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Easton's theorem and large cardinals from the optimal hypothesis

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ABSTRACT

The equiconsistency of a measurable cardinal with Mitchell order $o(\kappa) = \kappa^{++}$ with a measurable cardinal such that $2^\kappa = \kappa^{++}$ follows from the results by W. Mitchell (1984) [13] and M. Gitik (1989) [7]. These results were later generalized to measurable cardinals with 2^κ larger than κ^{++} (see Gitik, 1993 [8]).

In Friedman and Honzik (2008) [5], we formulated and proved Easton's (1970) theorem [4] in a large cardinal setting, using slightly stronger hypotheses than the lower bounds identified by Mitchell and Gitik (we used the assumption that the relevant target model contains $H(\mu)$, for a suitable μ, instead of the cardinals with the appropriate Mitchell order).

In this paper, we use a new idea which allows us to carry out the constructions in Friedman and Honzik (2008) [5] from the optimal hypotheses. It follows that the lower bounds identified by Mitchell and Gitik are optimal also with regard to the general behavior of the continuum function on regulars in the context of measurable cardinals.

1. Introduction

In the early 1970s, W. Mitchell introduced a new classification of large cardinals based on the notion of measurability. For normal κ-complete ultrafilters U and W over κ, he defined what is now called *Mitchell order* $U \lhd W$ iff U is an element of the ultrapower of the universe V by the ultrafilter W. The order $\lhd$ is irreflexive and well-founded. It follows that one can assign to each normal κ-complete ultrafilter over κ its $\lhd$-rank by $o(U) = \sup\{o(W)+1 \mid W \lhd U\}$, and to each cardinal κ its *Mitchell order* $o(\kappa) = \sup\{o(U)+1 \mid U$ is a normal κ-complete ultrafilter over $\kappa\}$. One can further show that if $2^\kappa = \kappa^+$, then $o(\kappa) \leqslant \kappa^{++}$.

The hypothesis that there exists a measurable cardinal κ such that $o(\kappa) = \kappa^{++}$ was shown to have the optimal consistency strength for a variety of propositions. In particular, it is the optimal large cardinal hypothesis for the failure of GCH at a measurable cardinal and the failure of SCH. The failure of GCH at a measurable was first forced in the mid 1970s by J. Silver (unpublished; see [3] for an account), assuming the existence of a κ^{++}-supercompact cardinal κ. In the early 1980s, Mitchell developed a core model for sequences of measures, see [13], and showed that if there is a measurable cardinal where GCH fails, then there exists an inner model with $o(\kappa) = \kappa^{++}$ for some κ. Thus, Silver's result provided an upper bound and Mitchell's result a lower bound for the consistency strength for the failure of GCH at a measurable.

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In the late 1980s (unpublished; see [3] for an account), H. Woodin made a substantial improvement with regard to the strength of the large cardinal hypothesis needed to construct a model where GCH fails. Assuming GCH, he started with the existence of an elementary embedding $j : V \to M$ with critical point κ such that

$$^{\kappa}M \subseteq M, \text{ and for some } f : \kappa \to \kappa,\, j(f)(\kappa) = \kappa^{++}.^{3} \tag{1.1}$$

The consistency strength of the existence of such j is far weaker than that of the existence of a κ^{++}-supercompact cardinal κ and seemed promisingly close to the lower bound $o(\kappa) = \kappa^{++}$ as identified by Mitchell. It was M. Gitik who finally showed in [7] that these two notions – (1.1) and the existence of κ with $o(\kappa) = \kappa^{++}$ – are in fact equiconsistent. Gitik's idea was to transform by forcing the chain of normal κ-complete ultrafilters under the Mitchell order into a commuting chain of (non-normal) κ-complete ultrafilters under the Rudin–Keisler order; such a commutative system of ultrafilters generates via the direct limit the elementary embedding j used by Woodin. Thus, when all these results are combined, it was shown that the failure of GCH at a measurable, and also the failure of SCH (by subsequent singularization by means of the Prikry forcing), are both equiconsistent with the existence of a measurable cardinal κ of Mitchell order κ^{++} in a model satisfying GCH.

Woodin's assumption (1.1) is a weakening of the properties holding for an embedding witnessing that κ is an $H(\kappa^{++})$-strong cardinal (also called $\mathscr{P}_2(\kappa)$-hypermeasurable or 2-strong cardinal); see Definition 2.1. The difference between (1.1) and an $H(\kappa^{++})$-strong embedding is quite substantial: under GCH, the condition that $H(\kappa^{++})$ of V is included in M for instance implies that there are many measurable cardinals below κ. On the other hand, it is implicit in Gitik's construction in [7] that a cardinal κ as in (1.1) can be the least measurable cardinal.

Let us briefly explain why the difference between (1.1) and an $H(\kappa^{++})$-strong embedding is immaterial for Woodin's argument while it matters for more general arguments, as the one in [5]. This paragraph also serves as a quick review of the lifting method, see [3] for more details. Woodin's construction uses Silver's original idea of "lifting" an embedding $j : V \to M$ to a generic extension for some forcing notion $\mathbb{P}$, where j^* is a lifting of j with respect to a $\mathbb{P}$-generic filter G if $j^* : V[G] \to M[H]$ is elementary, j^* extends j, and H is $j(\mathbb{P})$-generic over M. A sufficient condition for the existence of such a lift, identified by Silver, see Fact 2.4, is to find H as above which satisfies $j[G] \subseteq H$, i.e. the point-wise image of G is included in H. If j^* is definable in $V[G]$, then j^* witnesses the measurability of κ in $V[G]$. Fix an embedding j as in (1.1) but assume for simplicity that $f : \kappa \to \kappa$ is particularly simple, i.e. $f(\alpha) = \alpha^{++}$ for each regular $\alpha < \kappa$ which by elementarity implies $(\kappa^{++})^M = \kappa^{++}$. A natural way to force the failure of GCH, starting with GCH and this j, is to iterate in reverse Easton fashion the Cohen forcing $\mathrm{Add}(\alpha, \alpha^{++})$ which adds α^{++}-many Cohen subsets to each inaccessible cardinal $\alpha \leqslant \kappa$; this is the forcing $\mathbb{P}$ both Silver, and Woodin used.[4] If one looks at $j(\mathbb{P})$, one notices that $j(\mathbb{P})$ is equal to $\mathbb{P}$ up to κ, and is trivial in the interval (κ, μ), where μ is the least inaccessible cardinal in M above κ, and then again is non-trivial in the interval $[\mu, j(\kappa)]$. Woodin's argument was a major improvement on Silver's method because he devised a way of finding a generic for the stage $j(\kappa)$ of $j(\mathbb{P})$ without assuming the supercompactness of κ. With regard to the difference between (1.1) and $H(\kappa^{++})$-strength, notice that since μ must be greater than κ^{++} of M, which is the real κ^{++}, $j(\mathbb{P})$ is trivial in the interval $(\kappa, \kappa^{++}]$. Consequently, the requirement for $H(\kappa^{++})$ being in M does not play a role in building the $j(\mathbb{P})$-generic H over M, and for this particular argument, both hypotheses are equally good.

Let us now turn to the present paper. In [5], we generalized the original argument of W. Easton [4] concerning the continuum function on regular cardinals to a large cardinal context, focusing mainly on measurable cardinals. In this setting, it became necessary to control the powers of not only the inaccessible cardinals α below a given large cardinal κ, but also of the successor cardinals. For this reason we used the slightly stronger assumption of $H(F(\kappa))$-strength. For instance, the construction in [5] does not work with the weaker hypothesis of (1.1) if we aim to force $2^{\alpha} = \alpha^{++}$ for *every* regular cardinal $\alpha \leqslant \kappa$. The reason is that now $j(\mathbb{P})$ is non-trivial at both κ^{+} and κ^{++} of M, and $H(\kappa^{++})$ belonging to M seemed essential to procure the desired generic filter for the Cohen forcing at κ^{++} in the sense of M (see the paragraph just before Claim 3.3 and Observation 3.5 for a more precise statement of the problems involved; these problems do not apply to κ^{+} because M is closed under κ-sequences in V).[5]

A natural question arises whether the results in [5] can be proved from the optimal assumptions along the lines of (1.1) (see Section 5 where the optimal assumptions are generalized to Mitchell order on extenders to account for cases where $2^{\kappa} > \kappa^{++}$). In this paper we show that this indeed is possible.

This does not seem all that surprising – after all, the set of successor cardinals is small in any normal ultrafilter and so controlling the behavior of the continuum function at successors should not have implications for the optimal large-cardinal strength needed. However, an intuition is not the same as a proof. The principal method of the proof – the lifting argument – does seem to require some degree of correspondence between $H(\kappa^{++})$ of M and the real $H(\kappa^{++})$ (to stay with our typical example of κ^{++}). This presents a technical challenge with surprising connections to general forcing-related problems (see

[3] In fact, such f can be forced to exist by Woodin's *fast function forcing* (see [9] for an argument); thus it suffices to assume $^{\kappa}M \subseteq M$ and $j(\kappa) > \kappa^{++}$ in (1.1) above.

[4] We are sweeping some details under the rug here; Woodin actually needed to add some extra forcing to resolve certain technical difficulties with the lifting, so he worked with a forcing more complicated than just $\mathbb{P}$ above.

[5] There is a technical point here; if $(\kappa^{++})^M$ is strictly less than κ^{++} for an embedding as in (1.1), which can easily happen, then one can hope that the generic for $j(\mathbb{P})$ at $(\kappa^{++})^M$ can be obtained more easily. This may be true, but in any case, the real κ^{++} is a regular cardinal in M, and so the forcing $j(\mathbb{P})$ is non-trivial in the interval $[(\kappa^{++})^M, \kappa^{++}]$, and we face the same kind of problem as described above.

the discussion following Question 2 in the last section). Inspired by U. Abraham's paper [1], we have solved this problem by artificially adding a sufficient degree of correspondence between $H(\kappa^{++})^M$ and $H(\kappa^{++})$ by means of forcing, which allows us to lift the original embedding.

The paper is organized as follows. In Section 2 we define notions we are going to use and state some useful propositions. Section 3 contains the main results of the paper, formulated for the special (but typical) case of forcing $2^\alpha = \alpha^{++}$ for every regular cardinal $\alpha \leqslant \kappa$, while preserving the measurability of κ. In Section 4, we generalize the technique of Section 3 to a larger class of Easton functions. In Section 5, we use the notion of Mitchell order on extenders to generalize the results still more to situations where $2^\kappa = \kappa^{+n}$ for $n \in \omega, n \geqslant 2$. In the last Section 6, we state some open problems.

2. Preliminaries

Our forcing conventions are standard, following for instance [10]. We use the terms "κ-closed" and "κ-distributive" to mean "$<\kappa$-closed" and "$<\kappa$-distributive", in keeping with the convention regarding chain conditions.

Let us give precise definitions of the notions which we have mentioned in Section 1.

Definition 2.1. We say that κ is an *$H(\theta)$-strong cardinal*, where $\kappa < \theta$ and θ is a cardinal, if there exists an elementary embedding j from V into some transitive class M with critical point κ such that $j(\kappa) > \theta$, and $H(\theta)$ is included in M.

At the suggestion of a referee, we explicitly include "$H(\theta)$" in the name of the large cardinal concept in Definition 2.1 in order to distinguish it from the related concept of an α-strong cardinal as defined for instance in [12] or [10].[6] We prefer the H-hierarchy because it is less dependent on the continuum function which is closely tied to V-hierarchy.

If GCH is assumed, and θ is regular (this is sufficient for our purposes here), then the elementary embedding witnessing the $H(\theta)$-strength of κ can be taken to have the additional property that $M = \{j(f)(\alpha) \mid f : \kappa \to V \wedge \alpha < \theta\}$, $\theta < j(\kappa) < \theta^+$, and M is closed under κ-sequences in V (such a j is called an *extender ultrapower embedding*).

If we omit the condition that $H(\theta)$ is included in M, we obtain a weaker notion: if M is closed under κ-sequences and $j(\kappa) > \theta$, we get a large cardinal concept called *θ-tallness* in [9]. For our purposes, we find it useful to work with nicer embeddings than the tall ones.[7]

Definition 2.2. Assume GCH. We say that $j : V \to M$ with critical point κ is a *κ^{++}-correct embedding* if j satisfies:

(i) M is closed under κ-sequences in V,
(ii) $\kappa^{++} = (\kappa^{++})^M$.

Note that (ii) implies $\kappa^{++} < j(\kappa)$, and so a κ^{++}-correct cardinal is κ^{++}-tall. If j is κ^{++}-correct, one can use the usual extender ultrapower construction to get an even better embedding.

Definition 2.3. We call j a *κ^{++}-correct extender embedding* if j satisfies conditions (i)–(ii) in Definition 2.2, and moreover:

(iii) $M = \{j(f)(\alpha) \mid f : \kappa \to V \wedge \alpha < \kappa^{++}\}$.

We say that κ is *κ^{++}-correct* if there is a κ^{++}-correct embedding with critical point κ.

It is shown in [7] that if V satisfies GCH and $j : V \to M$ with critical point κ is as in (1.1), then there is a generic extension V^* satisfying GCH such that κ is κ^{++}-correct in V^*. Hence, we can use the assumption of κ^{++}-correctness in our arguments because it has the same consistency strength as the existence of κ with $o(\kappa) = \kappa^{++}$.

We now provide a quick review of the results relevant to lifting of embeddings.

Fact 2.4. *Let $\mathbb{P}$ be a forcing notion and $j : V \to M$ an embedding with critical point κ. Then the following hold* (*for proofs, see* [3]):

(i) (*Silver*) *Assume G is $\mathbb{P}$-generic over V and H is $j(\mathbb{P})$-generic over M such that $j[G] \subseteq H$. Then there exists an elementary embedding $j^* : V[G] \to M[H]$ such that $j^* \restriction V = j$, and $H = j^*(G)$. We say that j lifts to $V^{\mathbb{P}}$.*
(ii) *If j is moreover an extender ultrapower embedding, $\mathbb{P}$ is a κ^+-distributive forcing notion and G is $\mathbb{P}$-generic over V, then the filter G^* in $j(\mathbb{P})$ defined as*

$$G^* = \left\{q \;\middle|\; \exists p \in G,\ j(p) \leqslant q\right\}$$

is $j(\mathbb{P})$-generic over M.
(iii) *If $j : V \to M$ is an extender ultrapower embedding, so is $j^* : V[G] \to M[H]$.*

[6] Here, κ is called α-strong if $V_{\kappa+\alpha}$ is included in the target model.

[7] To our knowledge, an embedding as in Definition 2.2 does not yet have a specific name; we propose one here. Note that we give the definition just for $\theta = \kappa^{++}$ but a generalization to larger cardinals is straightforward.

3. The crucial step: κ^{++}

Theorem 3.1 captures the main idea of this paper. Theorem 4.1 and Corollary 4.2 are direct applications of Theorem 3.1 based on results in [6] and [5].

Theorem 3.1. *Assume GCH and let $j : V \to M$ be a κ^{++}-correct extender embedding with critical point κ. Then there exists a cofinality-preserving forcing notion $\mathbb{P}$ such that if G is $\mathbb{P}$-generic, the following hold in $V[G]$:*

(i) *$2^\alpha = \alpha^{++}$ for every regular cardinal $\alpha < \kappa$ which is the double successor of an inaccessible cardinal $\beta < \kappa$ (where α is the double successor of β if $\alpha = \beta^{++}$).*
(ii) *The embedding j lifts to $j^* : V[G] \to M[j^*(G)]$, and j^* is a κ^{++}-correct extender embedding in $V[G]$.*

Proof. The proof of the theorem will follow from Lemmas 3.2 and 3.4, with Claims 3.3 and 3.6 providing the key ingredients.

For a regular cardinal α and an ordinal $\beta > 0$ we write $\mathrm{Add}(\alpha, \beta)$ to denote the usual Cohen forcing which adds β-many Cohen subsets of α: a condition p belongs to $\mathrm{Add}(\alpha, \beta)$ if and only if p is a function from a subset of $\alpha \times \beta$ to 2 of size less than α. Wherever we need, we use other equivalent representations (for instance we can view $\mathrm{Add}(\alpha, \beta)$ as adding β-many new Cohen functions from α to α).

Let us now define the forcing $\mathbb{P}$. $\mathbb{P}$ will be a two-stage iteration $\mathbb{P}^0 * \dot{\mathbb{P}}^1$, where $\dot{\mathbb{P}}^1$ is a $\mathbb{P}^0$-name in M:

(1) $\mathbb{P}^0$ is an iteration of length κ with Easton support, $\mathbb{P}^0 = \langle (\mathbb{P}^0_\xi, \dot{Q}_\xi) \mid \xi < \kappa \rangle$, where $\dot{Q}_\xi$ is a name for the trivial forcing unless ξ is an inaccessible cardinal $< \kappa$, in which case

$$\mathbb{P}^0_\xi \Vdash \text{“}\dot{Q}_\xi \text{ is the forcing } \mathrm{Add}\big(\xi^+, \xi^{++}\big) * \dot{\mathrm{Add}}\big(\xi^{++}, \xi^{+4}\big),\text{”} \tag{3.1}$$

where $\mathrm{Add}(\xi^+, \xi^{++})$ is viewed as a product forcing which adds ξ^{++}-many Cohen functions from ξ^+ to ξ^+, and $\dot{\mathrm{Add}}(\xi^{++}, \xi^{+4})$ is viewed as (a name for) a forcing adding ξ^{+4}-many Cohen subsets of ξ^{++}.
(2) Notice that $\mathbb{P}^0$ is an element of M. $\dot{\mathbb{P}}^1$ is defined in M to be a $\mathbb{P}^0$-name which satisfies:

$$M \models \mathbb{P}^0 \Vdash \text{“}\dot{\mathbb{P}}^1 \text{ is the forcing } \mathrm{Add}\big(\kappa^+, \kappa^{++}\big) * \dot{\mathrm{Add}}\big(\kappa^{++}, 1\big),\text{”} \tag{3.2}$$

where $\mathrm{Add}(\kappa^+, \kappa^{++})$ is viewed as a product forcing which adds κ^{++}-many Cohen functions from κ^+ to κ^+, and $\dot{\mathrm{Add}}(\kappa^{++}, 1)$ is viewed as (a name for) a forcing adding a single Cohen subset of κ^{++}.

Lemma 3.2 *(GCH). $\mathbb{P}$ is a cofinality-preserving forcing notion over V.*

Proof. The forcing $\mathbb{P}^0$ is cofinality-preserving by standard arguments. Let G_κ be a $\mathbb{P}^0$-generic filter over V; then G_κ is also $\mathbb{P}^0$-generic over M. In order to verify that $\mathbb{P}$ is cofinality-preserving, it suffices to check that the forcing $(\dot{\mathbb{P}}^1)^{G_\kappa}$ defined in $M[G_\kappa]$ preserves cofinalities when forced over $V[G_\kappa]$. Notice first that $\mathrm{Add}(\kappa^+, \kappa^{++})$ of $M[G_\kappa]$ is the same as $\mathrm{Add}(\kappa^+, \kappa^{++})$ of $V[G_\kappa]$: this is because $\mathbb{P}^0$ has the κ-cc, and hence by standard arguments $M[G_\kappa]$ is still closed under κ-sequences in $V[G_\kappa]$. Let g be $\mathrm{Add}(\kappa^+, \kappa^{++})^{V[G_\kappa]}$-generic over $V[G_\kappa]$. Then by the previous sentence, g is also $\mathrm{Add}(\kappa^+, \kappa^{++})^{M[G_\kappa]}$-generic over $M[G_\kappa]$. Work in $M[G_\kappa * g]$ and let Q^* denote the forcing $\mathrm{Add}(\kappa^{++}, 1)$ of $M[G_\kappa * g]$. In the key Claim 3.3 we show that Q^* behaves properly over $V[G_\kappa * g]$ and this is enough to finish the proof of Lemma 3.2.

Note that Claim 3.3 in non-trivial: if the original M misses some subsets of κ^+ from V, then Q^* is a proper subset of $\mathrm{Add}(\kappa^{++}, 1)^{V[G_\kappa * g]}$, and hence cannot be κ^{++}-closed over $V[G_\kappa * g]$. Incidentally, there is a good reason why we attempt to force with Q^* over $V[G_\kappa * g]$: if M misses some subset of κ^+ from V, then no $\mathrm{Add}(\kappa^{++}, 1)^{V[G_\kappa * g]}$-generic filter can ever be Q^*-generic as by density this missing subset occurs as a segment in the $\mathrm{Add}(\kappa^{++}, 1)^{V[G_\kappa * g]}$-generic. We are left with the option of forcing directly with Q^* if we wish to lift the embedding; see Lemma 3.4.

Claim 3.3. *The forcing Q^* is κ^{++}-distributive over $V[G_\kappa * g]$.*

Proof. We will argue that the preparatory forcing $\mathrm{Add}(\kappa^+, \kappa^{++})$ ensures that Q^*, which is κ^{++}-closed over $M[G_\kappa * g]$, is still κ^{++}-distributive over $V[G_\kappa * g]$.

Let us work in $V[G_\kappa * g]$. Assume that $p \in Q^*$ is a condition and $\dot{f}$ is a name for a function from κ^+ to the ordinals:

$$p \Vdash \dot{f} : \kappa^+ \to \mathrm{ORD}. \tag{3.3}$$

We will show that there exists $q \leqslant p$ which decides all values of $\dot{f}$.

Write $H(\kappa^{++})$ of $M[G_\kappa * g]$ as $L_{\kappa^{++}}[B]$ for some subset B of κ^{++}, B in $M[G_\kappa * g]$. This is possible because by GCH in M and the chain condition of the forcing, $H(\kappa^{++})$ of $M[G_\kappa * g]$ has size κ^{++} in $M[G_\kappa * g]$. Fix an elementary submodel N of some large enough $H(\theta)^{V[G_\kappa * g]}$ which has size κ^+, is transitive below κ^{++}, is closed under κ-sequences and contains as elements B, Q^*, p and $\dot{f}$. We will show that p has an extension $q \leqslant p$ which hits all dense subsets of Q^* which belong to N; this will imply that q decides all values of $\dot{f}$ as required.

Let β be the ordinal $N \cap \kappa^{++}$ and let π be the transitive collapse of N to $\bar{N}$. Then $\pi(Q^*)$, which is equal to $Q^* \cap N$, belongs to $M[G_\kappa * g]$ because Q^* is definable in $L_{\kappa^{++}}[B]$, and so by π being an isomorphism, $\pi(Q^*)$ is definable in $L_{\pi(\kappa^{++})}[\pi(B)] = L_\beta[B \cap \beta]$. It suffices to extend $\pi(p) = p$ to a condition q which hits all dense subsets of $\pi(Q^*)$ which belong to $\bar{N}$.

For $\gamma < \kappa^{++}$, let $g \restriction \gamma$ denote $\{q \in g \mid q \restriction \gamma = q\}$. Pick some $\gamma < \kappa^{++}$ such that $\bar{N}$ is in $V[G_\kappa * g \restriction \gamma]$, and $\pi(Q^*)$ as well as some enumeration $\langle p^*_\xi \mid \xi < \kappa^+\rangle$ of $\pi(Q^*)$ are in $M[G_\kappa * g \restriction \gamma]$. Such a γ exists by the κ^{++}-cc of the forcing $\mathrm{Add}(\kappa^+, \kappa^{++})$ and the fact that $\bar{N}$ is a transitive set of size κ^+. Let h be the generic function $\kappa^+ \to \kappa^+$ at the coordinate γ in g. So h is $\mathrm{Add}(\kappa^+, 1)$-generic over $V[G_\kappa * g \restriction \gamma]$. Note that h belongs to $M[G_\kappa * g]$.

Define inductively in $M[G_\kappa * g]$ a decreasing sequence of conditions $\langle p_\xi \mid \xi < \kappa^+\rangle$ with $p_0 = p$, $p_\lambda = \bigcup_{\xi<\lambda} p_\xi$ for λ a limit ordinal $< \kappa^+$, and:

$$p_{\xi+1} = \begin{cases} p^*_{h(\xi)} & \text{if } p^*_{h(\xi)} \text{ extends } p_\xi, \\ p_\xi & \text{otherwise.} \end{cases}$$

Since all the parameters used in this construction, i.e. the sequence $\langle p^*_\xi \mid \xi < \kappa^+\rangle$, and $h, \pi(Q^*), p$, are in $M[G_\kappa * g]$, so is the whole sequence $\langle p_\xi \mid \xi < \kappa^+\rangle$. Let q be the greatest lower bound of this sequence, $q = \bigcup_{\xi<\kappa^+} p_\xi$. Since $\langle p_\xi \mid \xi < \kappa^+\rangle$ is in $M[G_\kappa * g]$, $q \in Q^*$.

We will show in $V[G_\kappa * g \restriction \gamma][h]$ that the sequence $\langle p_\xi \mid \xi < \kappa^+\rangle$ is $(\bar{N}, \pi(Q^*))$-generic. This already implies that q decides all the values of $\dot{f}$: For each $\xi < \kappa^+$, the set

$$D_\xi = \{p \in \pi(Q^*) \mid p \text{ decides } \pi(\dot{f})(\xi)\}$$

is a dense open set in $\pi(Q^*)$, which is an element of $\bar{N}$. If p_ζ for some $\zeta < \kappa^+$ meets D_ξ, then $p_\zeta = \pi^{-1}(p_\zeta)$ decides the value of $\dot{f}(\alpha)$, and so does $q \leqslant p_\zeta$.

The $(\bar{N}, \pi(Q^*))$-genericity is proved by using the generic h. Let D be a dense open set in $\pi(Q^*)$ which is an element of $\bar{N}$. We will show in $V[G_\kappa * g \restriction \gamma][h]$ that there is some p_ξ which meets D. To this end, it suffices to show that

$$\bar{D} = \{q \mid q \Vdash \text{“}\exists \xi < \kappa^+ \; p_\xi \in D\text{”}\}$$

is dense in $\mathrm{Add}(\kappa^+, 1)$ in $V[G_\kappa * g \restriction \gamma]$. Given a condition q, extend q first to some q' such that $\mathrm{dom}(q') = \delta$ for some $\delta < \kappa^+$; then q' decides the construction of $\langle p_\xi \mid \xi < \kappa^+\rangle$ up to δ (because it decides h up to δ): for some $p' \in \pi(Q^*)$, $q' \Vdash p_\delta = p'$. Pick $p'' \leqslant p'$ in D. In the enumeration $\langle p^*_\xi \mid \xi < \kappa^+\rangle$, p'' is some condition p^*_η. Set $q'' = q' \cup \{\langle \delta, \eta\rangle\}$. Then $q'' \Vdash$ “$p_{\delta+1}$ extends p_δ and meets D”, and so $q'' \leqslant q$ is in $\bar{D}$. It follows that $\bar{D}$ is dense and the proof of Claim 3.3 is finished. □

This shows that $\mathbb{P}$ is cofinality-preserving over V and ends the proof of Lemma 3.2. □

We now show that the embedding j can be lifted to $V^{\mathbb{P}}$.

Lemma 3.4. *The embedding j lifts to $V^{\mathbb{P}}$.*

Proof. Let $G = G_\kappa * g * g'$ be a $\mathbb{P}$-generic over V, where G_κ is $\mathbb{P}^0$-generic, g is $\mathrm{Add}(\kappa^+, \kappa^{++})^{M[G_\kappa]}$-generic over $V[G_\kappa]$, and g' is $\mathrm{Add}(\kappa^{++}, 1)^{M[G_\kappa * g]}$-generic over $V[G_\kappa * g]$. We need to find a $j(\mathbb{P})$-generic H over M such that $j[G] \subseteq H$.

As $H(\kappa)$ is included in M, $j(\mathbb{P}^0)_\kappa = \mathbb{P}^0$, and so we start building H by plugging in G_κ as the $j(\mathbb{P}^0)_\kappa$-generic over M.

The next forcing in $j(\mathbb{P})$ above κ is $Q = \mathrm{Add}(\kappa^+, \kappa^{++}) * \dot{\mathrm{Add}}(\kappa^{++}, \kappa^{+4})$ as defined in $M[G_\kappa]$. We need to find in $V[G]$ a Q-generic over $M[G_\kappa]$. By the definition of $\dot{\mathbb{P}}^1$, g is $\mathrm{Add}(\kappa^+, \kappa^{++})^{M[G_\kappa]}$-generic over $V[G_\kappa]$ (and hence over $M[G_\kappa]$). To complete the construction of a Q-generic, it remains to find some h which will be $\mathrm{Add}(\kappa^{++}, \kappa^{+4})^{M[G_\kappa * g]}$-generic over $M[G_\kappa * g]$.

When we look at the generics at our disposal, the natural candidate for h is the generic filter g'. Clearly, g' will need to be modified because it is only $\mathrm{Add}(\kappa^{++}, 1)^{M[G_\kappa * g]}$-generic over $V[G_\kappa * g]$, but not $\mathrm{Add}(\kappa^{++}, \kappa^{+4})^{M[G_\kappa * g]}$-generic over $V[G_\kappa * g]$. Note that there is a good reason for this apparent deficiency of g': While Claim 3.3 shows that $\mathrm{Add}(\kappa^{++}, 1)^{M[G_\kappa * g]}$ is sufficiently distributive over $V[G_\kappa * g]$, the forcing $\mathrm{Add}(\kappa^{++}, \kappa^{+4})^{M[G_\kappa * g]}$ never is, in fact it collapses κ^{++}:

Observation 3.5. *Let γ be an ordinal $< j(\kappa)$ which has V-cofinality κ^+, and whose cofinality in M is $> \kappa^+$. Then the forcing $\mathrm{Add}(\kappa^{++}, \gamma)^{M[G_\kappa * g]}$ collapses κ^{++} to κ^+ if forced over $V[G_\kappa * g]$.*

Proof. First notice that every M-regular cardinal in the interval $(\kappa^{++}, j(\kappa)]$ has V-cofinality κ^+: if μ is such a cardinal, then the set $\{\sup(j(f)[\kappa^{++}] \cap \mu) \mid f : \kappa \to \kappa \text{ in } V\}$ is cofinal in μ and has size κ^+ by the GCH in V. It follows that $\gamma = (\kappa^{+4})^M$ obeys the hypothesis of the observation.

Fix X to be a cofinal subset of γ of order type κ^+. Now, for each $\zeta \in \kappa^{++}$ and every $p \in \mathrm{Add}(\kappa^{++}, \gamma)^{M[G_\kappa * g]}$, one can find $q \leqslant p$ and $\xi \in X$ such that q at the coordinate ξ codes ζ in the sense that it contains ζ-many 1's followed by 0. Hence it is dense that every $\zeta \in \kappa^{++}$ is coded at some element $\xi \in X$. □

We now state a general claim which concerns κ^{++}-correct extender ultrapower embeddings under GCH. Assume $k : V \to M$ is a κ^{++}-correct extender ultrapower embedding and γ is an ordinal in the closed interval $[\kappa^{++}, j(\kappa^+)]$. We say that a bijection $\pi : \gamma \to \kappa^{++}$ is *locally M-correct* if for every $X \subseteq \gamma$ which is in M and has in M size $\leqslant \kappa^{++}$, the restriction $\pi \restriction X$ is also in M.

Claim 3.6. *Assume GCH and let $k : V \to M$ be a κ^{++}-correct extender ultrapower embedding. Let γ be an ordinal in the closed interval $[\kappa^{++}, j(\kappa^+)]$. Then:*

(i) *There exists in V a locally M-correct bijection $\pi : \gamma \to \kappa^{++}$.*
(ii) *Furthermore, if $\mathbb{R}$ is a forcing notion in M and $\mathbb{R}$ has the κ^{+3}-cc in M, then the bijection π in* (i) *is $M^{\mathbb{R}}$-locally correct.*

Proof. (i) We can assume that γ is at least $(\kappa^{+3})^M$ because otherwise γ has size κ^{++} in M, and so there exists a bijection in M between γ and κ^{++}.

In M, choose some regular cardinal θ greater than $k(\kappa^+)$ and consider the structure $H = (H(\theta), <)$, where $<$ is some well-order of $H(\theta)$. List all $f : \kappa \to [\kappa]^{\leqslant\kappa}$ in V as $\langle f_i \mid i < \kappa^+\rangle$. For $\beta < \kappa^+$ define S_β to consist of those ordinals less than γ which are definable in $H(\theta)$ from elements of $\{k(f_i) \mid i < \beta\} \cup \kappa^{++}$.

If X in M is a subset of γ of size κ^{++} in M, then X is contained in some S_β by the following argument: We can choose i so that $X = k(f_i)(\alpha)$ for some $\alpha < \kappa^{++}$ and therefore X is definable in $H(\theta)$ from $k(f_i)$ and α; then the $<$-least κ^{++}-enumeration of X is also definable in $H(\theta)$ from those parameters and each element of X is definable from $k(f_i)$ together with parameters $< \kappa^{++}$, as it is the δ-th element of that enumeration for some $\delta < \kappa^{++}$.

Now thin out if necessary the sequence $\langle S_\beta \mid \beta < \kappa^+\rangle$ to a sequence $\langle T_\beta \mid \beta < \kappa^+\rangle$ so that

$$T'_\beta = T_\beta \setminus \bigcup_{i<\beta} T_i$$

has size κ^{++} in M for each β. This is possible because we assumed that γ was at least κ^{+3} of M. For each β let π_β denote a bijection in M between T'_β and κ^{++} and define a bijection π' between γ and $\kappa^+ \times \kappa^{++}$ by:

$$\pi'(\delta) = \big(\beta, \pi_\beta(\delta)\big),$$

where δ belongs to T'_β (there is a unique β satisfying this). Finally, compose this π' with any bijection τ in M between $\kappa^+ \times \kappa^{++}$ and κ^{++}. Then $\pi = \tau \circ \pi'$ is as required.

(ii) Let F be $\mathbb{R}$-generic over M. If X is a subset of γ in $M[F]$ which has size $\leqslant \kappa^{++}$ in $M[F]$, then by the κ^{+3}-cc of $\mathbb{R}$ there is some $X' \supseteq X$ in M which has size $\leqslant \kappa^{++}$ in M. Then the desired result follows by application of (i).

This ends the proof of Claim 3.6. □

Note that the inverse function π^{-1} may not be "locally M-correct" in the sense of Claim 3.6 even for subsets $X \subseteq \kappa^{++}$ of size κ^+ in M. Indeed, if $\langle c_\xi \mid \xi < \kappa^+\rangle$ is cofinal in $(\kappa^{+4})^M$, then for $X = \{c_\xi \mid \xi < \kappa^+\}$, the set $\pi[X]$ may be in M (for instance when κ is $H(\kappa^{++})$-strong), while $\pi^{-1}[\pi[X]] = X$ is certainly not in M.

We now show that Claim 3.6 can be used to stretch the $\mathrm{Add}(\kappa^{++}, 1)$-generic g' over $V[G_\kappa * g]$ to an $\mathrm{Add}(\kappa^{++}, \kappa^{+4})^{M[G_\kappa * g]}$-generic over $M[G_\kappa * g]$.

Let $Q^* = \mathrm{Add}(\kappa^{++}, 1)^{M[G_\kappa * g]}$, and $\tilde{Q} = \mathrm{Add}(\kappa^{++}, \kappa^{+4})^{M[G_\kappa * g]}$.

Claim 3.7. *There exists in $V[G_\kappa * g * g']$ a $\tilde{Q}$-generic h over $M[G_\kappa * g]$.*

Proof. Let $\pi^* : \kappa^{++} \times (\kappa^{+4})^M \to \kappa^{++}$ be a bijection obtained by composing the bijection π from Claim 3.6 with any bijection in M between $\kappa^{++} \times (\kappa^{+4})^M$ and $(\kappa^{+4})^M$. Then π^* is locally $M[G_\kappa * g]$-correct in the sense of Lemma 3.6(ii), applied to subsets of $\kappa^{++} \times (\kappa^{+4})^M$ of size $\leqslant \kappa^{++}$ in M. For $p \in \tilde{Q}$, write p^* to denote the image of p under π^*: $\mathrm{dom}(p^*) = \pi^*[\mathrm{dom}(p)]$, and for each (ξ, ζ) in the domain of p, $p^*(\pi^*(\xi, \zeta)) = p(\xi, \zeta)$. By the local $M[G_\kappa * g]$-correctness of π^*, each p^* is in $M[G_\kappa * g]$, and hence is a condition in Q^*:

$$\{p^* \mid p \in \tilde{Q}\} \subseteq Q^*.$$

Note that the inclusion is proper because Q^* is κ^{++}-distributive over $V[G_\kappa * g]$, while $\tilde{Q}$ is not (see Observation 3.5).

Let us set

$$h = \{p \mid p^* \in g'\}.$$

We show that h is as required. First note that h is a filter: if p^* and q^* are in g', then $p^* \cup q^* = (p \cup q)^*$, and so $p \cup q$ is in h. Upward closure is obvious.

To finish the proof, we show that h meets every relevant maximal antichain. Assume A lies in $M[G_\kappa * g]$ and is a maximal antichain in $\tilde{Q}$, and so in particular A has size $\leqslant \kappa^{++}$ in $M[G_\kappa * g]$. Let us denote $\mathrm{dom}(A) = \bigcup\{\mathrm{dom}(p) \mid p \in A\}$.

Let us write $A^* = \{p^* \mid p \in A\}$ and $\mathrm{dom}(A^*) = \bigcup\{\mathrm{dom}(p^*) \mid p^* \in A^*\}$; then A^* is an antichain in Q^* and $\pi^* \restriction \mathrm{dom}(A)$ is in $M[G_\kappa * g]$ by the local $M[G_\kappa * g]$-correctness of π^*. To show that h is as required, it suffices to show that A^* is a maximal antichain in Q^*. Let q be any condition in Q^*; since q is in $M[G_\kappa * g]$, the intersection $\mathrm{dom}(q) \cap \mathrm{dom}(A^*)$ is in $M[G_\kappa * g]$. Since $\pi^* \restriction \mathrm{dom}(A)$ is in $M[G_\kappa * g]$, the set $(\pi^* \restriction \mathrm{dom}(A))^{-1}[\mathrm{dom}(q) \cap \mathrm{dom}(A^*)]$ is also in $M[G_\kappa * g]$. If q' denotes the condition in $\tilde{Q}$ with the domain $(\pi^* \restriction \mathrm{dom}(A))^{-1}[\mathrm{dom}(q) \cap \mathrm{dom}(A^*)]$ defined by $q'(\xi, \zeta) = q(\pi^*(\xi, \zeta))$, then there exists by the maximality of A some $p \in A$ compatible with q'. It follows that $p^* \in A^*$ is compatible with q because it is compatible with q on $\mathrm{dom}(p^*) \cap \mathrm{dom}(q)$. Thus A^* indeed maximal, and h meets A as required. This ends the proof of Claim 3.7. □

By Claim 3.7, we can conclude that $G_\kappa * g * h$ is $j(\mathbb{P}^0)_{\kappa+1}$-generic over M. The iteration $j(\mathbb{P}^0)$ in the interval $(\kappa+1, j(\kappa))$ is κ^{+++}-distributive in $M[G_\kappa * g * h]$, and so all the relevant dense open sets in $M[G_\kappa * g * h]$ can be met in κ^+-many steps, using the extender representation of M (see [5] for details). Let the resulting generic be denoted as $\tilde{h}$. Then $G_\kappa * g * h * \tilde{h}$ is $j(\mathbb{P}^0)$-generic over M, and we can partially lift to

$$j' : V[G_\kappa] \to M[G_\kappa * g * h * \tilde{h}].$$

It remains to lift j' to $\mathbb{P}^1 = \mathrm{Add}(\kappa^+, \kappa^{++}) * \dot{\mathrm{Add}}(\kappa^{++}, 1)$ of $M[G_\kappa * g]$. By Claim 3.3, $\mathbb{P}^1$ is κ^+-distributive over $V[G_\kappa]$, and therefore by Fact 2.4(ii), the filter $\tilde{\tilde{h}}$ generated by the j' image of $g * g'$ is $j'(\mathbb{P}^1)$-generic over $M[G_\kappa * g * h * \tilde{h}]$:

$$\tilde{\tilde{h}} = \{q \mid \exists p \in g * g', j'(p) \leqslant q\}.$$

If we define $H = G_\kappa * g * h * \tilde{h} * \tilde{\tilde{h}}$, then H is as required:

$$j^* : V\left[G_\kappa * g * g'\right] \to M[H].$$

This ends the proof of Lemma 3.4. □

Theorem 3.1 now follows from Lemma 3.2, Lemma 3.4, and Fact 2.4(iii). □

Claim 3.3 implies that if the GCH holds and j is a κ^{++}-correct extender embedding, then in a cofinality-preserving extension this j lifts to a κ^{++}-correct extender embedding with the Cohen forcing at κ^{++} in the target model well-behaved over the universe; this is stated in Corollary 3.8 below.

Corollary 3.8 *(GCH). Let $j : V \to M$ be a κ^{++}-correct extender embedding with critical point κ. Let $\mathbb{R}$ be an iteration of length $\kappa+1$ with Easton support which adds ξ^{++}-many Cohen subsets to each ξ^+, where ξ is an inaccessible cardinal less or equal κ. If G is $\mathbb{R}$-generic, then the following hold*:

(i) *GCH holds in $V[G]$;*
(ii) *j lifts to $j^* : V[G] \to M[j^*(G)]$;*
(iii) *$\mathrm{Add}(\kappa^{++}, 1)^{M[j^*(G)]}$ is κ^{++}-distributive over $V[G]$.*

Proof. (i) is obvious.

(ii) follows by an easy lifting argument: $j^*(G)$ is of the form $G_\kappa * g * h * \tilde{h}$, where $G = G_\kappa * g$ (G_κ is the generic filter for $\mathbb{R}$ below κ and g is the generic filter for $\mathrm{Add}(\kappa^+, \kappa^{++})^{V[G]}$), h is $j(\mathbb{R})$-generic over $M[G_\kappa * g]$ in the interval $(\kappa^+, j(\kappa)^+)$, and $\tilde{h}$ is obtained from g by application of Fact 2.4(ii).

(iii) follows by application of Claim 3.3 to $\mathrm{Add}(\kappa^{++}, 1)^{M[G]}$ in $V[G]$, while noticing that $\mathrm{Add}(\kappa^{++}, 1)^{M[G]}$ is the same forcing as $\mathrm{Add}(\kappa^{++}, 1)^{M[j^*(G)]}$ by κ^{++}-distributivity of $j(\mathbb{R})$ above κ^+. □

The idea behind the proof of Corollary 3.8 is that the generic filter g for $\mathrm{Add}(\kappa^+, \kappa^{++})$ of $V[G_\kappa]$ adds to $M[G_\kappa]$ just the right subsets of κ^+, which then become conditions in $\mathrm{Add}(\kappa^{++}, 1)$ of $M[G_\kappa * g]$, to make sure that $\mathrm{Add}(\kappa^{++}, 1)$ of $M[G_\kappa * g]$ is still distributive over $V[G]$. We do not know whether this step of adding new conditions is in fact necessary; it may be, although we do not credit it with high probability, that whenever $j : V \to M$ is a κ^{++}-correct extender ultrapower embedding, then $\mathrm{Add}(\kappa^{++}, 1)^M$ is κ^{++}-distributive over V. See the last section for some open questions regarding this topic.

4. Easton's theorem and large cardinals from the optimal hypothesis

Theorem 4.1. *Assume GCH and let $j : V \to M$ be a κ^{++}-correct extender embedding with critical point κ. Then there exists a cofinality-preserving forcing notion $\mathbb{R}$ such that if G is $\mathbb{R}$-generic, the following holds*:

(i) $2^\alpha = \alpha^{++}$ *for every regular cardinal* $\alpha \leqslant \kappa$.
(ii) *The embedding* j *lifts to* $j^* : V[G] \to M[j^*(G)]$, *and* j^* *is a* κ^{++}*-correct extender embedding in* $V[G]$. *In particular,* κ *is still measurable.*

Proof. Let $I(\kappa)$ denote the set of all inaccessible cardinals $< \kappa$, and $R(\kappa)$ the set of all regular cardinals $< \kappa$. Set $B = \{\alpha \in R(\kappa) \mid \exists \beta \in I(\kappa), \alpha = \beta \text{ or } \alpha = \beta^+\} \cup \{\kappa\}$, and $A = R(\kappa) \setminus B$. Then $A \cup B$ is the set of all regular cardinals $\leqslant \kappa$.

We define $\mathbb{R}$ as a two-stage iteration $\mathbb{R}_A * \dot{\mathbb{R}}_B$. $\mathbb{R}_A$ will be a cofinality-preserving forcing which will force the failure of GCH at every element in A. In $V^{\mathbb{R}_A}$, $\dot{\mathbb{R}}_B$ will be a cofinality-preserving forcing which will violate GCH at the remaining regular cardinals $\leqslant \kappa$, i.e. at the elements in B.

The definition of $\mathbb{R}_A$ is a modification of $\mathbb{P}$, as defined in Theorem 3.1. $\mathbb{R}_A$ is a two-stage iteration $\mathbb{R}^0_A * \dot{\mathbb{R}}^1_A$, where:

(1) $\mathbb{R}^0_A$ is an iteration of length κ with Easton support, $\mathbb{R}^0_A = \langle (\mathbb{R}^0_A)_\xi, \dot{Q}_\xi) \mid \xi < \kappa\rangle$, where $\dot{Q}_\xi$ is a name for a trivial forcing unless ξ is a limit cardinal $< \kappa$, in which case there are two possibilities:
   (a) If ξ is regular (and hence inaccessible), then

$$\big(\mathbb{R}^0_A\big)_\xi \Vdash \text{“}\dot{Q}_\xi \text{ is the forcing } \big[\mathrm{Add}\big(\xi^+, \xi^{++}\big) * \dot{\mathrm{Add}}\big(\xi^{++}, \xi^{+4}\big)\big] \times \prod_{\xi^{++} < \gamma < \xi^{+\omega}} \mathrm{Add}\big(\gamma, \gamma^{++}\big),\text{”} \tag{4.1}$$

   where $\mathrm{Add}(\xi^+, \xi^{++})$ is viewed as a product forcing which adds ξ^{++}-many Cohen functions from ξ^+ to ξ^+, $\dot{\mathrm{Add}}(\xi^{++}, \xi^{+4})$ is viewed as (a name for) a forcing adding ξ^{+4}-many Cohen subsets of ξ^{++}, and $\prod_{\xi^{++} < \gamma < \xi^{+\omega}} \mathrm{Add}(\gamma, \gamma^{++})$ is the standard product, which adds γ^{++}-many Cohen subsets to each regular cardinal γ such that $\xi^{++} < \gamma < \xi^{+\omega}$ (where $\xi^{+\omega}$ is the least limit cardinal above ξ).
   (b) If ξ is singular, then

$$\big(\mathbb{R}^0_A\big)_\xi \Vdash \text{“}\dot{Q}_\xi \text{ is the forcing } \textstyle\prod_{\xi < \gamma < \xi^{+\omega}} \mathrm{Add}\big(\gamma, \gamma^{++}\big),\text{”} \tag{4.2}$$

   where $\prod_{\xi^{++} < \gamma < \xi^{+\omega}} \mathrm{Add}(\gamma, \gamma^{++})$ is the standard product.
(2) Notice that $\mathbb{R}^0_A$ is an element of M. $\dot{\mathbb{R}}^1_A$ is defined in M to be an $\mathbb{R}^0_A$-name which satisfies:

$$M \models \mathbb{R}^0_A \Vdash \text{“}\dot{\mathbb{R}}^1_A \text{ is the forcing } \mathrm{Add}\big(\kappa^+, \kappa^{++}\big) * \dot{\mathrm{Add}}\big(\kappa^{++}, 1\big),\text{”} \tag{4.3}$$

   where $\mathrm{Add}(\kappa^+, \kappa^{++})$ is viewed as a product forcing which adds κ^{++}-many Cohen functions from κ^+ to κ^+, and $\dot{\mathrm{Add}}(\kappa^{++}, 1)$ is viewed as (a name for) a forcing adding a single Cohen subset of κ^{++}.

By standard arguments, see [5], and Claim 3.3 applied in the present context, the forcing $\mathbb{R}_A$ is cofinality-preserving. By [5], and an easy modification of Theorem 3.1, j lifts to a κ^{++}-correct extender embedding j' in $V^{\mathbb{R}_A}$: in the proof generalizing the proof of Theorem 3.1, one just needs to take into account the product $\prod_{\kappa^{++} < \gamma < \kappa^{+\omega}} \mathrm{Add}(\gamma, \gamma^{++})$ at stage κ of the iteration $j(\mathbb{R}^0_A)$. However, since in $M^{j(\mathbb{R}^0_A)_\kappa}$, $\mathrm{Add}(\kappa^+, \kappa^{++}) * \dot{\mathrm{Add}}(\kappa^{++}, \kappa^{+4})$ has the κ^{+3}-cc and the product $\prod_{\kappa^{++} < \gamma < \kappa^{+\omega}} \mathrm{Add}(\gamma, \gamma^{++})$ is κ^{+3}-closed, it follows by Easton's lemma that the generics for these two forcings are mutually generic. Accordingly, an $\mathrm{Add}(\kappa^+, \kappa^{++}) * \dot{\mathrm{Add}}(\kappa^{++}, \kappa^{+4})$-generic over $M^{j(\mathbb{R}^0_A)_\kappa}$ is obtained as in Theorem 3.1, while a $\prod_{\kappa^{++} < \gamma < \kappa^{+\omega}} \mathrm{Add}(\gamma, \gamma^{++})$-generic is obtained by a standard construction using the κ^{+3}-distributivity of the forcing.

If G_A denotes an $\mathbb{R}_A$-generic, then the following holds in $V[G_A]$:

(i) GCH holds in $V[G_A]$ at every inaccessible cardinal $\alpha \leqslant \kappa$ and at the successors of these inaccessible cardinals.
(ii) $2^\alpha = \alpha^{++}$ for every regular cardinal $\alpha < \kappa$ other than those specified in (i).
(iii) There exists in $V[G_A]$ a κ^{++}-correct extender embedding $j' : V[G_A] \to M[j'(G_A)]$ which is a lifting of the original j.

In $V[G_A]$, we define $\mathbb{R}_B$ as follows.

$\mathbb{R}_B$ is an iteration of length $\kappa + 1$ with Easton support, $\mathbb{R}_B = \langle (\mathbb{R}_B)_\xi, \dot{Q}_\xi) \mid \xi < \kappa + 1\rangle$, where $\dot{Q}_\xi$ is a name for a trivial forcing unless ξ is an inaccessible cardinal $\leqslant \kappa$, in which case there are two cases:

(a) If $\xi < \kappa$, then

$$(\mathbb{R}_B)_\xi \Vdash \text{“}\dot{Q}_\xi \text{ is the forcing } \mathrm{Sacks}\big(\xi, \xi^{++}\big) \times \mathrm{Add}\big(\xi^+, \xi^{+3}\big),\text{”} \tag{4.4}$$

   where $\mathrm{Sacks}(\xi^+, \xi^{++})$ is the generalized Sacks product forcing at ξ which adds ξ^{++}-many new subsets of ξ (see [11], and [6] for details), and $\mathrm{Add}(\xi^+, \xi^{++})$ is viewed as adding ξ^{+3}-many Cohen subsets of ξ^+.
(b) If $\xi = \kappa$, then

$$(\mathbb{R}_B)_\xi \Vdash \text{“}\dot{Q}_\xi \text{ is the forcing } \mathrm{Sacks}\big(\xi, \xi^{++}\big) \times \mathrm{Add}\big(\xi^+, \xi^{++}\big).\text{”} \tag{4.5}$$

By standard results, see [5], $\mathbb{R}_B$ is cofinality-preserving over $V[G_A]$ (here, it is important that $\mathrm{Add}(\xi^+, \xi^{+3})$ is still ξ^+-distributive over $\mathrm{Sacks}(\xi, \xi^{++})$).

Let G_B be $\mathbb{R}_B$-generic over $V[G_A]$. Using the "tuning-fork" argument in the original paper [6], together with [5], one can show that j' lifts to $V[G_A][G_B]$. Notice here that it is sufficient to add just κ^{++}-many Cohen subset of κ^+, cf. (4.5), in order to lift, and so GCH holds in $V[G_A][G_B]$ above κ (if so desired).

If we set $G = G_A * G_B$, then $V[G]$ is as required. □

We can achieve even more generality, along the lines [4] and [5]. We say that a proper-class function F from regular cardinals into cardinals is an *Easton function*, if for all regular cardinals κ, λ:

(i) $\kappa < \lambda \to F(\kappa) \leqslant F(\lambda)$,
(ii) $\mathrm{cf}(F(\kappa)) > \kappa$.

A cardinal μ is said to be a *closure point* of F if $F(\nu) < \mu$ for every regular cardinal $\nu < \mu$.

We say that F is *realized* in some cofinality-preserving extension $V^{\mathbb{R}}$ if F is the continuum function in $V^{\mathbb{R}}$ on regular cardinals.

Corollary 4.2. *Assume GCH and let $j : V \to M$ be a κ^{++}-correct embedding with critical point κ. If an Easton function F satisfies*:

(i) *κ is a closure point of F, $F(\kappa) = \kappa^{++}$, and*
(ii) *the set $\{\alpha < \kappa \mid \alpha$ is a regular cardinal and $F(\alpha) \geqslant \alpha^{++}\}$ contains all regulars in a closed unbounded set,*

then there exists a cofinality-preserving forcing $\mathbb{R}$ such that the Easton function F is realized in $V^{\mathbb{R}}$, and j lifts to $V^{\mathbb{R}}$; in particular κ is still measurable in $V^{\mathbb{R}}$.

Proof. This is just like the relevant part of [5], with the arguments in Theorems 3.1 and 4.1 added to be able to prove this result from the optimal hypothesis of a κ^{++}-correct embedding. □

Let us note that the condition (ii) implies that $j(F)(\kappa) \geqslant \kappa^{++}$ for any κ^{++}-correct embedding, which is actually all that is needed from (ii) in the proof.

5. Mitchell order on extenders

It is known that Woodin's construction for κ^{++} from the assumption (1.1) naturally generalizes to κ^{+n}-tall cardinals for $n < \omega$ (see [9] for an argument).

Similarly, the technique in this paper generalizes to all $n < \omega$.

By results in [8], the existence of a measurable cardinal κ with $2^\kappa = \kappa^{+n}$ is equiconsistent with the existence of a measurable cardinal κ with $o(\kappa) = \kappa^{+n}$. Note that for $n > 2$, the Mitchell order of κ is counted in terms of extenders, not measures. Thus for $n > 2$, the assumption $o(\kappa) = \kappa^{+n}$ means that there is a coherent sequence of length κ^{+n} of $H(\kappa^{+n-1})$-strong extenders at κ (where an extender at κ is $H(\kappa^{+n-1})$-strong if the associated extender ultrapower embedding is $H(\kappa^{+n-1})$-strong). Generalizing the construction in [7], the assumption $o(\kappa) = \kappa^{+n}$ for $n < \omega$ implies that there exists a generic extension V^* satisfying GCH and an elementary embedding $j : V^* \to M$ such that:

(i) M is closed under κ-sequences in V^*,
(ii) $H(\kappa^{+n-1})$ of V^* is included in M,
(iii) $(\kappa^{+n})^M = \kappa^{+n}$.

Without giving the details, we just mention that the construction in this paper can be used to show that if $j : V \to M$ is as in (i)–(iii) and GCH holds in V, then Corollary 4.2 holds for $F(\kappa) = \kappa^{+n}$.

In fact, one can attempt to generalize Corollary 4.2 to $o(\kappa) = \kappa^{+\beta}$ for infinite β's. The situation with $\beta \geqslant \omega$ is a little bit more involved than with $n < \omega$ (see [8]), but we believe that the technique in this paper should be useful. See the next section for open questions.

6. Open questions

Question 1. For which $\beta \geqslant \omega$ can we obtain the analogue of Corollary 4.2 with $F(\kappa) = \kappa^{+\beta}$?

Question 2. Is there a κ^{++}-correct embedding $j : V \to M$ such that $\mathrm{Add}(\kappa^{++}, 1)^M$ is *not* κ^{++}-distributive over V?

An obvious strategy of attack to answer Question 2 in the affirmative is to devise a forcing $\mathbb{R}$, lift j to $j^* : V[G] \to M[j^*(G)]$, where G is $\mathbb{R}$-generic, and show that $\mathrm{Add}(\kappa^{++}, 1)$ of $M[j^*(G)]$ collapses κ^{++} when forced over $V[G]$ (so in particular, it cannot be κ^{++}-distributive). This reminds one of an argument which dates back to Baumgartner and his forcing for specializing an ω_1-Aronszajn tree: one can find two proper forcings P and Q living in a ground model V^*, with P being the forcing $\mathrm{Add}(\omega_1, 1)$, and Q a three-stage iteration featuring a "specialization" forcing, such that P collapses ω_1 when forced over V^{*Q} (see for instance [14], p. 827). The analogy here is that if $j : V \to M$ is an embedding, then we can equate V with V^{*Q}, and M with V^* in the example above. However, such "specialization" forcings are often hard to generalize to larger cardinals (see for instance [2,15]).

Lastly, there is nothing special about the Cohen forcing $\mathrm{Add}(\kappa^{++}, 1)^M$ and the assumption of κ^{++}-correctness in Question 2, except that we needed this in our present proof. In general, one can ask the analogue of Question 2 for some other forcing $P \in M$ and an elementary embedding $j : V \to M$.

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