Available at WWW.MATHEMATICSWEB.ORG POWERED BY SCIENCE DIRECT®

Journal of Functional Analysis 205 (2003) 380–400

JOURNAL OF Functional Analysis

http://www.elsevier.com/locate/jfa

Linear maps on the space of all bounded observables preserving maximal deviation

Lajos Molnár* and Mátyás Barczy

Institute of Mathematics and Informatics, University of Debrecen, 4010 Debrecen, P.O. Box 12, Hungary

Received 29 August 2002; revised 10 March 2003; accepted 14 May 2003

Communicated by D. Sarason

Abstract

In this paper we determine all the bijective linear maps on the space of bounded observables which preserve a fixed moment or the variance. Nonlinear versions of the corresponding results are also presented.

Keywords: Bounded observable; Mean value; Moments; Variance; Deviation; Linear preservers; Isometry

1. Introduction and statements of the results

In the Hilbert space formalism of quantum mechanics there are several structures of linear operators which play distinguished role in the theory. These are, among others, the following. The Jordan algebra $B_s(H)$ of all self-adjoint bounded linear operators on the Hilbert space H which are called bounded observables, the lattice $P(H)$ of all projections (i.e., self-adjoint idempotents) on H called quantum events, the convex set $S(H)$ of all positive trace-class operators on H with trace 1 called (mixed) states, and the so-called effect algebra $E(H)$ of all positive bounded linear operators which are majorized by the identity I. These structures play essential role in the probabilistic aspects of quantum theory.

Just as in the case of any algebraic structure in mathematics in general, the study of the automorphisms of the above-mentioned structures is of remarkable

*Corresponding author.
E-mail addresses: molnarl@math.klte.hu (L. Molnár), barczy@math.klte.hu (M. Barczy).

0022-1236/03/$ - see front matter
doi:10.1016/S0022-1236(03)00213-1

importance. One can find an interesting unified treatment of those automorphisms in [7]. In our recent papers [1,16] we presented some results on the local behaviour of the automorphisms in question, while in [17–19] we have started to study how these automorphisms can be characterized by their preserving properties.

The systematic study of preserver problems (more precisely, linear preserver problems, so-called LPPs) constitutes a part of matrix theory. In fact, this area represents one of the most active research fields in matrix theory (we refer only to two survey papers [13,14]). In the last decades considerable attention has also been paid to the infinite-dimensional case as well, i.e., to linear preserver problems concerning algebras of linear operators on general Hilbert spaces or Banach spaces (once again, we only refer to a survey paper [6]). From the point of view of the present paper, the most important point is that the solutions of linear preserver problems provide, in most of the cases, important new information on the automorphisms of the underlying algebras (matrix algebras, or more generally, operator algebras) as they show how those automorphisms are determined by their various preserving properties. These properties mainly concern a certain important numerical quantity or a set of them corresponding to operators (e.g., norm, spectrum), or they concern a distinguished set of operators (e.g., the set of projections), or they concern an important relation among operators (e.g., commutativity). This kind of results may help to better understand the behaviour of the automorphisms of the underlying algebras.

In our above-mentioned papers [17–19] we have started to study the automorphisms of Hilbert space effect algebras and those of the Jordan algebra of bounded observables from a similar, preserver point of view. There we have considered transformations which preserve quantities, or relations, or properties that all have physical meaning. For example, as for observables, in [18] we determined all bijective transformations (no linearity was assumed) of $B_s(H)$ that preserve the order (which is just the usual order among self-adjoint operators). In [19] we described the general form of those bijections of $B_s(H)$ which preserve commutativity (in quantum theory the expression compatibility is used in the place of commutativity) and are multiplicative on commuting pairs of operators.

We now turn to the content of the present paper. In classical probability theory the mean value (or, more generally, the moments) and the variance are among the most important characteristics of a random variable. Therefore, it is not surprising that the same is true for the quantum mechanical variables, i.e., for the observables. The main aim of this paper is to show that the preservation of any of those quantities more or less completely characterizes the automorphisms among the linear transformations of $B_s(H)$.

In what follows, let H be a complex Hilbert space. Let $A \in B_s(H)$ and pick a unit vector $\varphi \in H$. The mean value $m(A, \varphi)$ of the observable A in the (pure) state represented by φ is defined as

$$m(A, \varphi) = \langle A\varphi, \varphi \rangle.$$

So, unlike in classical probability, in quantum theory there is a set of mean values of a single variable. We intend to determine all the bijective linear transformations ϕ of $B_s(H)$ which preserve this set in the sense that

$$\begin{aligned}\{m(\phi(A),\varphi):\ \varphi\in H, ||\varphi||=1\} &= \{\langle \phi(A)\varphi,\varphi\rangle:\ \varphi\in H, ||\varphi||=1\}\\ &= \{\langle A\varphi,\varphi\rangle:\ \varphi\in H, ||\varphi||=1\}\\ &= \{m(A,\varphi):\ \varphi\in H, ||\varphi||=1\}\end{aligned}$$

holds for every $A\in B_s(H)$. Clearly, the set of all mean values of an observable $A\in B_s(H)$ is equal to the numerical range of the operator A. So, the above problem can be reformulated as the linear preserver problem concerning the numerical range on $B_s(H)$. Obviously, it is a more general problem to preserve the numerical radius $w(.)$ instead of the numerical range. It is well-known that for a self-adjoint operator A this former quantity $w(A)$ is equal to the operator norm $||A||$. Hence, we easily arrive at the problem of describing the surjective linear isometries of $B_s(H)$. The solution of this problem is well known in the literature. For example, one can consult the paper [8]. The corresponding result reads as follows.

Theorem 1. *Let $\phi: B_s(H)\to B_s(H)$ be a bijective linear map which preserves the operator norm, that is, suppose that*

$$||\phi(A)|| = ||A|| \quad (A\in B_s(H)).$$

Then there is an either unitary or antiunitary operator U on H such that ϕ is either of the form

$$\phi(A) = UAU^* \quad (A\in B_s(H)) \tag{1}$$

or of the form

$$\phi(A) = -UAU^* \quad (A\in B_s(H)). \tag{2}$$

(By an antiunitary operator we mean a norm preserving conjugate-linear bijection of the underlying Hilbert space H.) Although this is not a new result, in Section 2 we present the sketch of a short proof that applies preserver techniques.

Observe that the above statement is a self-adjoint analogue of a well-known result of Kadison [12] on the surjective isometries of C^*-algebras and also that of a result of Brešar and Šemrl [5] describing the form of all bijective linear maps of the algebra of all bounded linear operators on a Banach space which preserve the spectral radius (recall that the norm of a self-adjoint operator is equal to its spectral radius). However, there is no doubt, those results are much deeper than the one we have formulated above.

With the help of Theorem 1 we can describe the bijective linear maps of $B_s(H)$ which preserve the set of mean values. In fact, as the second possibility (2) can be excluded, we obtain that the maps in question are exactly the automorphisms of the

Jordan algebra $B_s(H)$ (cf. [7]). Moreover, observe that using the same result Theorem 1 we can solve also the problem of preserving a fixed moment of bounded observables. For any $n\in\mathbb{N}$, the nth moment of an observable $A\in B_s(H)$ is the set

$$\{m(A^n,\varphi):\ \varphi\in H, ||\varphi||=1\}=\{\langle A^n\varphi,\varphi\rangle:\ \varphi\in H, ||\varphi||=1\}.$$

Now, the solution of the mentioned problem immediately follows as one can refer to the equality

$$\sup\{|\langle A^n\varphi,\varphi\rangle|:\ \varphi\in H, ||\varphi||=1\}=w(A^n)=||A^n||=||A||^n$$

which holds for every self-adjoint operator A on H.

Beside moments, the other very important probabilistic character of an observable is its variance. Just as with mean values, we have variance with respect to every (pure) state. Let $A\in B_s(H)$ and $\varphi\in H,\ ||\varphi||=1$. The variance $var(A,\varphi)$ of A in the state φ is defined by

$$\begin{aligned} var(A,\varphi) &= m((A-m(A,\varphi)I)^2,\varphi)\\ &= \langle (A-\langle A\varphi,\varphi\rangle I)^2\varphi,\varphi\rangle\\ &= \langle A^2\varphi,\varphi\rangle-\langle A\varphi,\varphi\rangle^2. \end{aligned}$$

We intend to determine all bijective linear maps on $B_s(H)$ which preserve the set of variances of observables. It is obvious that every linear map ϕ on $B_s(H)$ which preserves this set, i.e., which satisfies

$$\{var(\phi(A),\varphi):\ \varphi\in H, ||\varphi||=1\}=\{var(A,\varphi):\ \varphi\in H, ||\varphi||=1\}$$

for every $A\in B_s(H)$, also preserves the quantity

$$||A||_v=\sup_{||\varphi||=1} var(A,\varphi)^{1/2}, \tag{3}$$

i.e., satisfies

$$||\phi(A)||_v=||A||_v$$

for every $A\in B_s(H)$. The quantity $||A||_v$ is called the maximal deviation of the observable $A\in B_s(H)$. In its definition (3) we have used the square root of the variances since, as it will be clear from Lemma 1, the so-obtained quantity is a semi-norm on $B_s(H)$ which is quite convenient to handle.

Observe that every automorphism of $B_s(H)$ (see [7]) as well as its negative preserves the maximal deviation and that perturbations by scalar operators also do not change this quantity. Our result that follows (which can be considered as the main result of the paper) states that from these two types of transformations we can construct all the linear preservers under consideration.

Theorem 2. *Let $\phi : B_s(H) \to B_s(H)$ be a bijective linear map which preserves the maximal deviation, that is, suppose that*

$$||\phi(A)||_v = ||A||_v \quad (A \in B_s(H)).$$

Then there exist an either unitary or antiunitary operator U on H and a linear functional $f : B_s(H) \to \mathbb{R}$ such that ϕ is either of the form

$$\phi(A) = UAU^* + f(A)I \quad (A \in B_s(H)) \tag{4}$$

or of the form

$$\phi(A) = -UAU^* + f(A)I \quad (A \in B_s(H)). \tag{5}$$

Unlike with the transformations preserving the set of mean values, for the bijective linear maps on $B_s(H)$ which preserve the set of variances, the second possibility (5) above can obviously occur. Hence, we obtain that every such preserver is "an automorphism of $B_s(H)$ or its negative perturbed by a scalar operator valued linear transformation".

Since, from the physical point of view, to assume the linearity of the considered transformations on the space of observables sometimes seems to be a strong assumption that can be quite difficult to check in the particular cases, in the remaining results we formulate nonlinear versions of Theorems 1 and 2 as follows. First observe that

$$d_m(A, B) = \sup_{||\varphi||=1} |m(A - B, \varphi)| = ||A - B|| \quad (A, B \in B_s(H))$$

defines a metric on $B_s(H)$, while

$$d_v(A, B) = \sup_{||\varphi||=1} var(A - B, \varphi)^{1/2} = ||A - B||_v \quad (A, B \in B_s(H))$$

defines a semi-metric on $B_s(H)$. Both d_m and d_v represent certain stochastic distances between bounded observables. Using the first two results and the celebrated Mazur–Ulam theorem on surjective nonlinear isometries of normed spaces [15], we can prove the following statements which show how close the stochastic isometries with respect to either d_m or d_v are to the automorphisms of the Jordan algebra $B_s(H)$.

Theorem 3. *Let $\phi : B_s(H) \to B_s(H)$ be a bijective transformation (linearity is not assumed) with the property that*

$$d_m(\phi(A), \phi(B)) = d_m(A, B) \quad (A \in B_s(H)).$$

Then there is an either unitary or antiunitary operator U on H and a fixed operator $X \in B_s(H)$ such that ϕ is either of the form

$$\phi(A) = UAU^* + X \quad (A \in B_s(H))$$

or of the form

$$\phi(A) = -UAU^* + X \quad (A \in B_s(H)).$$

The last result of the paper describes the form of all "stochastic isometries" with respect to the semi-metric d_v.

Theorem 4. *Let $\phi : B_s(H) \to B_s(H)$ be a bijective transformation (linearity is not assumed) with the property that*

$$d_v(\phi(A), \phi(B)) = d_v(A, B) \quad (A \in B_s(H)).$$

Then there exist an either unitary or antiunitary operator U on H, a fixed operator $X \in B_s(H)$, and a functional $f : B_s(H) \to \mathbb{R}$ (not linear in general) such that ϕ is either of the form

$$\phi(A) = UAU^* + X + f(A)I \quad (A \in B_s(H))$$

or of the form

$$\phi(A) = -UAU^* + X + f(A)I \quad (A \in B_s(H)).$$

2. Proofs

We first remark that in what follows, whenever we speak about the preservation of an object or relation we always mean that this is preserved in both directions.

We now present a short proof of Theorem 1.

Sketch of the proof of Theorem 1. Let $\phi : B_s(H) \to B_s(H)$ be a surjective linear isometry. Clearly, ϕ preserves the extreme points of the unit ball of $B_s(H)$ which are well-known (and easily seen) to be exactly the self-adjoint unitaries, i.e., the operators of the form $2P - I$ where P is a projection. Now, one can readily prove that among those extreme points, I and $-I$ are distinguished by the following property. The extreme point U is either I or $-I$ if and only if we have $||U - V|| \in \{0, 2\}$ for every extreme point V. Therefore, we get $\phi(\{I, -I\}) = \{I, -I\}$. Clearly, we can suppose without loss of generality that $\phi(I) = I$. In that case we obtain that ϕ preserves the projections. This gives us that ϕ is a Jordan automorphism of $B_s(H)$, that is, it satisfies the equality $\phi(AB + BA) = \phi(A)\phi(B) + \phi(B)\phi(A)$ for every $A, B \in B_s(H)$ (cf. [3] or [6]). Therefore, we have that ϕ is of the form

$$\phi(A) = UAU^* \quad (A \in B_s(H))$$

with some unitary or antiunitary operator U on H (see, for example, [7]). □

The proof of Theorem 2 is much more difficult than the one given above and is based on the following series of lemmas. Our first observation below will prove to be fundamental from the viewpoint of the proof of Theorem 2 that we are going to present. It states that the maximal deviation of an operator T is equal to the so-called factor norm of T in the factor Banach space $B_s(H)/\mathbb{R}I$. (In particular, this result implies that the function $T \mapsto ||T||_v$ is a semi-norm on $B_s(H)$.) Denote by $\overline{T}$ the equivalence class of T in $B_s(H)/\mathbb{R}I$. The factor norm $||\overline{T}||$ of T is defined by

$$||\overline{T}|| = \inf_{\lambda \in \mathbb{R}} ||T + \lambda I||.$$

As the spectral radius and the operator norm of a self-adjoint operator are the same, it easily follows that $||\overline{T}||$ is equal to the half of the diameter of the spectrum $\sigma(T)$ of T.

Lemma 1. *For all $T \in B_s(H)$ we have $||T||_v = ||\overline{T}|| = diam(\sigma(T))/2$.*

Proof. As we have already verified that $||\overline{T}|| = diam(\sigma(T))/2$, we have to prove only the first equality. For a scalar operator T, both quantities $||T||_v$ and $||\overline{T}||$ are 0. Otherwise, we can assume that $0 \leqslant T \leqslant I$ and that $\{0,1\} \subset \sigma(T) \subset [0,1]$. This is because the factor norm and the maximal deviation of T are invariant under adding scalar operators and they are absolute homogeneous. In this case we have $||\overline{T}|| = \frac{1}{2}$.

First we prove the easier inequality $||T||_v \leqslant ||\overline{T}||$. For any $\lambda \in \mathbb{R}$ we have

$$\begin{aligned} ||T||_v^2 = ||T + \lambda I||_v^2 &= \sup_{||\varphi||=1} (\langle (T + \lambda I)^2 \varphi, \varphi \rangle - \langle (T + \lambda I)\varphi, \varphi \rangle^2) \\ &\leqslant \sup_{||\varphi||=1} \langle (T + \lambda I)^2 \varphi, \varphi \rangle = ||(T + \lambda I)^2|| = ||T + \lambda I||^2. \end{aligned}$$

This yields $||T||_v \leqslant ||T + \lambda I||$ for all $\lambda \in \mathbb{R}$ which implies that $||T||_v \leqslant ||\overline{T}||$.

Now, we turn to the less obvious inequality $\frac{1}{2} = ||\overline{T}|| \leqslant ||T||_v$. Let E_T be the spectral measure corresponding to T. Since 0 and 1 are in the spectrum of T, it follows that for any $0 < \delta \leqslant \frac{1}{2}$, the measures of $]-\delta, \delta[\cap \sigma(T)$ and $]1-\delta, 1+\delta[\cap \sigma(T)$ under E_T are mutually orthogonal nonzero projections. At this stage δ is not fixed, we shall specify it later. Denote these projections by P_0 and P_1, respectively.

Let x be a unit vector in the range of P_0 and y be a unit vector in the range of P_1. Define $\varphi = (x + y)/\sqrt{2}$. Then $\varphi \in H$ is a unit vector and we assert that the following inequality holds:

$$\sqrt{\langle T^2 \varphi, \varphi \rangle - \langle T\varphi, \varphi \rangle^2} \geqslant \sqrt{\frac{(1-2\delta)^2}{2} - \frac{(1+2\delta)^2}{4}}. \tag{6}$$

To see this, first observe that $Tx = TP_0x$ and $Ty = TP_1y$. Since

$$TP_0 = \int_{]-\delta,\delta[\cap\sigma(T)} t\, d\, E_T(t),$$

we deduce $||TP_0|| \leqslant \delta$. This yields that

$$||Tx|| \leqslant \delta.$$

A similar argument shows that $||Ty - y|| = ||TP_1y - P_1y|| \leqslant \delta$. Since $||y|| = 1$, this gives us that

$$1 - \delta \leqslant ||Ty|| \leqslant 1 + \delta.$$

Now, to prove (6) we estimate $\langle T^2\varphi, \varphi\rangle = ||T\varphi||^2$ from below and $\langle T\varphi, \varphi\rangle^2$ from above. Since $T\varphi = (Tx + Ty)/\sqrt{2}$, we have

$$||T\varphi|| \geqslant \frac{-||Tx|| + ||Ty||}{\sqrt{2}} \geqslant \frac{-\delta + 1 - \delta}{\sqrt{2}}$$

and thus we get

$$\langle T^2\varphi, \varphi\rangle = ||T\varphi||^2 \geqslant \frac{(1-2\delta)^2}{2}. \tag{7}$$

Using the equality $TP_0 = P_0T$ and the fact that P_0 and P_1 are mutually orthogonal projections, we have

$$\langle Tx, y\rangle = \langle TP_0x, P_1y\rangle = \langle P_0Tx, P_1y\rangle = \langle Tx, P_0P_1y\rangle = 0.$$

This also implies that $\langle Ty, x\rangle = 0$. Therefore, we infer

$$\langle T\varphi, \varphi\rangle = \frac{1}{2}(\langle Tx, x\rangle + \langle Ty, y\rangle).$$

Since $|\langle Tx, x\rangle| \leqslant ||Tx|| \leqslant \delta$ and $|\langle Ty, y\rangle| \leqslant ||Ty|| \leqslant 1 + \delta$, we obtain

$$\langle T\varphi, \varphi\rangle^2 \leqslant \frac{(1+2\delta)^2}{4}.$$

This inequality together with (7) gives (6).

Now, for an arbitrary $\varepsilon > 0$, choosing δ such that it satisfies

$$\sqrt{\frac{(1-2\delta)^2}{2} - \frac{(1+2\delta)^2}{4}} \geqslant \frac{1}{2} - \varepsilon,$$

it follows from what we have already proved that we can pick a unit vector $\varphi\in H$ for which

$$||T||_v\geqslant\sqrt{\langle T^2\varphi,\varphi\rangle-\langle T\varphi,\varphi\rangle^2}\geqslant\frac{1}{2}-\varepsilon.$$

This gives us that $||T||_v\geqslant\frac{1}{2}=||\overline{T}||$ which completes the proof of the lemma. □

Remark 1. As we have seen, the quantity $||T||_v=||\overline{T}||$ is exactly the half of the diameter of the spectrum of T. Therefore, if $T\geqslant 0$ and $0\in\sigma(T)$, then $||T||_v=||\overline{T}||\leqslant\frac{1}{2}$ if and only if $0\leqslant T\leqslant I$.

This observation will be used in the proof of our next lemma which determines the extreme points of the (closed) $\frac{1}{2}$-ball of the Banach space $B_s(H)/\mathbb{R}I$.

Lemma 2. *The extreme points of the ball* $\{\overline{A}\in B_s(H)/\mathbb{R}I : ||\overline{A}||\leqslant\frac{1}{2}\}$ *are the classes of nontrivial projections, that is, the elements* $\overline{P}\in B_s(H)/\mathbb{R}I$, *where* P *is a nontrivial projection* $(P\neq 0,I)$ *on* H.

Proof. The point in the proof is to reduce the problem concerning classes of operators to a problem concerning single operators.

First, we check that the classes of nontrivial projections are extreme points of the ball in question. Suppose that P is a nontrivial projection and

$$\overline{P}=\mu\overline{T}+(1-\mu)\overline{S},$$

where $0<\mu<1$, $||\overline{T}||\leqslant\frac{1}{2}$, $||\overline{S}||\leqslant\frac{1}{2}$, $T,S\in B_s(H)$. Adding scalar operators if necessary, we can suppose that $T,S\geqslant 0$, $0\in\sigma(T)$, $0\in\sigma(S)$. Clearly,

$$P=\mu T+(1-\mu)S+\lambda I$$

holds for some $\lambda\in\mathbb{R}$.

We claim that $\lambda=0$. If $\varphi\in H$ is a unit vector in the kernel of P, we infer that

$$0=\langle P\varphi,\varphi\rangle=\mu\langle T\varphi,\varphi\rangle+(1-\mu)\langle S\varphi,\varphi\rangle+\lambda.$$

Since $\langle T\varphi,\varphi\rangle\geqslant 0$ and $\langle S\varphi,\varphi\rangle\geqslant 0$, the above equality yields $\lambda\leqslant 0$.

It follows from $\sigma(P)=\{0,1\}$ that $||\overline{P}||=\frac{1}{2}$. We compute

$$\frac{1}{2}=||\overline{P}||=||\mu\overline{T}+(1-\mu)\overline{S}||\leqslant\mu||\overline{T}||+(1-\mu)||\overline{S}||\leqslant(\mu+1-\mu)\frac{1}{2}=\frac{1}{2},$$

from which we deduce that $||\overline{T}|| = ||\overline{S}|| = \frac{1}{2}$. Using Remark 1 we get $0 \leqslant T, S \leqslant I$. So, if φ is a unit vector in the range of P, then we have

$$1 = \langle P\varphi, \varphi \rangle = \mu \langle T\varphi, \varphi \rangle + (1-\mu)\langle S\varphi, \varphi \rangle + \lambda \leqslant \mu + (1-\mu) + \lambda,$$

which gives us that $\lambda \geqslant 0$. Therefore, it follows that $\lambda = 0$ as we have claimed.

Consequently, we have $P = \mu T + (1-\mu)S$. This means that P is a nontrivial convex combination of two elements of the operator interval $[0, I]$. However, it is well-known that the extreme points of this operator interval are exactly the projections. Hence, we get $P = T = S$. This proves that the classes of nontrivial projections are really extreme points.

It remains to prove that these classes are the only extreme points. In order to see this, let B be a self-adjoint operator with $||\overline{B}|| = \frac{1}{2}$ which is not a nontrivial projection. We show that $\overline{B}$ is not an extreme point of the ball in question. Clearly, just as above, we can assume that $B \geqslant 0$ and $0 \in \sigma(B)$. Then we have $0 \leqslant B \leqslant I$. As $||\overline{B}|| = \frac{1}{2}$, it also follows that $1 \in \sigma(B)$. We are going to show that there exist two operators B_1, B_2 in the operator interval $[0, I]$ such that $B = (B_1 + B_2)/2$ and $B \neq B_1, B_2$. In the present situation this will imply that $\overline{B} \neq \overline{B_1}, \overline{B_2}$. Then, as $\overline{B} = (\overline{B_1} + \overline{B_2})/2$, $||\overline{B_1}||, ||\overline{B_2}|| \leqslant \frac{1}{2}$ (see Lemma 1), we can infer that $\overline{B}$ is not an extreme point. So, in order to construct such operators B_1, B_2, choose $\lambda_0 \in \sigma(B) \cap]0, 1[$. (The existence of such a λ_0 follows from the facts that B is not a nontrivial projection and that $||\overline{B}|| \neq 0$.) Now, one can easily find continuous real valued functions $f_1, f_2 : [0,1] \to [0,1]$ such that $(f_1 + f_2)/2$ is the identity on $[0,1]$ and $f_1(\lambda_0) \neq \lambda_0 \neq f_2(\lambda_0)$. Defining $B_1 = f_1(B), B_2 = f_2(B)$, it follows from the properties of the continuous function calculus that we obtain operators with the desired properties. This completes the proof of the lemma. □

In what follows, we intend to characterize the unitary equivalence of nontrivial projections P, Q by means of some correspondence between the classes $\overline{P}$ and $\overline{Q}$ that can be expressed in terms of the metric induced by the factor norm. The first step in this direction is made in the following lemma.

Lemma 3. *Let P and Q be projections on H. Suppose that P is nontrivial and $||\overline{P} - \overline{Q}|| < \frac{1}{2}$. Then P is unitarily equivalent to Q.*

Proof. First observe that $Q \neq 0, I$. In fact, in the opposite case we would have $||\overline{P}|| < 1/2$. But this means that the diameter of $\sigma(P)$ is less than 1, which gives us that P is a trivial projection, a contradiction.

Because of the definition of the factor norm there exists a $\mu \in \mathbb{R}$ such that $||P - (Q + \mu I)|| < \frac{1}{2}$. Let R be a projection of rank at most 2 whose range contains a unit vector from the range of P and a unit vector from the range of Q, respectively. The operators RPR and $R(Q + \mu I)R$ are of finite rank, 1 is the largest eigenvalue of RPR and $1 + \mu$ is the largest eigenvalue of $R(Q + \mu I)R$. Indeed, to prove for example this

last statement, observe that

$$R(Q+\mu I)R \leqslant R(I+\mu I)R = (1+\mu)R \leqslant (1+\mu)I.$$

This shows that the spectrum of $R(Q+\mu I)R$ is a subset of the interval $]-\infty, 1+\mu]$. On the other hand, $1+\mu$ is an eigenvalue of the operator $R(Q+\mu I)R$ since the range of R contains a unit vector from the range of Q.

By Weyl's perturbation theorem (see, for example, [2, Corollary III.2.6]) we deduce

$$|\mu| = |1-(1+\mu)| \leqslant ||RPR - R(Q+\mu I)R||$$

$$\leqslant ||R||\,||P-(Q+\mu I)||\,||R|| < \frac{1}{2},$$

and so we have

$$||P-Q|| \leqslant ||P-(Q+\mu I)|| + |\mu| < \frac{1}{2} + \frac{1}{2} = 1.$$

But it is a well-known result that if the distance between two projections in the operator norm is less than 1, then they are unitarily equivalent. This completes the proof of the lemma. □

A useful solution of the problem mentioned before Lemma 3 is given in the next result.

Lemma 4. *Let P and Q be projections on H and suppose that P is nontrivial. Then P is unitarily equivalent to Q if and only if there exists a continuous function $\varphi : [0,1] \to \overline{P(H)}$ such that $\varphi(0) = \overline{P}$ and $\varphi(1) = \overline{Q}$.*

(Here $\overline{P(H)}$ denotes the set of classes in $B_s(H)/\mathbb{R}I$ which correspond to projections.)

Proof. The necessity is easy to see. Indeed, this follows from the well-known fact that if P, Q are equivalent projections then they can be connected by a continuous curve (continuity is meant in the operator norm topology) in the set of projections and from the fact that the operator norm majorizes the factor norm.

Now, conversely, suppose that there exists a continuous mapping $\varphi : [0,1] \to \overline{P(H)}$ such that $\varphi(0) = \overline{P}$ and $\varphi(1) = \overline{Q}$. As φ is defined on a compact set, it is uniformly continuous. Hence, we can choose a positive δ such that

$$||\varphi(t) - \varphi(s)|| < \frac{1}{2} \quad \text{if } |s-t| < \delta, \ \ s, t \in [0,1].$$

It follows that there exist projections $P_1, \ldots, P_n$ with the property that

$$||\overline{P} - \overline{P_1}|| < \frac{1}{2}, \ldots, ||\overline{P_n} - \overline{Q}|| < \frac{1}{2}.$$

By Lemma 3, we obtain that P and P_1 are unitarily equivalent (and, consequently, P_1 is nontrivial). Using this argument again and again we can conclude that P is unitarily equivalent to Q. $\square$

The meaning of our last lemma which follows is a metric characterization of the equality of nontrivial projections in $B_s(H)$ with respect to the semi-norm $||.||_v$. Denote by $F_s(H)$ the set of all finite rank elements in $B_s(H)$.

Lemma 5. *Let P and Q be nontrivial projections on H such that*

$$||P + A||_v = ||Q + A||_v$$

holds for all $A \in F_s(H)$. Then we have $P = Q$.

Proof. [1] Let R be a rank-1 subprojection of the projection P. Then the diameter of the spectrum of $P + R$ is 2, so by Lemma 1 we have

$$1 = ||P + R||_v = ||Q + R||_v,$$

that is, the diameter of $\sigma(Q + R)$ is also equal to 2. Since $0 \leqslant Q + R \leqslant 2I$, thus $\sigma(Q + R)$ is a subset of the closed interval $[0, 2]$. Therefore, we have $0, 2 \in \sigma(Q + R)$.

It is well-known that the spectrum of any normal operator coincides with its approximate point spectrum. Consequently, we can find unit vectors x_n in H ($n \in \mathbb{N}$) such that

$$||Qx_n + Rx_n - 2x_n|| \to 0 \quad \text{as } n \to \infty.$$

This yields that

$$||Qx_n + Rx_n|| \to 2. \tag{8}$$

Denote $u_n = Qx_n$ and $v_n = Rx_n$. We have $||u_n|| \leqslant 1, ||v_n|| \leqslant 1$. Since v_n is in the range of R which is one dimensional, there must exist a convergent subsequence of (v_n). Without any loss of generality, we can assume that this subsequence is (v_n) itself. So, there exists a vector v in the range of R such that $||v_n - v|| \to 0$. Since

$$|\,||u_n + v|| - ||u_n + v_n||\,| \leqslant ||v - v_n|| \to 0$$

[1] We remark that in his/her report the referee presented a more elementary proof of this lemma which uses only matrix (finite-dimensional) arguments.

and $||u_n + v_n|| \to 2$, we have $||u_n + v|| \to 2$. On the other hand, by the parallelogram identity we obtain

$$||u_n - v||^2 = 2||u_n||^2 + 2||v||^2 - ||u_n + v||^2.$$

Therefore, we have

$$\limsup_{n\to\infty} ||u_n - v||^2 \leqslant 2 + 2 - 4 = 0,$$

which implies that $||u_n - v|| \to 0$. So, both $(u_n), (v_n)$ converge to v. Taking (8) into account, it is clear that $v \neq 0$.

Since the sequence (u_n) is in the range of Q which is a closed subspace, it follows that its limit v also belongs to this range. But v generates the range of R and hence R is a subprojection of Q. So, we have proved the following: every rank-1 subprojection of P is a subprojection of Q. Therefore, P is a subprojection of Q. Changing the role of P and Q, we get that Q is also a subprojection of P and hence we obtain $P = Q$. □

Now, we are in a position to prove our main result.

Proof of Theorem 2. The brief summary of the proof is as follows. Our transformation ϕ which preserves the maximal deviation induces a surjective linear isometry Φ on the factor space $B_s(H)/\mathbb{R}I$. This Φ necessarily preserves the extreme points of the $\frac{1}{2}$-ball which points are well characterized in Lemma 2. This implies a certain preserving property of the original transformation ϕ. Namely, we obtain that ϕ preserves the operators of the form "nontrivial projection + scalar $\cdot I$". This will imply that ϕ preserves the commutativity on $F_s(H) + \mathbb{R}I$. Extending ϕ from this set to its complex linear span $F(H) + \mathbb{C}I$ ($F(H)$ stands for the set of all finite rank bounded linear operators on H), we obtain a complex-linear transformation which preserves normal operators. Applying the technique of the proof of a nice result of Brešar and Šemrl given in [4], we can conclude the proof in the case when $\dim H \geqslant 3$. If $\dim H = 2$, then rather surprisingly we can reduce our problem quite easily to Wigner's classical unitary–antiunitary theorem. So, this is the plan what we now carry out.

Define a map $\Phi : B_s(H)/\mathbb{R}I \to B_s(H)/\mathbb{R}I$ in the following way

$$\Phi(\overline{A}) = \overline{\phi(A)} \quad (A \in B_s(H)).$$

The transformation ϕ is a linear bijection of $B_s(H)$ which preserves the maximal deviation. By Lemma 1, we easily obtain that ϕ preserves the scalar operators and then that Φ is a well-defined linear bijection on $B_s(H)/\mathbb{R}I$ which preserves the factor norm. It follows that Φ preserves all closed balls around $\overline{0}$ as well as their extreme points. Therefore, by Lemma 2, we deduce that ϕ preserves the set of all operators of the form $P + \lambda I$, where P is a nontrivial projection and $\lambda \in \mathbb{R}$.

We shall show that ϕ preserves the commutativity on $F_s(H)+\mathbb{R}I$. Let P' and Q' be mutually orthogonal projections. We known that there exist projections P,Q,R and real numbers $\lambda_1,\lambda_2,\lambda_3$ such that

$$\phi(P')=P+\lambda_1 I,$$

$$\phi(Q')=Q+\lambda_2 I,$$

$$\phi(P'+Q')=R+\lambda_3 I.$$

By the linearity of ϕ this implies that $P+Q=R+tI$ for some real number t (in fact, $t=\lambda_3-\lambda_1-\lambda_2$). We assert that P and Q are either commuting or the projections P,Q,R are unitarily equivalent to each other.

In order to prove this, we distinguish the following cases.

Case I: Suppose that R is scalar. Then $P+Q$ is also scalar which implies that P,Q commute.

Case II: Suppose that R is not scalar, that is, R is a nontrivial projection. Consider the orthogonal decomposition of H induced by the range and the kernel of R. Every operator has a matrix representation with respect to this decomposition. As for $P+Q$, we can write

$$P+Q=R+tI=\begin{bmatrix}(1+t)I & 0\\ 0 & tI\end{bmatrix}. \tag{9}$$

The inequality $0\leqslant P+Q\leqslant 2I$ implies that $0\leqslant t\leqslant 1$. According to the possible values of t we have the following sub-cases.

Case II/1: Suppose that $t=0$. Then $P+Q=R$ is a projection and hence $(P+Q)^2=P+Q$. From this equality we easily deduce $PQ=QP=0$ which implies that P,Q commute.

Case II/2: Suppose that $t=1$. Then $P+Q=R+I$, which implies that $R+(I-Q)=P$ is a projection. Just as above, we obtain that $R,I-Q$ are commuting projections. This implies that R,Q commute and, finally, it follows from the equality $R+(I-Q)=P$ that P,Q also commute.

Case II/3:[2] Suppose that $0<t<1$. In this case we use the result that any two projections in generic position (i.e., with no common eigenvectors) are unitarily equivalent (see [9,11]). As the spectrum of $P+Q=R+tI$ is contained in $\{t,1+t\}$, the numbers 0,1,2 are not in the spectrum of $P+Q$. This implies that P,Q are in a generic position and hence they are unitarily equivalent. Similarly, as the spectrum of $R-P$ is contained in $\{-t,1-t\}$ which does not contain -1,0,1, we infer that P,R are in a generic position and hence they are unitarily equivalent. It follows that the projections P,Q,R are pairwise unitarily equivalent. What does this mean for our original projections P',Q'? Obviously, in the present case P,Q,R are nontrivial. Using Lemma 4 and the isometric property of Φ with respect to the factor norm,

[2]The main part of the argument in this case was suggested by the referee. Due to his/her idea, the original proof could be reduced from 3 pages to few lines.

we obtain that the projections $P', Q', P'+Q'$ are pairwise unitarily equivalent. But if P', Q' are nonzero mutually orthogonal finite rank projections, then this cannot happen.

Therefore, we have proved that for any finite rank projections P', Q' with $P'Q' = Q'P' = 0$ it follows that $\phi(P')\phi(Q') = \phi(Q')\phi(P')$. If we pick operators $A, B \in F_s(H)$ which commute, then they can be diagonalized simultaneously. Using the just proved property of ϕ one can easily deduce that $\phi(A), \phi(B)$ also commute.

We show that

$$\phi(F_s(H) + \mathbb{R}I) = F_s(H) + \mathbb{R}I.$$

If $\dim H < \infty$, this is obvious. So, let H be infinite dimensional. Pick a nonzero finite rank projection P'. Then $\phi(P') = P + \lambda I$ holds for some nontrivial projection P and real number λ. If P is of finite rank or of finite corank, then we obtain $\phi(P') \in F_s(H) + \mathbb{R}I$. So, let us see what happens if P is of infinite rank and infinite corank.

First suppose that $\dim \operatorname{rng} P \leqslant \dim \operatorname{rng} P^\perp$. Then we can find nontrivial projections P_1 and P_2 such that $P = P_1 + P_2$ and P, P_1, P_2 are mutually unitarily equivalent. Now, referring to Lemma 4, there are nontrivial projections P'_1, P'_2 such that

$$P' + \mu I = P'_1 + P'_2$$

holds for some $\mu \in \mathbb{R}$ and the projections P', P'_1, P'_2 are mutually unitarily equivalent. So, the projections P'_1, P'_2 are of finite rank and we see that on the right-hand side of the equality above there is a finite rank operator. This gives us that μ must be zero and then we have $P' = P'_1 + P'_2$. Like in the argument given in Case II/1, we obtain that P'_1, P'_2 are mutually orthogonal projections. We now conclude that, because of unitary equivalence and orthogonality, the equality $P' = P'_1 + P'_2$ is untenable which is a contradiction.

Next, suppose that $\dim \operatorname{rng} P \geqslant \dim \operatorname{rng} P^\perp$. Then we can apply the argument above for $P^\perp$ to find nontrivial projections P_1 and P_2 such that $P^\perp = P_1 + P_2$ and $P^\perp, P_1, P_2$ are mutually unitarily equivalent. This implies that there are nontrivial projections P'_1, P'_2 such that

$$P'^\perp + \nu I = P'_1 + P'_2 \tag{10}$$

holds for some $\nu \in \mathbb{R}$ and the projections $P'^\perp, P'_1, P'_2$ are mutually unitarily equivalent. (Observe that, as $\Phi(\overline{P'}) = \overline{P}$, we have $\Phi(\overline{P'^\perp}) = \overline{P^\perp}$.) It follows that the projections P'_1, P'_2 are of finite corank and hence their ranges have nonempty intersection. Therefore, we obtain that 2 belongs to the spectrum of the operator $P'_1 + P'_2$, and by (10) this implies that $\nu = 1$. Now, Eq. (10) can be rewritten in the form

$$P' = (I - P'_1) + (I - P'_2) = P'^\perp_1 + P'^\perp_2,$$

where the nontrivial projections $P', P_1'^{\perp}, P_2'^{\perp}$ are pairwise unitarily equivalent. Just as in the previous paragraph we arrive at a contradiction.

Therefore, we have $\phi(P') \in F_s(H) + \mathbb{R}I$ for every finite rank projection P'. Applying the spectral theorem for self-adjoint finite rank operators, it follows that $\phi(F_s(H) + \mathbb{R}I) \subset F_s(H) + \mathbb{R}I$. As ϕ^{-1} has the same properties as ϕ, considering the above relation for ϕ^{-1} in the place of ϕ, we conclude that

$$\phi(F_s(H) + \mathbb{R}I) = F_s(H) + \mathbb{R}I.$$

To sum up what we have already proved, it has turned out that ϕ when restricted onto $F_s(H) + \mathbb{R}I$ is a bijective linear map which preserves commutativity. Consider the complex unital algebra $F(H) + \mathbb{C}I$. As the real and imaginary parts of an operator in $F(H) + \mathbb{C}I$ belong to $F_s(H) + \mathbb{R}I$, one can readily verify that the map $\widetilde{\phi} : F(H) + \mathbb{C}I \to F(H) + \mathbb{C}I$ defined by

$$\widetilde{\phi}(A + iB) = \phi(A) + i\phi(B) \quad (A, B \in F_s(H) + \mathbb{R}I)$$

is a bijective complex-linear transformation. It is an elementary fact that a bounded linear operator is normal if and only if its real and imaginary parts are commuting. As ϕ preserves commutativity between self-adjoint finite rank operators, it follows that $\widetilde{\phi}$ preserves normality. If $\dim H \geqslant 3$, then this latter preserving property is strong enough to imply that $\widetilde{\phi}$ is of a certain particular form. In fact, there is a nice result of Brešar and Šemrl [4, Theorem 2] which, in the case when $\dim H \geqslant 3$, characterizes the bijective linear mappings on $B(H)$ that preserve normal operators. Although the algebra on which our transformation $\widetilde{\phi}$ is defined differs from $B(H)$ in general, it is not hard to see that the technique used in [4] can be applied to our present situation as well. This gives us the following two possibilities for the form of $\widetilde{\phi}$:

(i) there exist a unitary operator U on H, a linear functional $f : F(H) + \mathbb{C}I \to \mathbb{C}$ and a scalar $c \in \mathbb{C}$ such that

$$\widetilde{\phi}(T) = cUTU^* + f(T)I \quad (T \in F(H) + \mathbb{C}I),$$

(ii) there exist an antiunitary operator U on H, a linear functional $f : F(H) + \mathbb{C}I \to \mathbb{C}$ and a scalar $c \in \mathbb{C}$ such that

$$\widetilde{\phi}(T) = cUT^*U^* + f(T)I \quad (T \in F(H) + \mathbb{C}I).$$

Concerning ϕ, this means that there is an either unitary or antiunitary operator U on H, a real-linear function $f : F_s(H) + \mathbb{R}I \to \mathbb{C}$, and a constant $c \in \mathbb{C}$ such that

$$\phi(A) = cUAU^* + f(A)I \quad (A \in F_s(H) + \mathbb{R}I).$$

As $\phi(A)$ is self-adjoint, we have

$$\overline{c}UAU^* + \overline{f(A)}I = cUAU^* + f(A)I \tag{11}$$

for every $A \in F_s(H) + \mathbb{R}I$. If A is not a scalar operator, then it follows from this equality that $\overline{c} = c$. Next, we obtain from (11) that f is real valued. As ϕ preserves maximal deviation, we obtain that $|c| = 1$. Therefore, $c = \pm 1$ and we have the desired form for our transformation ϕ on $F_s(H) + \mathbb{R}I$. It remains to show that the same formula holds also on the whole space $B_s(H)$.

In order to see this, observe that composing ϕ by the transformation $A \mapsto cU^*AU$, we can assume without loss of generality that

$$\phi(A) = A + l(A)I$$

holds for every $A \in F_s(H) + \mathbb{R}I$, where $l : F_s(H) + \mathbb{R}I \to \mathbb{R}$ is a linear functional. Let P be a nontrivial projection on H. We know that $\phi(P) = Q + \mu I$ for some nontrivial projection Q and real number μ. Pick an arbitrary $A \in F_s(H)$. Since $\phi(A)$ is a scalar perturbation of A, we have

$$||Q + A||_v = ||\phi(P) + A||_v = ||\phi(P) + \phi(A)||_v = ||\phi(P + A)||_v = ||P + A||_v.$$

Since this holds true for every self-adjoint finite rank operator A, it follows from Lemma 5 that $Q = P$. This gives us that $\phi(P) - P \in \mathbb{R}I$ which holds also in the case when P is trivial. So, we have $\Phi(\overline{P}) = \overline{P}$ for every projection P. Since the linear transformations $A \mapsto \Phi(\overline{A})$ and $A \mapsto \overline{A}$ are continuous (on $B_s(H)$ we consider the operator norm while $B_s(H)/\mathbb{R}I$ is equipped with the factor norm), they are equal on the projections, it follows from the spectral theorem of self-adjoint operators and from the properties of the spectral integral that we have $\Phi(\overline{A}) = \overline{A}$ for every $A \in B_s(H)$. This gives us that

$$\phi(A) - A \in \mathbb{R}I \quad (A \in B_s(H))$$

which obviously implies that there is a linear functional $h : B_s(H) \to \mathbb{R}$ such that

$$\phi(A) = A + h(A)I \quad (A \in B_s(H)).$$

This completes the proof in the case when $dim\, H \geqslant 3$.

As the statement of the theorem is trivial for $dim\, H = 1$, it remains to consider the case when $dim\, H = 2$. In this case the nontrivial projections are exactly the rank-one projections. Pick a rank-one projection P. We know that there is a rank-one projection P' such that $\phi(P)$ is equal to the sum of P' and a scalar operator. It is easy to see that this P' is unique. (In fact, one can prove independently from the dimension of H that in the class of every nontrivial projection there is only one projection.) Therefore, we can denote $P' = \psi(P)$ and obtain a bijective transformation ψ on the set of all rank-one projections. We assert that ψ has the property that

$$tr\, PQ = tr\, \psi(P)\psi(Q) \tag{12}$$

holds for arbitrary rank-one projections P, Q on H. Here tr denotes the usual trace functional. As ϕ preserves the maximal deviation, this will clearly follow from

the equality

$$||P - Q||_v = \sqrt{1 - tr\,PQ} \tag{13}$$

that we are going to prove now. In fact, observe that the maximal deviation and the trace functional are invariant under the transformations $A \mapsto VAV^*$, where V is any unitary operator. Therefore, we can assume that

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

while Q is an arbitrary self-adjoint idempotent 2 by 2 matrix. It is easy to check that Q is of the form

$$Q = \begin{bmatrix} a & \sqrt{a(1-a)}e^{i\theta} \\ \sqrt{a(1-a)}e^{-i\theta} & 1-a \end{bmatrix},$$

where a, θ are real numbers and $0 \leqslant a \leqslant 1$. We have that the eigenvalues of $P - Q$ are $\pm\sqrt{1-a}$ and hence obtain that $||P - Q||_v = \sqrt{1-a}$. On the other hand, it is trivial to check that $tr\,PQ = a$. This results in the desired equality (13).

So, we have a bijective transformation ψ on the set of all rank-one projections which satisfies (12). Wigner's classical theorem on quantum mechanical symmetries (the so-called unitary–antiunitary theorem) describes the form of exactly such transformations in the case of general Hilbert spaces. We obtain that there exists an either unitary or antiunitary operator U on H such that

$$\psi(P) = UPU^*$$

holds for every rank-one projection P. As $\phi(P)$ differs from $\psi(P)$ only by a scalar operator, we obtain that $\phi(P) - UPU^* \in \mathbb{R}I$. By linearity this gives us that $\phi(A) - UAU^*$ is a scalar operator for every $A \in B_s(H)$. Now, one can easily complete the proof in the case when $dim\,H = 2$. □

Remark 2. As it is seen, preserving commutativity has played important role in our proof above. In fact, preserver problems of this kind are among the most fundamental ones in the theory of LPPs. To mention one of the most well-known results of this type which concerns operator algebras, we refer to [20].

Proof of Theorem 3. This follows immediately from Theorem 1 using the following important result of Mazur and Ulam [15]. If $\mathscr{V}$ is a real normed vector space and $T : \mathscr{V} \to \mathscr{V}$ is a bijective map which preserves the distance on $\mathscr{V}$ (i.e., T satisfies $||T(x) - T(y)|| = ||x - y||$ $(x, y \in \mathscr{V})$), then T can be written in the form $T(x) = L(x) + x_0$ $(x \in \mathscr{V})$, where $L : \mathscr{V} \to \mathscr{V}$ is a bijective linear isometry and $x_0 \in \mathscr{V}$ is a fixed vector. □

As for the proof of Theorem 4, we have to work more than in the previous proof as $||.||_v$ is only a semi-norm.

Proof of Theorem 4. Considering the map $A \mapsto \phi(A) - \phi(0)$, it is obvious that we can assume that ϕ sends 0 to 0. In what follows we use this assumption.

Consider the linear functional $\lambda I \mapsto \lambda$ on $\mathbb{R}I$. Extend it to a linear functional l of the whole vector space $B_s(H)$. (We do not need any kind of continuity of l, so no need to use Hahn–Banach theorem.) Define the transformation $\phi_1 : B_s(H) \to B_s(H)$ in the following way:

$$\phi_1(A) = \phi(A) - l(\phi(A))I + l(A)I \quad (A \in B_s(H)).$$

We assert that $\phi_1 : B_s(H) \to B_s(H)$ is a bijective linear map, it preserves the distance (with respect to the semi-metric d_v) and for every $A \in B_s(H)$, $\phi(A)$ and $\phi_1(A)$ differs only in a scalar operator. If this is really the case, then we can apply Theorem 2 for ϕ_1 and we are done. So, it remains to prove that ϕ_1 has the mentioned properties. As the last two ones are obvious from the definition, we have to prove only that ϕ_1 is linear and bijective. We begin with the linearity. As ϕ preserves the distance with respect to d_v and we have supposed that $\phi(0) = 0$, it follows that ϕ preserves the scalar operators (in fact, scalar operators can be characterized by the equality $||A||_v = 0$; see Lemma 1). Next, it is easy to show that the formula

$$\Phi(\overline{A}) = \overline{\phi(A)} \quad (A \in B_s(H))$$

defines a bijective isometry (distance preserving map) on $B_s(H)/\mathbb{R}I$ with respect to the factor norm. We only prove the isometric property. Indeed,

$$||\Phi(\overline{A}) - \Phi(\overline{B})|| = ||\phi(A) - \phi(B)||_v = ||A - B||_v = ||\overline{A} - \overline{B}||$$

holds for every $A, B \in B_s(H)$. Since $\Phi(\overline{0}) = \overline{\phi(0)} = \overline{0}$, by Mazur–Ulam theorem we obtain that Φ is linear. Thus, for any $A, B \in B_s(H)$ we have

$$\Phi(\overline{A} + \overline{B}) = \Phi(\overline{A}) + \Phi(\overline{B}),$$

that is

$$\overline{\phi(A + B)} = \overline{\phi(A) + \phi(B)}.$$

This gives us that $\phi(A + B) - (\phi(A) + \phi(B))$ is a scalar operator, say

$$\phi(A + B) - (\phi(A) + \phi(B)) = \lambda I.$$

We compute

$$\phi(A + B) - (\phi(A) + \phi(B)) = \lambda I = l(\lambda I)I = l(\phi(A + B) - (\phi(A) + \phi(B)))I.$$

This implies that

$$\phi(A+B) - l(\phi(A+B))I = \phi(A) - l(\phi(A))I + \phi(B) - l(\phi(B))I.$$

Adding $l(A+B)I = l(A)I + l(B)I$ to this equality, we obtain the additivity of ϕ_1. The homogeneity can be proved in a similar way.

We next show that ϕ_1 is injective. Suppose that

$$0 = \phi_1(A) = \phi(A) - l(\phi(A))I + l(A)I$$

holds for some $A \in B_s(H)$. Then $\phi(A)$ is a scalar operator, say $\phi(A) = \lambda I$, and this implies that A is also scalar, say $A = \mu I$. It follows from the above equation that

$$0 = \lambda I - l(\lambda I)I + l(\mu I)I = (\lambda - \lambda + \mu)I$$

which yields $\mu = 0$, i.e., we have $A = 0$. This proves the injectivity of ϕ_1.

Finally, we prove that ϕ_1 is surjective. To show this, first observe that, by the definition of ϕ_1 and the surjectivity of ϕ, the range of ϕ_1 and $\mathbb{R}I$ generate the whole space $B_s(H)$. So, if ϕ_1 is not surjective, then we have $rng\,\phi_1 \cap \mathbb{R}I = \{0\}$. This means that the only scalar operator in the range of ϕ_1 is 0. Now, as $\phi(I)$ is a scalar operator, it follows that $\phi_1(I)$ is also scalar. As $\phi_1(I) \in rng\,\phi_1$, we obtain that $\phi_1(I) = 0$, which, by the injectivity of ϕ_1 implies that $I = 0$, a contradiction. Therefore, ϕ_1 must be surjective. So, we have proved all the asserted properties of ϕ_1 and hence the proof of the theorem is complete. □

3. An open problem

To conclude the paper we give another interpretation of our main result Theorem 2. Namely, in view of Lemma 1, our theorem describes the form of all bijective linear transformations of $B_s(H)$ which preserve the diameter of the spectrum. This result is in a close connection with the result of our paper [10] where we have determined all the linear bijections of $C(X)$ (the algebra of all continuous complex valued functions on the first countable compact Hausdorff space X) which preserve the diameter of the range of functions. In fact, in $C(X)$ the spectrum of an element f is exactly its range. As the result in [10] seems to attract considerable interest among some researchers in the field of function algebras, and there is so much interest in preserver problems on operator algebras which concern the spectrum, we would like to pose the following open problem.

Problem. Determine all the bijective linear transformations on $B(H)$, the algebra of all bounded linear operators on the Hilbert space H, which preserve the diameter of the spectrum.

Observe that our result Theorem 2 solves the corresponding problem for $B_s(H)$. Regarding the mentioned facts, we believe that this is a prosperous and quite deep problem which deserves some attention.

Acknowledgments

The authors are very grateful to the referee for his/her important observations and remarks that helped to improve the presentation of the paper.

This paper was written when L.M. held a Humboldt Research Fellowship. He is very grateful to the Alexander von Humboldt Foundation for providing ideal conditions for research and also to his host Werner Timmermann (TU Dresden, Germany) for very warm hospitality. L.M. also acknowledges support from the Hungarian National Foundation for Scientific Research (OTKA), Grant T030082, T031995, and from the Ministry of Education, Hungary, Grant FKFP 0349/2000.

References

[1] M. Barczy, M. Tóth, Local automorphisms of the sets of states and effect on a Hilbert space, Rep. Math. Phys. 48 (2001) 289–298.
[2] R. Bhatia, Matrix Analysis, Springer, 1997.
[3] M. Brešar, P. Šemrl, Mappings which preserve idempotents, local automorphisms and local derivations, Canad. J. Math. 45 (1993) 483–496.
[4] M. Brešar, P. Šemrl, Normal-preserving linear mappings, Canad. Math. Bull. 37 (1994) 306–309.
[5] M. Brešar, P. Šemrl, Linear maps preserving the spectral radius, J. Funct. Anal. 142 (1996) 360–368.
[6] M. Brešar, P. Šemrl, Linear preservers on $B(X)$, Banach Cent. Publ. 38 (1997) 49–58.
[7] G. Cassinelli, E. De Vito, P. Lahti, A. Levrero, Symmetry groups in quantum mechanics and the theorem of Wigner on the symmetry transformations, Rev. Math. Phys. 8 (1997) 921–941.
[8] T. Dang, Y. Friedman, B. Russo, Affine geometric proofs of the Banach Stone theorems of Kadison and Kaup, Rocky Mountain J. Math. 20 (1990) 409–428.
[9] C. Davis, Separation of two linear subspaces, Acta Sci. Math. (Szeged) 19 (1958) 172–187.
[10] M. Győry, L. Molnár, Diameter preserving linear bijections of $C(X)$, Arch. Math. 71 (1998) 301–310.
[11] P.R. Halmos, Two subspaces, Trans. Amer. Math. Soc. 144 (1969) 381–389.
[12] R.V. Kadison, Isometries of operator algebras, Ann. Math. 54 (1951) 325–338.
[13] C.K. Li, S. Pierce, Linear preserver problems, Amer. Math. Monthly 108 (2001) 591–605.
[14] C.K. Li, N.K. Tsing, Linear preserver problems: a brief introduction and some special techniques, Linear Algebra Appl. 162–164 (1992) 217–235.
[15] S. Mazur, S.M. Ulam, Sur les transformations isemétriques des espaces vectoriels normés, C.R. Acad. Sci. Paris 194 (1932) 946–948.
[16] L. Molnár, Local automorphisms of some quantum mechanical structures, Lett. Math. Phys. 58 (2001) 91–100.
[17] L. Molnár, Characterizations of the automorphisms of Hilbert space effect algebras, Comm. Math. Phys. 223 (2001) 437–450.
[18] L. Molnár, Order-automorphisms of the set of bounded observables, J. Math. Phys. 42 (2001) 5904–5909.
[19] L. Molnár, Conditionally multiplicative maps on the set of all bounded observables preserving compatibility, Linear Algebra Appl. 349 (2002) 197–201.
[20] M. Omladič, On operators preserving commutativity, J. Funct. Anal. 66 (1986) 105–122.