sigma$. Under this hypothesis we can think of $F$, $G$, and $H$ as matrices with coefficients in $R$. Matrices will be written on the left, so that $F$ is $n \times n$, $G$ is $n \times m$, and $H$ is $p \times n$. Later on we will usually omit most adjectives and we will simply say that $\Sigma = (R^n, R^m, R^p, F, G, H)$ or $\Sigma = (F, G, H)$ is an $R$-system.

**Definition 1.2.** The dual system, $\Sigma'$, of the $R$-system $\Sigma = (F, G, H)$ is the $R$-system given by the triple $\Sigma' = (F', H', G')$ where $F'$, $H'$, and $G'$ are the dual $R$-homomorphisms of $F$, $H$, and $G$ respectively.

Associated with $\Sigma = (F, G, H)$ we have for each nonnegative integer $i$ the $R$-linear maps $F^iG$ from $R^m$ to $R^n$ and $HF^i$ from $R^n$ to $R^p$. The reachabil-
ity map $G$ of $\Sigma$ from $\bigoplus_{i>0} R^m$ into $R^n$ is given by $G = \bigoplus_{i>0} F^i G$, and the observability map $\tilde{H}$ of $\Sigma$ from $R^n$ into $\prod_{i>0} R^p$ is given by

$$\tilde{H}(x) = (H(x), H F(x), H F^2(x), \ldots)$$

for every $x$ in $R^n$.

**Definition 1.3.** The $R$-system $\Sigma = (F, G, H)$ is reachable if $G$ is surjective. The $R$-system $\Sigma = (F, G, H)$ is observable if $\tilde{H}$ is injective.

By the Cayley-Hamilton theorem over $R$, the $R$-linear map $G$ is determined by the $n \times (mn)$ matrix

$$c = [G, FG, \ldots, F^{m-1} G],$$

and the $R$-linear map $\tilde{H}$ is determined by the $(pn) \times n$ matrix

$$\tilde{H} = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix},$$

where $n$ is the rank of $\Sigma$.

**Theorem 1.4.** Let $\Sigma = (F, G, H)$ be an $R$-system with rank $n$. The following statements are equivalent:

(i) $\Sigma$ is reachable.

(ii) The columns of $G$ generate $R^n$.

(iii) The ideal $\mathcal{Q}_n(G)$ generated by all the $n \times n$ minors of the matrix $G$ is the unit ideal.

**Theorem 1.5.** Let $\Sigma = (F, G, H)$ be an $R$-system of rank $n$. The following statements are equivalent:

(i) $\Sigma$ is observable.

(ii) The ideal $\mathcal{Q}_n(H)$ generated by all the $n \times n$ minors of $H$ has zero annihilator (i.e., $\mathcal{Q}_n(H)$ is a finitely generated faithful ideal of $R$).

**Corollary 1.6.** If $\Sigma = (F, G, H)$ is a reachable $R$-system, then $\Sigma^t = (F^t, H^t, G^t)$ is an observable $R$-system.

The proofs of the above results can be found in [1, pp. 55, 58].
2. THE CHARACTERIZATION THEOREM

**Definition 2.1.** Let $R$ be a ring. We say that the duality principle holds for $R$ if for every system $\Sigma = (F, G, H)$ over $R$ the following statements are equivalent:

(i) The system $\Sigma = (F, G, H)$ is observable over $R$.
(ii) The system $\Sigma' = (F', G', H')$ is reachable over $R$.

**Remark 2.2.** If $R$ is a noetherian ring that is equal to its own total quotient ring, then the duality principle holds for $R$ (see [2]).

Now we can state the duality theorem as follows:

**Theorem 2.3.** Let $R$ be a ring. The following statements are equivalent:

(i) The duality principle holds for $R$.
(ii) Every finitely generated faithful ideal of $R$ contains a unit.

**Proof.** (ii) \(\Rightarrow\) (i): Let $\Sigma = (F, G, H)$ be an $R$-system. If $\Sigma$ is $R$-observable and $H$ is the $(pn) \times n$ matrix 

$$
\tilde{H} = \begin{bmatrix}
H \\
HF \\
\vdots \\
HF^{n-1}
\end{bmatrix},
$$

then, by Theorem 1.5, $\mathcal{U}_n(\tilde{H})$ is a finitely generated faithful ideal of $R$. By statement (ii) one has $\mathcal{U}_n(\tilde{H}) = R$. Therefore, by Theorem 1.4, $\Sigma'$ is $R$-reachable.

(i) \(\Rightarrow\) (ii): Let $I$ be a finitely generated faithful ideal of $R$, with $I \neq R$. Let $x_1, \ldots, x_n$ be a system of generators of $I$. Let $\Sigma$ be the $R$-system

$$
\Sigma : R^m \xrightarrow{G} R \xrightarrow{F} R \xrightarrow{H} R^p,
$$

where $F$ and $G$ are arbitrary $R$-homomorphisms and $H$ is given by the $p \times 1$
matrix

\[ H = \begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_p
\end{bmatrix}. \]

Since \( \tilde{H} = [H] \), we have

\[ \mathcal{V}_1(\tilde{H}) = \mathcal{V}_1(H) = I. \]

Therefore, by theorem 1.5, the R-system \( \Sigma \) is R-observable. On the other hand we have

\[ \mathcal{V}_1(H^t) = \mathcal{V}_1(H) = I \subsetneq R, \]

and hence, by Theorem 1.4, \( \Sigma' \) is not R-reachable.

**Corollary 2.4.** If the duality principle holds for the ring \( R \), then \( R \) is a total quotient ring.

**Proof.** Let \( r \) be a nonzero divisor of \( R \). Then the principal ideal \( I = rR \) is faithful. By Theorem 2.3, one has \( I = R \). Therefore \( r \) is a unit of \( R \).

**Corollary 2.5.** Let \( R \) be a noetherian ring. The following statements are equivalent:

(i) The duality principle holds for \( R \).

(ii) \( R \) is equal to its own total quotient ring.

**Proof.** It follows from Remark 2.2 and Corollary 2.4.

**Corollary 2.6.** Let \( R \) be a ring such that for every finitely generated ideal \( I \) of \( R \), \( I \neq R \), the height of \( I \) is zero (i.e., \( I \) is contained in a minimal prime ideal of \( R \)). Then the duality principle holds for \( R \).

**Proof.** Let \( I \) be a finitely generated ideal of \( R \). If \( I \neq R \), then \( I \) is contained in a minimal prime ideal of \( R \). Therefore \( I \) is not faithful (see [5, p. 63, Exercise 8]. Hence by Theorem 2.3 the duality principle holds for \( R \).
REMARK 2.7. In particular, if $R$ is a zero-dimensional ring (i.e., the Krull dimension of $R$ is zero), then the duality principle holds for $R$.

Now we can state duality theorem for reduced rings.

PROPOSITION 2.8. Let $R$ be a reduced ring. The following statements are equivalent:

(i) The duality principle holds for $R$.
(ii) For every finitely generated ideal $I$ of $R$, $I \neq R$, the height of $I$ is zero.

Proof. (ii) $\Rightarrow$ (i): It follows from Corollary 2.6.
(i) $\Rightarrow$ (ii): Let $I$ be a finitely generated ideal of $R$, with $I \neq R$. Then, by Theorem 2.3, $I$ is not faithful. Let $s$ be a nonzero element of $R$ such that $sI = (0)$. Since $R$ is a reduced ring, there exists a minimal prime ideal $\mathfrak{p}$ of $R$ such that $s \notin \mathfrak{p}$. Hence $sI = (0) \subseteq \mathfrak{p}$ implies $I \subseteq \mathfrak{p}$. Thus the height of $I$ is zero.

REFERENCES


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