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An Extension of Carlson's Theorem for Analytic Functions

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Let $f=u+iv$ be an entire function of exponential type less than π. If $u=0$ on two lines of lattice points $\{n\}$ and $\{n+i\}$, then $f(z)\equiv 0$ [1]. This result has been improved by Trembinska [6], who showed that if $\{f(n)\}\in l^1$, then $f(z)\equiv 0$ provided that either $u(n+i)=0$, $-\infty<n<\infty$, or $v(n+i)=0$, $-\infty<n<\infty$; in other words, the hypothesis that u vanishes at the integers can be replaced by the weaker condition that $\sum |f(n)|$ converges. Here we raise the question of finding complementary sequences M and N of integers with the property that if f, of exponential type less than π, satisfies $\{f(n)\}\in l^1$, then $f\equiv 0$ provided that $\operatorname{Re} f(n+i)=0$ when $n\in M$ and $\operatorname{Im} f(n+i)=0$ when $n\in N$.

We show here that, under the additional hypothesis that $\operatorname{Re} f(n)=0$, the pair $M=\{\text{negative integers}\}$, $N=\{\text{nonnegative integers}\}$ has the required property.

It was shown in [5] that if $f(z)=u(z)+iv(z)$ and $V(x)=\sum_{n=-\infty}^{\infty} v(n)\, e^{inx}$, $-\pi<x<\pi$, then $\operatorname{Re} f(m+i)=u(m+i)$ is the mth Fourier coefficient of $iS(x)\,V(x)$, where $S(x)$ is the 2π-periodic continuation of $\sinh x$, $-\pi<x<\pi$, and $v(m+i)$ is the mth Fourier coefficient of $C(x)\,V(x)$, where $C(x)$ is the 2π-periodic continuation of $\cosh x$, $-\pi<x<\pi$. If, then, $u(m+i)=0$ for $m<0$ and $v(m+i)=0$ for $m\geqslant 0$, we have

$$V(x)\,S(x)=\sum_{n=0}^{\infty} q_n\, e^{inx}=B(x),$$

$$V(x)\,C(x)=\sum_{n=-\infty}^{-1} p_n\, e^{inx}=A(x).$$

Consequently $V(x)\,C(x)$ and $V(x)\,S(x)$ are the boundary functions of $A(z)$ and $B(z)$, which are analytic in the lower and upper half planes, respectively.

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Now define $V(z)=A(z)/C(z)$ in the lower half plane and $V(z)=B(z)/S(z)$ in the upper half plane. Then $A(z)/C(z)$ is analytic in the lower half plane except along the lines $x=k\pi$ ($k=\pm 1, \pm 3,\ldots$) and $x=(k+\frac{1}{2})\pi$ ($k=0, \pm 1, \pm 2,\ldots$), and $B(z)/S(z)$ is analytic in the upper half plane except along the lines $x=j\pi$ ($j=0, \pm 1, \pm 2,\ldots$). These functions coincide on the intervals $k\pi<x<(k+2)\pi$, $k=\pm 1, \pm 3,\ldots$, and are therefore analytic continuations of each other over these intervals. We now have $A(z)$ analytic in the lower half plane and $B(z)$ analytic in the lower half plane except along the lines $x=k\pi$ ($k=\pm 1, \pm 3,\ldots$) and $(k+\frac{1}{2})\pi$ ($k=0, \pm 1, \pm 2,\ldots$).

We have $A(z)=C(z)\,B(z)/S(z)$ in the lower half plane. Now C has zeros at the points $-(k+\frac{1}{2})\pi i+2m\pi$ ($k=0, 1, 2,\ldots$; $m=0, \pm 1, \pm 2,\ldots$). On the other hand, $S(z)\neq 0$ at these points, and therefore $A(z)=0$ at these points.

For $y<0$ we also have

$$|A(z)| \leqslant \sum_{n=1}^{\infty} |p_{-n}\, e^{-inz}| = \sum_{n=1}^{\infty} |p_{-n}|\, e^{-n|y|}.$$

But since $A(x)=\sum_{n=-\infty}^{-1} p_n\, e^{inx}$ is the product of $V(x)$ and $C(x)$, for $n<0$ we have

$$p_n=\frac{\sinh \pi}{\pi} \sum_{k=-\infty}^{\infty} (-1)^k \frac{v(n-k)}{1+k^2},$$

$$\sum_{n=1}^{\infty} |p_{-n}| \leqslant \frac{\sinh \pi}{\pi} \sum_{n=-\infty}^{-1} \sum_{k=-\infty}^{\infty} \frac{|v(n-k)|}{1+k^2}$$

$$\leqslant \frac{\sinh \pi}{\pi} \sum_{k=-\infty}^{\infty} \frac{1}{1+k^2} \sum_{n=-\infty}^{-1} |v(n-k)|$$

$$\leqslant C(0) \sum_{n=-\infty}^{\infty} |v(n)| < \infty.$$

Hence A is bounded in the lower half plane. In each half strip $y<0$, $m\pi<x<(m+1)\pi$, the function A has zeros of imaginary part $-(k+\frac{1}{2})\pi$. Such a function must vanish identically. Let us suppose the contrary.

If we apply Jensen's theorem [4, p. 125] to a disk Δ of radius R, centered at $z=-iR$, the number of zeros of A in Δ does not exceed

$$\frac{1}{2\pi} \int_{|z+iR|<R} \log |A(re^{i\theta})|\, d\theta - \log |A(-iR)|.$$

Since $|A(z)|$ is bounded in the lower half plane, we cannot have $R^{-1} \log |A(-iR)| \to -\infty$ as $R\to\infty$ [3, p. 169, Problem III 326]. Hence we have $\log |A(-iR)| > -\lambda R$ for some λ and $R=R_n\to\infty$. Applying Jensen's theorem with $R=R_n$, we see that if $|A(z)| \leqslant K$ ($K>1$), the number of zeros of A in Δ_n is $O(R_n \log K)$. This is impossible, for the number of zeros of A in Δ_n exceeds

$$2[R_n/(2^{3/2}\pi)][R_n\pi/2^{1/2}]$$

(where $[x]$ is the integral part of x). Since $[x] > x - 1$, it follows that the number of zeros of A in Δ_n exceeds $(\pi^{-1}R_n - 2)^2$. (The argument is the same as that of Gauss's proof that a disk of radius R contains $\pi R^2 + O(R)$ lattice points [2, pp. 270–271].) Therefore $A(z) \equiv 0$, all $v(n) = 0$, and finally $f(z) \equiv 0$.

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