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On the Lattice of $\mathfrak{F}$-Subnormal Subgroups*

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1. INTRODUCTION

All groups considered are finite.

It is well known that the set of all subnormal subgroups of a group G is a lattice. Now, assume that $\mathfrak{F}$ is a subgroup-closed saturated formation containing the class of all nilpotent groups. It is known that the intersection of two $\mathfrak{F}$-subnormal subgroups of a soluble group G is an $\mathfrak{F}$-subnormal subgroup of G (cf. [6, 5, 2]). One might wonder if the set of all $\mathfrak{F}$-subnormal subgroups of a soluble group is a lattice. The answer is negative in general (see [2]), but there exist subgroup-closed saturated formations containing properly the class of all nilpotent groups for which the lattice property holds.

In this paper, we obtain the exact description of the subgroup-closed saturated formations $\mathfrak{F}$ of soluble groups such that the set of all $\mathfrak{F}$-subnormal subgroups is a lattice for every soluble group.

2. PRELIMINARIES

In this section we collect some definitions and notations as well as some known results.

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First recall that if $\mathfrak{F}$ is a saturated formation and G is a group, a maximal subgroup M of G is said to be $\mathfrak{F}$-normal in G if the primitive group $G/M_G \in \mathfrak{F}$, and $\mathfrak{F}$-abnormal otherwise ($M_G = \bigcap \{M^g \mid g \in G\}$). A subgroup H of a group G is called $\mathfrak{F}$-subnormal in G if either $H = G$ or there exists a chain $H = H_n < H_{n-1} < \cdots < H_0 = G$ such that H_{i+1} is a maximal $\mathfrak{F}$-normal subgroup of H_i, for every $i = 0, ..., n-1$.

If M is a maximal subgroup of a group G such that $M_G = 1$, we will say that M is a core-free maximal subgroup of G, and if $\mathfrak{X}$ is a class of groups we denote char $\mathfrak{X} = \{p \in \mathbb{P}/C_p \in \mathfrak{X}\}$, where C_p denotes the cyclic group of order p. Recall that the boundary $b(\mathfrak{X})$ of a class of groups $\mathfrak{X}$ consists of all groups G satisfying $G \notin \mathfrak{X}$ and $G/N \in \mathfrak{X}$ for all $1 \neq N \lhd G$.

If π is a set of prime numbers, let $\mathfrak{S}$ and $\mathfrak{S}_\pi$ denote the classes of soluble and soluble π-groups, respectively. $\mathfrak{N}$ denotes the class of all nilpotent groups.

The following results will turn out to be crucial in the proof of our main result.

(2.1) LEMMA [8, Theorem 1.3.11]. *If H is a subnormal subgroup of a finite group G, then* Soc(G) *normalizes H.*

(2.2) LEMMA [2]. *Let $G \in \mathfrak{N}\mathfrak{F}$, where $\mathfrak{F}$ is a saturated formation, and let E be an $\mathfrak{F}$-maximal subgroup of G satisfying $G = EF(G)$. Then E is an $\mathfrak{F}$-normalizer of G.*

(2.3) LEMMA [3, 1]. *Let G be a group and let $\mathfrak{F}$ be a saturated formation. If $G^{\mathfrak{F}}$ is abelian, then $G^{\mathfrak{F}}$ is complemented in G and any two complements in G of $G^{\mathfrak{F}}$ are conjugate. The complements are the $\mathfrak{F}$-normalizers of G.*

(2.4) LEMMA [4, Hilfssatz 1.3]. *Let H be a group with a unique minimal normal subgroup M, where M is a q-group. If p is a prime distinct from q then H has a faithful irreducible representation over $GF(p)$.*

(2.5) LEMMA [7, Lemma 1.1]. *Let $\mathfrak{F}$ be a subgroup-closed saturated formation. If H is $\mathfrak{F}$-subnormal in G and $H \leqslant U \leqslant G$, then H is $\mathfrak{F}$-subnormal in U.*

For details about formations the reader is referred to [4].

3. THE LATTICE OF $\mathfrak{F}$-SUBNORMAL SUBGROUPS

(3.1) LEMMA. *Let G be a group and let H be an $\mathfrak{F}$-subnormal subgroup of G, where $\mathfrak{F}$ is a subgroup-closed saturated formation. Then $H^{\mathfrak{F}}$ is subnormal in G.*

Proof. We argue by induction on the order of G. Let N be a minimal normal subgroup of G. Then HN/N is $\mathfrak{F}$-subnormal in G, so that $H^{\mathfrak{F}}N$ is subnormal in G. If HN is a proper subgroup of G, then $H^{\mathfrak{F}}$ is subnormal in HN by Lemma 2.5. Therefore $H^{\mathfrak{F}}$ is subnormal in G and the lemma is true. So we can assume $G = HN$ for each minimal normal subgroup N of G. This implies that H is contained in a core-free $\mathfrak{F}$-normal maximal subgroup of G. But then G is an $\mathfrak{F}$-group and $H^{\mathfrak{F}} = 1$ is subnormal in G.

(3.2) LEMMA. *Let $\mathfrak{C} = \{\pi_i : i \in I\}$ be a partition of π, a set of prime numbers, and let $\mathfrak{F}$ be the saturated formation of soluble groups locally defined by the formation function f given by $f(p) = \mathfrak{S}_{\pi_i}$, if $p \in \pi_i$ and $i \in I$, and $f(q) = \emptyset$, if $q \notin \pi$. Then G is an $\mathfrak{F}$-group if and only if G is a soluble π-group with a normal Hall π_i-subgroup, for every $i \in I$.*

Proof. Assume that $G \in \mathfrak{F}$. We see that G has a normal Hall π_i-subgroup for each $i \in I$ by induction on $|G|$. Let N be a minimal normal subgroup of G and let $p \in \pi_i$ the prime divisor of $|N|$. If H is a Hall π_i-subgroup of G, then $N \leqslant H$ and H/N is a Hall π_i-subgroup of G/N. By induction we deduce that H is a normal subgroup of G. Now, let A be a Hall π_j-subgroup of G with $j \neq i$. Then $A \cap N = 1$ and since AN/N is a Hall π_j-subgroup of G/N, we have that $AN \lhd G$. If $AN < G$, then $A \lhd AN$ and $A \lhd G$. Therefore we can assume that $G = AN$ and A is a maximal subgroup of G. Since $G \in \mathfrak{F}$, we have that $G/A_G \in \mathfrak{S}_{\pi_i}$, but then $A = A_G$ because $A/A_G \in \mathfrak{S}_{\pi_j}$, that is, A is a normal subgroup of G.

The converse is clear since the chief factors of a such group G are $\mathfrak{F}$-central.

(3.3) THEOREM. *Let $\mathfrak{F}$ be a subgroup-closed saturated formation of soluble groups containing $\mathfrak{N}$, the class of all nilpotent groups, and let f be the full and integrated local formation function defining $\mathfrak{F}$. Then $\mathfrak{F}$ satisfies the following condition:*

(∗) *"If H_1 and H_2 are two $\mathfrak{F}$-subnormal subgroups of $G \in \mathfrak{S}$, then $\langle H_1, H_2 \rangle$ is an $\mathfrak{F}$-subnormal subgroup of G", if and only if f can be described in the following way: "There exists a partition $\{\pi_i\}_{i \in I}$ of $\mathbb{P}$, the set of all prime numbers, such that $f(p) = \mathfrak{S}_{\pi_i}$, for every prime number $p \in \pi_i$ and for every $i \in I$".*

Proof. Assume that the formation $\mathfrak{F} = LF(f)$ as above satisfies (∗). It is well known that $f(p)$ is a subgroup-closed formation for every prime p (cf. [4, Hilfssatz 2.2]). We split the first part of the proof into the following steps:

(1) For each prime number $p \in \mathbb{P}$, every primitive group $G \in \mathfrak{F} \cap (b(f(p))$ is cyclic.

It is clear that G has a unique minimal normal subgroup N, and evidently N must be a q-group, where $p \neq q \in \mathbb{P}$. Therefore there exists an irreducible and faithful G-module V_p over $GF(p)$. We claim that G has a unique maximal subgroup M such that $M_G = 1$, which provides the result.

Assume that M_1 and M_2 are maximal subgroups of G, $M_1 \neq M_2$ and $(M_i)_G = 1$, $i = 1, 2$. Then $M_i \in f(p)$. Consider now the semidirect product $H = [V_p]G$, with respect to the action of G on V_p. Clearly $H \notin \mathfrak{F}$, so $H^{\mathfrak{F}} = V_p$ and G is not $\mathfrak{F}$-subnormal in H. But for $i = 1, 2$, $V_p M_i$ is $\mathfrak{F}$-normal maximal subgroup of H, and M_i is $\mathfrak{F}$-subnormal in $V_p M_i$, because $V_p M_i \in \mathfrak{S}_p f(p) = f(p) \subseteq \mathfrak{F}$, that is, M_i is $\mathfrak{F}$-subnormal in H. Since $\mathfrak{F}$ satisfies $(*)$, we have that $G = \langle M_1, M_2 \rangle$ is $\mathfrak{F}$-subnormal in H, which is a contradiction.

(2) If p and q are prime numbers and $q \in \mathrm{char}(f(p))$, then $p \in \mathrm{char}(f(q))$.

Assume that $C_p \notin f(q)$ and consider an irreducible and faithful C_q-module V_p over $GF(p)$. Then the semidirect product $[V_p]C_q$, with respect to the action of C_q on V_p, belongs to $\mathfrak{F} \cap b(f(q))$, which contradicts (1).

(3) If p and q are prime numbers and $p \in \mathrm{char}(f(q))$, then $\mathrm{char}(f(p)) = \mathrm{char}(f(q))$.

If $r \in \mathbb{P}$ and $r \in \mathrm{char}(f(q)) \backslash \mathrm{char}(f(p))$, then $r \neq q$ and $C_q \in f(r)$, because of (2). Consider now an irreducible and faithful C_q-module V_r over $GF(r)$. Then $[V_r]C_q \in \mathfrak{F} \cap b(f(p))$, a contradiction with (1).

(4) If $p, q \in \mathbb{P}$ and $p \in \mathrm{char}(f(q))$, then $\mathfrak{S}_p \subseteq f(q)$.

Since $f(q)$ is subgroup-closed, and a p-group of order p^n is isomorphic with a subgroup of the n-fold iterated wreath product $(\cdots(C_p \sim_{\mathrm{reg}} C_p)\cdots) \sim_{\mathrm{reg}} C_p = H_n$, it is enough to prove that $H_n \in f(q)$, $\forall n \in \mathbb{N}$.

Denote inductively $H_1 = C_p$ and $H_n = H_{n-1} \sim_{\mathrm{reg}} C_p$, for $n \geqslant 2$, and assume inductively that $H_{n-1} \in f(q)$. Since $Z(H_n)$ is cyclic, H_n has a unique minimal normal subgroup, and consequently there exists an irreducible and faithful H_n-module V_q over $GF(q)$. Consider the semidirect product $G = [V_q]H_n$, with respect to the action of H_n on V_q. If $(H_{n-1})^{\#}$ denotes the base group of H_n, then $H_n = (H_{n-1})^{\#} C_p$. Since $(H_{n-1})^{\#}$ and C_p are $f(q)$-groups, we have that $V_q(H_{n-1})^{\#}$ and $V_q C_p$ are $f(q)$-groups. So $(H_{n-1})^{\#}$ and C_p are $\mathfrak{F}$-subnormal in G. Consequently $G \in \mathfrak{F}$ and then $H_n \in f(q)$.

(5) If $p, q \in \mathbb{P}$ and $p \in \mathrm{char}(f(q))$, then $\mathfrak{S}_p f(q) = f(q)$.

Assume that G is a group of minimal order in $\mathfrak{S}_p f(q) \backslash f(q)$. Then G has a unique minimal normal subgroup N, $G/N \in f(q)$ and N is a p-group. If

$G \in \mathfrak{F}$, we may argue as in (1) to obtain that G has a unique maximal subgroup, that is, G is cyclic, and consequently $G \in f(q)$, a contradiction. Therefore $G \notin \mathfrak{F}$, in particular $N \not\leq \phi(G)$, and so there exists a maximal subgroup R of G such that $G = NR$, $R \in f(q)$ and $G^{\mathfrak{F}} = N$. Now R must be again a cyclic r-group, with $p \neq r \in \mathbb{P}$. Finally, from (3) and (4), we have that $G \in \mathfrak{S}_p \mathfrak{S}_r \subseteq \mathfrak{S}_p f(p) = f(p) \subseteq \mathfrak{F}$, which is a contradiction.

(6) If $p \in \mathbb{P}$ and $\pi = \mathrm{char}(f(p))$, then $f(p) = \mathfrak{S}_\pi$.

Since $f(p)$ is a subgroup-closed formation, it is clear that $f(p) \subseteq \mathfrak{S}_\pi$.

On the other hand, if $f(p) \neq \mathfrak{S}_\pi$, choose a group G of minimal order in $\mathfrak{S}_\pi \backslash f(p)$, and consider a minimal normal q-subgroup N of G, $q \in \pi$. Then $G/N \in f(p)$, that is, $G \in \mathfrak{S}_q f(p) = f(p)$, by (5), a contradiction.

Conversely, let $\mathfrak{E} = \{\pi_i : i \in I\}$ be a partition of $\mathbb{P}$, the set of all prime numbers, and let $\mathfrak{F}$ be the saturated formation of soluble groups locally defined by the integrated and full formation function f given by $f(p) = \mathfrak{S}_{\pi_i}$, if $p \in \pi_i$. We see that $\mathfrak{F}$ verifies the condition $(*)$.

Suppose not and take G of minimal order among the groups X having two $\mathfrak{F}$-subnormal subgroups A and B such that $\langle A, B \rangle$ is not $\mathfrak{F}$-subnormal in X. Then there exists two $\mathfrak{F}$-subnormal subgroups H and K of G such that $T = \langle H, K \rangle$ is not $\mathfrak{F}$-subnormal in G. The group G should have the following properties:

(1) G is a primitive group and T is a core-free maximal subgroup of G.

Take N a minimal normal subgroup of G. Since HN/N and KN/N are two $\mathfrak{F}$-subnormal subgroups of G/N, then $TN/N = \langle HN/N, KN/N \rangle$ is $\mathfrak{F}$-subnormal in G/N by minimality of G. Moreover, if $TN < G$ again T is $\mathfrak{F}$-subnormal in TN. Therefore T would be $\mathfrak{F}$-subnormal in G, a contradiction. So $TN = G$ for every minimal normal subgroup N of G. But then T is a core-free maximal subgroup of G, and G is a primitive group.

Suppose that p is the prime dividing the order of $N = \mathrm{Soc}(G)$. Let $i \in I$ such that $p \in \pi_i$.

(2) If L is an $\mathfrak{F}$-subnormal subgroup of G contained in T, then L is a π_i-group. In particular, H and K are π_i-groups.

By Lemma 3.1, $L^{\mathfrak{F}}$ is a subnormal subgroup of G and so Lemma 2.1 implies that $N \leqslant N_G(L^{\mathfrak{F}})$. On the other hand $N \cap L^{\mathfrak{F}} = 1$, since L is contained in T. So $L^{\mathfrak{F}} \leqslant C_G(N) = N$ and L is an $\mathfrak{F}$-group. Let N_0 be a minimal L-invariant subgroup of N. If LN_0 were not an $\mathfrak{F}$-group, then N_0 would be the $\mathfrak{F}$-residual of LN_0, but this is impossible because L is $\mathfrak{F}$-subnormal in LN_0. Therefore if $\mathrm{Soc}_L(N)$ denotes the product of all minimal L-invariant subgroups of N, we have that $L\,\mathrm{Soc}_L(N)$ is an $\mathfrak{F}$-group. On the other hand, if LN were not an $\mathfrak{F}$-group there would exist an $\mathfrak{F}$-maximal subgroup F

of LN containing $L\,\mathrm{Soc}_L(N)$. By Lemmas 2.2 and 2.3 we have that $LN = F(LN)^{\mathfrak{F}}$ and $F \cap (LN)^{\mathfrak{F}} = 1$. But $1 \neq (LN)^{\mathfrak{F}} \cap \mathrm{Soc}_L(N) \leqslant F \cap (LN)^{\mathfrak{F}}$, a contradiction. Therefore LN must be an $\mathfrak{F}$-group.

Now if $j \in I$ and $i \neq j$, we have that $O_{\pi_j}(L) \leqslant C_G(N) = N$, so $O_{\pi_j}(L) = 1$ and L is a π_i-group, by Lemma 3.2.

Among the pairs (A, B) of $\mathfrak{F}$-subnormal subgroups of G such that $\langle A, B \rangle$ is not $\mathfrak{F}$-subnormal in G, we take a pair (H, K) with $|H| + |K|$ maximal. Suppose $|K| \leqslant |H|$. Then:

(3) $\langle H, H^x \rangle$ is $\mathfrak{F}$-subnormal in G, for every $x \in G$.

Assume the result is not true. Then by the choice of (H, K), we have $|H| = |K|$. Among the elements g of G such that $\langle H, H^g \rangle$ is not $\mathfrak{F}$-subnormal in G, take $x \in G$ with $\langle H, H^x \rangle$ of minimal order. It is clear that $G = \langle H, H^x \rangle N$ and $R = \langle H, H^x \rangle$ is a core-free maximal subgroup of G. If $G = \langle H, x \rangle$, then $G = R\langle H, x \rangle^{\mathfrak{F}}$ and $x = tr$ with $t \in R$ and $r \in \langle H, x \rangle^{\mathfrak{F}}$. On the other hand, $G = \langle H, H^x, x \rangle = \langle H, H^t, r \rangle$. If $\langle H, H^t \rangle$ is a proper subgroup of R, then $\langle H, H^t \rangle$ is $\mathfrak{F}$-subnormal in G by the choice of R, and since $r \in \langle H, H^t, r \rangle^{\mathfrak{F}}$, we have $r \in \langle H, H^t \rangle$ by [2]. Therefore $x \in R$, a contradiction. Thus, $R = \langle H, H^t \rangle$. Consequently, without loss of generality, we can assume that $x \in R$.

For this subgroup R we have:

(a) $R^{\mathfrak{F}} \leqslant N_G(H)$.

Suppose that $R^{\mathfrak{F}}$ is not contained in $N_G(H)$ and let z be an element of $R^{\mathfrak{F}} \backslash N_G(H)$. Then H is a proper subgroup of $\langle H, H^z \rangle$ and there exists an $\mathfrak{F}$-normal maximal subgroup M of G such that $\langle H, H^z \rangle \leqslant M$. Then $\langle H, H^z \rangle$ is $\mathfrak{F}$-subnormal in M by the choice of G, so that $\langle H, H^z \rangle$ is $\mathfrak{F}$-subnormal in G and, by the choice of (H, K), we have that $R = \langle H, H^x, H^z \rangle$ is an $\mathfrak{F}$-subnormal subgroup of G, a contradiction.

(b) Every maximal subgroup M of R containing H is $\mathfrak{F}$-normal in R.

Let M be a maximal subgroup of R such that $H \leqslant M$. If M were not $\mathfrak{F}$-normal in R, M would be a supplement of $R^{\mathfrak{F}}$ in R. Applying (a), the normal closure H^R of H in R must be contained in M but this is impossible because $H^R = R$. Thus M is $\mathfrak{F}$-normal in R.

(c) R is a π_i-group.

Since H is a π_i-group, there exists a Hall π_i-subgroup A of R containing H. Then AN is a Hall π_i-subgroup of G. If A were a proper subgroup of R, there would exist a maximal subgroup M of R such that $H \leqslant A \leqslant M$. By (b), M is $\mathfrak{F}$-normal in R. So, $L = MN$ is an $\mathfrak{F}$-normal maximal subgroup of G containing AN. Assume $|G : L| = q^a$, q a prime number, and let $j \in I$ such that $q \in \pi_j$. Since L is $\mathfrak{F}$-normal in G, we have that G/L_G is a π_j-group.

On the other hand, $i \neq j$ because $|G : AN|$ is a π_i'-number. Therefore, R is contained in L_G and $L = G$, a contradiction.

Now, by (c), G is a π_i-group. Since $\mathfrak{S}_{\pi_i}$ is contained in $\mathfrak{F}$, we have that G is an $\mathfrak{F}$-group, a contradiction. Therefore (3) is true.

By the choice of (H, K) and applying (3), it is rather easy to see that $K \leqslant N_G(H)$. Then, $T = HK$ is a π_i-group by (2). This implies that G is a π_i-group, which provides the final contradiction.

(3.4) *Remark.* It becomes clear from our proof of the first part of the above theorem that the following statement holds:

> If $\mathfrak{F}$ is a subgroup-closed saturated formation of soluble groups, non-necessarily containing the class of all nillpotent groups, satisfying the condition $(*)$, then there exists a family of pairwise disjoint sets of primes $\{\pi_i : i \in I\}$ such that $\mathfrak{F}$ is locally defined by the integrated and full formation function f given by
>
> $$f(p) = \mathfrak{S}_{\pi_i}, \qquad \text{if} \quad p \in \pi_i,\ i \in I,$$
>
> and
>
> $$f(q) = \varnothing, \qquad \text{for each} \quad q \notin U\{\pi_i : i \in I\}.$$

Notice that in this case $\mathfrak{F}$ has not full characteristic.

Next we see that indeed the hypothesis of $\mathfrak{N} \subseteq \mathfrak{F}$, where $\mathfrak{N}$ is the class of all nilpotent groups, is unnecessary in the above theorem.

(3.5) THEOREM. *Let $\mathfrak{F}$ be a subgroup-closed saturated formation of soluble groups and let f be the full and integrated formation function defining $\mathfrak{F}$. Denote $\pi :=$ char $\mathfrak{F}$. Then the set of all $\mathfrak{F}$-subnormal subgroups is a lattice for every soluble group if and only if f can be described in the following way*:

> *"There exists a partition $\{\pi_i : i \in I\}$ of π such that $f(p) = \mathfrak{S}_{\pi_i}$, for every prime number $p \in \pi_i$ and for every $i \in I$, and $f(q) = \varnothing$, for every $q \notin \pi$."*

Proof. If the set of all $\mathfrak{F}$-subnormal subgroups is a lattice for every soluble group, then $\mathfrak{F}$ satisfies the condition $(*)$ in Theorem 3.3. By the above remark f can be described in the mentioned way.

Conversely, let $\{\pi_i : i \in I\}$ be a partition of π and let $\mathfrak{F}$ be the saturated formation of soluble groups locally defined by the integrated and full formation function f given by $f(p) = \mathfrak{S}_{\pi_i}$, for every prime number $p \in \pi_i$ and for every $i \in I$, and $f(q) = \varnothing$, for every $q \notin \pi$.We see that the set of all $\mathfrak{F}$-subnormal subgroups is a lattice for every soluble group.

Let $\mathfrak{H}$ be the saturated formation of soluble groups locally defined by the formation function h given by $h(p) = f(p)$, for every $p \in \pi$, and $h(q) = S_{\pi'}$, for each $q \notin \pi$.

It is clear that $\mathfrak{H}$ is a subgroup-closed saturated formation of soluble groups containing the class of all nilpotent groups. Applying Theorem 3.3 the set of all $\mathfrak{H}$-subnormal subgroups is a lattice for every soluble group.

Now consider a soluble group G and two $\mathfrak{F}$-subnormal subgroups H and K of G. It is clear that H and K are $\mathfrak{H}$-subnormal subgroups of G. Therefore $T = \langle H, K \rangle$ is $\mathfrak{H}$-subnormal in G. That is, if $T < G$, there exists a chain $T = T_0 < T_1 < \cdots < T_n = G$ of subgroups of G such that T_i is $\mathfrak{H}$-normal maximal subgroup of T_{i+1}, for $0 \leqslant i < n$. On the other hand, T has π-index in G. Therefore T_i has π-index in T_{i+1}, and then T_i is $\mathfrak{F}$-normal in T_{i+1}, for $0 \leqslant i < n$. This means that T is $\mathfrak{F}$-subnormal in G.

Next we see that $H \cap K$ is $\mathfrak{F}$-subnormal in G by induction on $|G|$. We distinguish two cases:

(1) H is a maximal subgroup of G. Since G/H_G is a π-group, we have that $H \cap K$ has π-index in K. Therefore $H \cap K$ has π-index in G because K is $\mathfrak{F}$-subnormal in G. Arguing as above, taking into account that $H \cap K$ is $\mathfrak{H}$-subnormal in G, we conclude that $H \cap K$ is $\mathfrak{F}$-subnormal in G.

(2) The general case. If $H < G$, let M be an $\mathfrak{F}$-normal maximal subgroup of G such that $H \leqslant M$. Applying case (1), we have that $M \cap K$ is $\mathfrak{F}$-subnormal in G. By induction and Lemma 2.5, $H \cap K$ is $\mathfrak{F}$-subnormal in M. Therefore $H \cap K$ is $\mathfrak{F}$-subnormal in G.

Consequently, if H and K are $\mathfrak{F}$-subnormal subgroups of G then $H \cap K$ and $\langle H, K \rangle$ are $\mathfrak{F}$-subnormal subgroups of G, i.e., the set of all $\mathfrak{F}$-subnormal subgroups of G is a lattice.

4. Some Applications

Let $\{\pi_i : i \in I\}$ be a family of pairwise disjoint sets of primes and put $\pi = \dot{\bigcup} \{\pi_i : i \in I\}$. In the sequel $\mathfrak{F}$ *denotes the saturated Fitting formation of soluble groups locally defined by the integrated and full formation function f given by*: $f(p) = \mathfrak{S}_{\pi_i}$, *if* $p \in \pi_i$ *and* $i \in I$, *and* $f(q) = \emptyset$, *if* $q \notin \pi$. It is clear that $\pi = \operatorname{char} \mathfrak{F}$.

(4.1) Theorem. *If H and K are two $\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroups of a soluble group G, then $\langle H, K \rangle \in \mathfrak{F}$. Consequently, if $G \in \mathfrak{S}_\pi$, the $\mathfrak{F}$-radical $G_\mathfrak{F}$ of G has the form*

$$G_\mathfrak{F} = \langle X \in \mathfrak{F} / X \text{ is } \mathfrak{F}\text{-subnormal in } G \rangle.$$

Proof. The second statement is a mere consequence of the first one.

Assume that the first statement is not true and take G of minimal order among the groups X having two $\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroups A and B such that $\langle A, B\rangle$ is not an $\mathfrak{F}$-group. Among the pairs (A, B) of $\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroups of G such that $\langle A, B\rangle$ is not an $\mathfrak{F}$-group, we choose a pair (H, K) with $|H| + |K|$ maximal.

Because of Lemma 2.5 and the choice of G, it must be $G = \langle H, K\rangle$. Since $\mathfrak{F}$ is a Fitting class, and since H and K are $\mathfrak{F}$-groups, we may assume that $N_G(H) < G$. Take $x \in G \backslash N_G(H)$. If $\langle H, H^x\rangle < G$, then $\langle H, H^x\rangle \in \mathfrak{F}$ by the minimal choice of G and Lemma 2.5. Moreover Theorem (3.5) implies that $\langle H, H^x\rangle$ is $\mathfrak{F}$-subnormal in G. Consequently, since $H < \langle H, H^x\rangle$ and because of the choice of (H, K), it follows that $G = \langle H, H^x, K\rangle \in \mathfrak{F}$, a contradiction.Therefore, $\langle H, H^x\rangle = G$. In particular, we may also deduce that $N_G(H)$ is the unique maximal subgroup of G containing H. Since H is $\mathfrak{F}$-subnormal in G, $N_G(H)$ is $\mathfrak{F}$-normal in G and then $G^{\mathfrak{F}} \leqslant N_G(H)$.

Again from the choice of G, it is clear that G is in the boundary of the saturated formation $\mathfrak{F}$. Consequently G must be a primitive group and if N denotes the socle of G, we have that $N = G^{\mathfrak{F}}$.

Let p be the prime dividing the order of N and let $i \in I$ such that $p \in \pi_i$. (Note that G is a π-group because H is an $\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroup of G.) Now, if $j \in I$ and $j \neq i$, we have that $O_{\pi_j}(H) \leqslant C_G(N) \leqslant N$. So $O_{\pi_j}(H) = 1$ and H is a π_i-group.

If we assume that G is not a π_i-group, then there exists a Hall π_i-subgroup A of G and a maximal subgroup L of G such that $H \leqslant A \leqslant L$. Suppose that $|G : L|$ is a q-power, q a prime number, and let $j \in I$ such that $q \in \pi_j$. Since L is $\mathfrak{F}$-normal in G (note that $L = N_G(H)$), we have that G/L_G is a π_j-group and $j \neq i$ because $|G : L|$ is a π_i'-number. Therefore $A \leqslant L_G$ and $G = \langle H, H^x\rangle \leqslant L_G$, a contradiction. Consequently G is π_i-group, in particular G is an $\mathfrak{F}$-group, which provides the final contradiction.

Nothing can be said about the relation between the $\mathfrak{F}$-radical and the $\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroups of an arbitrary soluble group. For instance, if $\mathfrak{F} = \mathfrak{S}_p$, one can find soluble groups G satisfying $1 = \langle X \in \mathfrak{F} / X$ is $\mathfrak{F}$-subnormal in $G\rangle < O_p(G) < G = O^p(G)$.

It is well known the Baer's characterization of the p-radical of a group, that is, a p-element x of a group G lies in $O_p(G)$ if, and only if, any two conjugates of x generate a p-subgroup of G. It is rather easy to derive from this result that a subgroup H of a group G is contained in $F(G)$ if, and only if, $\langle H, H^g\rangle$ is a nilpotent group, for every $g \in G$. As a consequence of Theorem 4.1 we see once more that our $\mathfrak{F}$ has an analogous behaviour to the class of nilpotent groups as the next theorem shows:

(4.2) Theorem. *For a subgroup H of a group $G \in \mathfrak{S}_\pi$, the following statements are equivalent*:

(i) *H is contained in the $\mathfrak{F}$-radical $G_{\mathfrak{F}}$ of G;*

(ii) *$\langle H, H^g \rangle$ is an $\mathfrak{F}$-group for every $g \in G$.*

Proof. (i) implies (ii). If $H \leqslant G_{\mathfrak{F}}$, then $\langle H, H^g \rangle$ is contained in the $\mathfrak{F}$-group $G_{\mathfrak{F}}$ and so $\langle H, H^g \rangle$ is an $\mathfrak{F}$-group for every $g \in G$.

(ii) implies (i). Evidently H is $\mathfrak{F}$-subnormal in $\langle H, H^g \rangle$ for every $g \in G$, and arguing as in [2, Th. 1, (3) $\rightarrow$ (1)] we deduce that H is $\mathfrak{F}$-subnormal in G. Since H is an $\mathfrak{F}$-group it follows that $H \leqslant G_{\mathfrak{F}}$ by Theorem 4.1.

(4.3) *Remark.* The above theorem does not hold for arbitrary subgroup-closed saturated Fitting formations of soluble groups. Take, for instance, $\mathfrak{F} = \mathfrak{N}^2$ the class of all groups with nilpotent length at most 2 and $G = \mathrm{Sym}(4)$. If H is a subgroup of G generated by a transposition, then $\langle H, H^g \rangle \in \mathfrak{F}$, for every $g \in G$, but H is not contained in $\mathrm{Alt}(4) = G_{\mathfrak{N}^2}$.

F. P. Lockett has studied in [9] the $\mathfrak{F}$-injectors of soluble π-groups, when $\mathfrak{F}$ is our Fitting formation and $\pi = \mathrm{char}\, \mathfrak{F}$. Exactly, he has obtained the following result:

(4.4) THEOREM [9, Th. 2.1.1]. *If $G \in \mathfrak{S}_\pi$, then the $\mathfrak{F}$-injectors of G are exactly the subgroups $X_{i \in I} V_{\pi_i}$, where $V_{\pi_i} \in \mathrm{Hall}_{\pi_i}(C_G(O_{\pi_i'}(F(G))))$.*

We ask ourselves whether the $\mathfrak{F}$-injectors obtained by Lockett have a good behaviour with respect to $\mathfrak{F}$-subnormal subgroups. The answer is given in the following theorem:

(4.5) THEOREM. *If $G \in \mathfrak{S}_\pi$ and V is an $\mathfrak{F}$-injector of G and H is an $\mathfrak{F}$-subnormal subgroup of G, then $V \cap H$ is an $\mathfrak{F}$-injector of H.*

Proof. Assume that the result is not true and let G be a counterexample of minimal order. Evidently we may suppose that H is an $\mathfrak{F}$-normal maximal subgroup of G. From Theorem 4.4 we know that $V = X_{i \in I} V_i$, where $V_i \in \mathrm{Hall}_{\pi_i}(C_G(O_{\pi_i'}(F(G))))$. Consequently, since $O_{\pi_i'}(F(G)) \leqslant X_{j \neq i} O_{\pi_j}(G)$ and since $O_{\pi_i}(G) \leqslant V_i$, we have $V_i \leqslant C_G(X_{j \neq i} O_{\pi_j}(G)) \leqslant C_G(O_{\pi_i'}(F(G)))$, that is, $V_i \in \mathrm{Hall}_{\pi_i}(C_G(X_{j \neq i} O_{\pi_j}(G))) = \mathrm{Hall}_{\pi_i}(C_G(O_{\pi_i'}(F(G))))$. Moreover there exists an $i \in I$ such that G/H_G is a π_i-group, because H is $\mathfrak{F}$-normal in G and so $O_{\pi_k}(G) = O_{\pi_k}(H) = O_{\pi_k}(H_G)$, for every $k \neq i$ and $V \cap H \in \mathrm{Hall}_{\pi_i}(C_H(x_{j \neq i} O_{\pi_j}(H)))$.

Since $O_{\pi_i}(H)$ centralizes $X_{j \neq i} O_{\pi_j}(G)$, there exists a Hall π_i-subgroup of $C_G(X_{j \neq i} O_{\pi_j}(G))$ containing $O_{\pi_i}(H)$, and now Theorem 4.4 implies that V_k centralizes $O_{\pi_i}(H)$ if $k \neq i$. Therefore we have that $V_k \leqslant C_H(X_{j \neq k} O_{\pi_j}(H))$ if $k \neq i$. Consequently, if $k \neq i$, $V_k \leqslant R_k \in \mathrm{Hall}_{\pi_k}(C_H(X_{j \neq k} O_{\pi_j}(H)))$. We show that $R_k \leqslant C_G(X_{j \neq k} O_{\pi_j}(G))$, and so $V_k = R_k$. To see that this is so, we only need to prove that R_k centralizes $O_{\pi_i}(G)$. But from $R_k \leqslant H_G$, we deduce that $[O_{\pi_i}(G), R_k] \leqslant H_G \cap O_{\pi_i}(G) \leqslant O_{\pi_i}(H)$ and hence $[O_{\pi_i}(G), R_k, R_k] = 1$. This implies that R_k is subnormal in $O_{\pi_i}(G)R_k$ and obviously $[O_{\pi_i}(G), R_k] = 1$.

On the other hand, we have that $V_i \cap H \leqslant C_H(X_{j \neq i} O_{\pi_j}(G)) = C_H(X_{j \neq i} O_{\pi_j}(H))$. Then, the result will be proved if we see that $V_i \cap H$ is here a Hall π_i-subgroup. Denote $C = C_G(X_{j \neq i} O_{\pi_j}(G))$ and let $V_i \cap H \leqslant R_i \in \mathrm{Hall}_{\pi_i}(H \cap C)$. Take $g \in C$ such that $R_i = H \cap V_i^g$. If $G = C$, then $C = G = H_G V_i$. Now $H \cap V_i^g = H \cap V_i^h = (H \cap V_i)^h$ for some $h \in H$, and clearly $H \cap V_i = R_i$. Hence we may assume that $C < G$. Since $C \lhd G$, we have that $C = C_C(X_{j \neq i} O_{\pi_j}(C))$ and $V \cap C = V_i \times (X_{k \neq i} V_k \cap C)$ is an $\mathfrak{F}$-injector of C by minimality of G. Moreover it is easy to prove that $H \cap C$ is $\mathfrak{F}$-subnormal in C, because $C^{\mathfrak{F}} \leqslant G^{\mathfrak{F}} \cap C \leqslant H \cap C$. Therefore, because of the minimal choice of G, it follows that $V \cap H \cap C = (V_i \cap H) \times (X_{k \neq i} V_k \cap C)$ is an $\mathfrak{F}$-injector of $H \cap C$. In particular, $V_i \cap H \in \mathrm{Hall}_{\pi_i}(C^*)$, where $C^* = C_{C \cap H}(X_{j \neq i} O_{\pi_j}(C \cap H))$. Analogously $V_i^g \cap H \in \mathrm{Hall}_{\pi_i}(C^*)$, because $V_i^g \times (X_{k \neq i} V_k)$ is also and $\mathfrak{F}$-injector of G. Consequently $|V_i^g \cap H| = |V_i \cap H|$ and $R_i = V_i \cap H$.

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