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Numerical solution of functional integral equations by the variational iteration method

Jafar Biazar [a], Mehdi Gholami Porshokouhi [b,*], Behzad Ghanbari [b], Mohammad Gholami Porshokouhi [c]

[a] Department of Mathematics, Faculty of Science Guilan University, P.O Box 1914, P.C. 41938, Rasht, Iran

[b] Member of Young Researchers Club, Department of Mathematics, Faculty of Science Islamic Azad University, Takestan Branch, Iran

[c] Member of Young Researchers Club, Department of Agricultural Machinery, Islamic Azad University, Takestan Branch, Iran

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ABSTRACT

In the present article, we apply the variational iteration method to obtain the numerical solution of the functional integral equations. This method does not need to be dependent on linearization, weak nonlinearity assumptions or perturbation theory. Application of this method in finding the approximate solution of some examples confirms its validity. The results seem to show that the method is very effective and convenient for solving such equations.

1. Introduction

The variational iteration method [1–3] has proved to be one of the useful techniques in solving numerous linear and nonlinear differential equations. For instance, see [1–6] and the references therein. An important advantage of the method is that it uses the initial conditions only and does not require the specific transformations for nonlinear terms as required by some existing techniques. Furthermore, we can apply the method directly needing no linearization, discretization or perturbation. The method also gives the exact solution of rapidly convergent successive approximations if such a solution exists. For concrete problems, we can use a few approximations for numerical purposes with a high degree of accuracy.

In this work, we aim to investigate the Fredholm functional integral equations of the second kind in the general form

$$y(x) + p(x)y\,(h(x)) + \int_a^b k\,(x,t)\,y(t)\mathrm{d}t = g(x), \quad a \le x \le b, \tag{1}$$

and also, the Volterra type integral equations of the second kind in the form of

$$y(x) + p(x)y\,(h(x)) + \int_a^x k\,(x,t)\,y(t)\,\mathrm{d}t = g(x), \quad a \le x \le b, \tag{2}$$

where $p(x)$, $h(x)$ and $g(x)$ are analytical known functions.

Since the integral equations appear frequently in modeling of physical phenomena, they have a major role in the fields of science and engineering, and a considerable amount of research work has been done in studying them. Some examples of this kind include Taylor series [7], Chebyshev polynomials [8], homotopy perturbation method [9], expansion method [10], and Lagrange interpolation [11].

* Corresponding author.
E-mail addresses: jafar.biazar@gmail.com (J. Biazar), m_gholami_p@yahoo.com (M. Gholami Porshokouhi), b.ghanbary@yahoo.com (B. Ghanbari).

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In the present study, we aim to employ the variational iteration method (VIM) to obtain the approximate solutions to the integral equations. The method is reliable and capable of providing analytic treatment for the functional integral equations. Unlike the Adomian decomposition method [12], where specific algorithms are normally used to determine the so-called Adomian polynomials, the VIM handles linear and nonlinear terms in a similar manner without any additional requirement or restriction. Another important advantage is that the VIM method is capable of greatly reducing the size of calculations while still maintaining high accuracy of the numerical solution.

This paper has been organized as follows: In Section 2, we illustrate the main idea of the variational iteration method. In Section 3, the variational iteration method is applied to the functional integral equations. Some numerical examples and comparisons between the results obtained in this work and the ones obtained in [10,11] are given in Section 4. Finally, conclusions are stated in the last section.

2. He's variational iteration method

Consider the following general nonlinear differential equation:

$$Lu(x) + Nu(x) = g(x), \tag{3}$$

where L is a linear operator, N is a nonlinear operator and $g(x)$ is a known analytical function. According to the variational iteration method, a correction functional can be constructed as

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\xi)\left[Ly_n(\xi) + N\hat{y}_n(\xi) - g(\xi)\right] d\xi, \tag{4}$$

where λ is a general Lagrange multiplier and the term $\hat{y}_n$ is considered as a restricted variation, i.e. $\delta\hat{y}_n = 0$. Making the above correction functional stationary, we obtain

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta\int_0^x \lambda(\xi)\left[Ly_n(\xi) + N\hat{y}_n(\xi) - g(\xi)\right] d\xi. \tag{5}$$

In order to identify the Lagrange multiplier, from Eq. (5) we have

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta\int_0^x \lambda(\xi)\left[Ly_n(\xi) - g(\xi)\right] d\xi. \tag{6}$$

In general, the Lagrange multiplier λ, can be readily identified by imposing the stationary condition $\delta y_{n+1}(x) = 0$ on the correction functional (6). After determining the Lagrange multiplier λ and selecting an appropriate initial function y_0, the successive approximations y_n of the solution y can be readily obtained.

Consequently, the solution of Eq. (3) is given by

$$y(x) = \lim_{n\to\infty} y_n(x).$$

In other words, the correction functional (4) will give successive approximations, and therefore the exact solution is obtained as the limit of the resulting successive approximations.

3. Analysis of the method

To apply the variational iteration method, let us differentiate Eqs. (1), (2) with respect to x, as follows, respectively:

$$y'(x) = g'(x) - \left(p'(x)y(h(x)) + p(x)h'(x)y'(h(x)) + \int_a^b \frac{\partial k(x,t)}{\partial x} y(t)\, dt\right)$$

$$y'(x) = g'(x) - \left(p'(x)y(h(x)) + p(x)h'(x)y'(h(x)) + \int_a^x \frac{\partial k(x,t)}{\partial x} y(t)\, dt + k(x,x)y(x)\right).$$

According to the variational iteration method, we can construct the following correction functionals as

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\xi)\left\{y_n'(\xi) - g'(\xi) + p'(\xi)y_n(h(\xi)) + p(\xi)h'(\xi)y_n'(h(\xi)) + \int_a^b \frac{\partial k(\xi,t)}{\partial \xi} y_n(t)\, dt\right\} d\xi$$

$$\begin{aligned} y_{n+1}(x) = {} & y_n(x) + \int_0^x \lambda(\xi)\left\{y'(\xi) - g'(\xi) + p'(\xi)y(h(\xi)) + p(\xi)h'(\xi)y'(h(\xi))\right. \\ & \left. + \int_a^\xi \frac{\partial k(\xi,t)}{\partial \xi} y(t)\, dt + k(\xi,\xi)y(\xi)\right\} d\xi. \end{aligned}$$

Making the above correction functionals stationary, we obtain

$$\begin{aligned}\delta y_{n+1}(x) &= \delta y_n(x)+\delta\int_0^x \lambda(\xi)\left\{y_n'(\xi)-g'(\xi)+p'(\xi)\tilde{y}_n(h(\xi))+p(\xi)h'(\xi)\tilde{y}_n'(h(\xi))\right.\\ &\quad\left.+\int_a^b \frac{\partial k(\xi,t)}{\partial\xi}\tilde{y}_n(t)\,\mathrm{d}t\right\}\mathrm{d}\xi\\ \delta y_{n+1}(x) &= \delta y_n(x)+\delta\int_0^x \lambda(\xi)\left\{y_n'(\xi)-g'(\xi)+p'(\xi)\tilde{y}_n(h(\xi))+p(\xi)h'(\xi)\tilde{y}_n'(h(\xi))\right.\\ &\quad\left.+\int_a^\xi \frac{\partial k(\xi,t)}{\partial\xi}\tilde{y}_n(t)\,\mathrm{d}t+k(\xi,\xi)\tilde{y}_n(\xi)\right\}\mathrm{d}\xi.\end{aligned}$$

Since $\tilde{y}_n$ is considered as a restricted variation, i.e. $\delta\tilde{y}_n=0$, these correction functionals can be written as

$$\delta y_{n+1}(x)=\delta y_n(x)+\delta\int_0^x \lambda(\xi)y_n'(\xi)\,\mathrm{d}\xi,$$

or equivalently

$$\delta y_{n+1}(x)=\delta y_n(x)+\lambda(\xi)\,\delta y_n(\xi)\,|_{\xi=x}-\int_0^x \delta y_n(\xi)\,\lambda'(\xi)\,\mathrm{d}\xi.$$

This yields the following stationary conditions:

$$1+\lambda(\xi)\,|_{\xi=x}=0,$$
$$\lambda'(\xi)=0.$$

Therefore, the Lagrange multiplier can be easily identified as

$$\lambda(\xi)=-1.$$

Consequently, we obtain the iteration formula

$$y_{n+1}(x)=y_n(x)-\int_0^x\left\{y_n'(\xi)-g'(\xi)+p'(\xi)y_n(h(\xi))+p(\xi)h'(\xi)y_n'(h(\xi))+\int_a^b\frac{\partial k(\xi,t)}{\partial\xi}y_n(t)\,\mathrm{d}t\right\}\mathrm{d}\xi, \tag{7}$$

in the case of Fredholm functional integral equations (1) and

$$\begin{aligned}y_{n+1}(x) &= y_n(x)-\int_0^x\left\{y_n'(\xi)-g'(\xi)+p'(\xi)y_n(h(\xi))+p(\xi)h'(\xi)y_n'(h(\xi))\right.\\ &\quad\left.+\frac{\mathrm{d}}{\mathrm{d}\xi}\int_a^\xi\frac{\partial k(\xi,t)}{\partial\xi}y_n(t)\,\mathrm{d}t\right\}\mathrm{d}\xi,\end{aligned} \tag{8}$$

in the case of Volterra functional integral equations (2).

Starting with initial approximation y_0 in (7) and (8), successive approximations y_n will be easily obtained.

4. Numerical examples and comparison discussions

In this section, we present a selection of examples to illustrate the efficiency of the VIM. For the sake of comparing purposes, we consider the same examples as used in [10,11]. In this paper, we also use the same computed error σ_n as was defined in [10,11]

$$\sigma_n=\left(\frac{1}{n}\sum_{j=0}^{n}e_n^2(x_j)\right)^{0.5}$$

where

$$e_n(x_j)=y_{\text{exact}}(x_j)-y_{\text{appr}}(x_j),$$

and

$$x_j=a+\left(\frac{b-a}{n}\right)j,\quad j=0,\ldots,n,$$

also y_{exact} is the exact solution of the problem and y_{appr} is the obtained approximate solution from VIM.

Table 1
Comparison of the absolute errors for Example 4.1.

n	$h(x) = x^2/2$		$h(x) = 0.8x$	
	VIM	NM [8]	VIM	NM [8]
2	1.4e−03	5.2e−02	0.1e−02	6.1e−02
3	7.9e−03	3.8e−02	8.6e−03	3.1e−02
4	2.5e−04	5.7e−03	3.2e−04	9.2e−03
5	6.3e−05	5.1e−04	4.8e−05	3.2e−04

Table 2
Comparison of the absolute errors for Example 4.2.

n	$h(x) = x/2$		$h(x) = x\ln(x+2.0)$	
	VIM	NM [7]	VIM	NM [7]
2	3.7e−03	3.9e−03	3.8e−04	6.4e−03
3	4.6e−04	2.2e−04	5.5e−05	8.2e−04
4	7.3e−07	9.2e−06	8.9e−06	7.6e−05
5	1.1e−07	4.1e−07	1.1e−06	5.8e−06

Example 4.1. Let us first consider the following Fredholm integral equation of the second kind [10]

$$y(x) + \mathrm{e}^{-x} y\left(h(x)\right) + \int_a^b \mathrm{e}^{x-t} y(t)\,\mathrm{d}t = g(x), \quad a \le x \le b, \tag{9}$$

where

$$g(x) = \mathrm{e}^x + \mathrm{e}^{h(x)-x} + (b-a)\,\mathrm{e}^x, \quad a = 0,\ b = 1.1.$$

It can be easily verified that the exact solution of (9) reads as $y(x) = \mathrm{e}^x$.
Using (7), the iteration scheme for this example is

$$y_{n+1}(x) = y_n(x) - \int_0^x \left\{ y_n'(\xi) - \mathrm{e}^{\xi} - \left(h'(\xi) - 1\right) \mathrm{e}^{h(\xi)-\xi} - 1.1\mathrm{e}^{\xi} - \mathrm{e}^{-\xi} y_n\left(h(\xi)\right) + \mathrm{e}^{-\xi}\left[y_n\left(h(\xi)\right)\right]' + \int_0^{1.1} \mathrm{e}^{\xi-t} y_n(t)\,\mathrm{d}t \right\} \mathrm{d}\xi. \tag{10}$$

Starting with $y_0 = 1 + x + \frac{x^2}{2}$ in (10), with the help of symbolic computations performed by Maple package the approximate solution of (9) can be obtained.

Table 1 compares the computed errors σ_n of the nth-order approximation of VIM with those of a numerical approach reported in [10] for different cases of $h(x)$ in Example 4.1.

Example 4.2. Consider the following functional Volterra integral equation of the second kind [11]

$$y(x) + x\mathrm{e}^{-x} y\left(h(x)\right) + \int_a^x \mathrm{e}^{x-t} y(t)\,\mathrm{d}t = g(x), \quad a \le x \le b, \tag{11}$$

where

$$g(x) = \mathrm{e}^x + x\mathrm{e}^{h(x)-x} + \mathrm{e}^x\,(x-a)\,, \quad a = 0,\ b = 1.1,$$

and the exact solution reads as $y(x) = \mathrm{e}^x$.

Applying VIM and from (8), the iteration scheme for this example is obtained as:

$$y_{n+1}(x) = y_n(x) - \int_0^x \left\{ y_n'(\xi) - 2\mathrm{e}^{\xi} - \left(1 + \xi\left(h'(\xi) - 1\right)\right) \mathrm{e}^{h(\xi)-\xi} - \xi\mathrm{e}^{\xi} + \left(\mathrm{e}^{-\xi} - \xi\mathrm{e}^{-\xi}\right) y_n\left(h(\xi)\right) + \xi\mathrm{e}^{-\xi} y_n'\left(h(\xi)\right) + \frac{\mathrm{d}}{\mathrm{d}\xi}\int_0^{\xi} \mathrm{e}^{\xi-t} y_n(t)\,\mathrm{d}t \right\} \mathrm{d}\xi. \tag{12}$$

Starting with $y_0 = 1 + x$ in iterative scheme (12), the approximate solution of (11) will be obtained.

In Table 2, we compare the computed errors σ_n of the nth-order approximation of VIM with those obtained in [11] for some cases of $h(x)$ in Example 4.2.

Table 3
Comparison of the absolute errors for Example 4.3.

n	$h(x) = x^2$		$h(x) = x/2$	
	VIM	NM [7]	VIM	NM [7]
2	1.8e−15	3.5e−16	1.7e−14	3.9e−16
3	4.7e−16	5.4e−16	6.4 e−15	2.9 e−16
4	5.5e−17	9.3e−16	8.7 e−16	4.0 e−16
5	1.1e−17	1.1e−15	1.4 e−16	3.2 e−16

Example 4.3. We now consider the following Volterra integral equation of the second kind

$$y(x) + \int_a^x e^{x-h(t)} y\,(h(t))\ dt = g(x),$$

where

$$g(x) = e^x + (x - a)e^x, \quad a = 0,\ b = 1.1.$$

The exact solution of this example was found to be [11]

$$y(x) = e^x.$$

The iteration scheme for this example, using (8), reads as

$$y_{n+1}(x) = y_n(x) - \int_0^x \left\{ y_n'(\xi) - e^{\xi} - \lambda e^{\xi} - \lambda \xi e^{\xi} + \lambda \frac{d}{d\xi} \int_0^{\xi} e^{\xi - h(t)} y_n\,(h(t))\ dt \right\} d\xi. \tag{13}$$

We start with an initial approximation $y_0(x) = 1+x$; by the iteration formula (13), we can obtain the next approximations of the solution.

In Table 3, we present the comparisons made between the computed errors σ_n of the nth-order approximation of VIM and those of a numerical approach proposed in [11] for different cases of $h(x)$ in Example 4.3.

It is noteworthy to be mentioned that in Examples 4.1–4.3, the subsequent components of y_n's were too long to be mentioned.

5. Conclusion

An efficient algorithm based on the variational iteration method has been successfully applied to the second order functional integral equations. As can be seen from Tables 1–3, by implementing only a few steps in the variational iteration method, the approximate analytical solutions with high accuracy can be obtained.

It can be concluded that the variational iteration method is a powerful and promising tool for solving such kinds of functional integral equations. This method can also be extended to the functional differential equations of the form

$$A_1(x)y''\,(h(x)) + A_2(x)y''(x) + B_1(x)y'\,(h(x)) + B_2(x)y'(x) + C_1(x)y\,(h(x)) + C_2(x)y(x) = g(x),$$
$$y(a) = \alpha, \qquad y'(a) = \beta$$

which are under study in our research group.

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