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# Coherence for star-autonomous categories

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**Abstract**

This paper presents a coherence theorem for star-autonomous categories exactly analogous to Kelly and Mac Lane's coherence theorem for symmetric monoidal closed categories. The proof of this theorem is based on a categorial cut-elimination result, which is presented in some detail.


*Keywords:* Symmetric monoidal closed categories; Star-autonomous categories; Cut elimination; Categorial coherence

## 1. Introduction

From the inception of proof nets in the late 1980s (see [16] and [8]), it could have been realized that they are connected with the graphs one finds in Kelly and Mac Lane's coherence theorem for symmetric monoidal closed categories of [17]. The earliest explicit reference for that we know about is [3] (see also [4]). It was also soon suggested that the multiplicative fragment of classical linear logic, which has an involutive negation that satisfies De Morgan laws, is closely related to the notion of star-autonomous category, which stems from [1] (see [18,21] and [2]).

Star-autonomous categories in the sense of [2] are symmetric monoidal closed categories that have an object $\perp$ such that the canonical natural transformation from the identity functor to the functor $(\_ \to \perp) \to \perp$ is a natural isomorphism (here $\_ \to \_$ is the internal hom-bifunctor). This notion is equivalent to the notion of symmetric linearly (alias weakly) distributive category with negation in the sense of [7] (Section 4, Definition 4.3). Establishing the equivalence of the two notions is rather arduous, as noted in [7] (Theorem 4.5; a proof may be found in [13], Chapter 3).

The aim of this paper is to present a coherence theorem for symmetric linearly distributive categories with negation, which is exactly analogous to Kelly and Mac Lane's coherence theorem for symmetric monoidal closed categories mentioned above. Like Kelly and Mac Lane's proof of [17], the proof of our coherence theorem is based on cut-elimination or similar results. We will not present all of them. Some of these results are in [12], and some in [13] and [14]. We will present in some detail only a cut-elimination theorem for symmetric linearly distributive categories with negation freely generated by a set of objects, on which our coherence theorem relies. This is a cut-elimination

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theorem that asserts not only that for every derivation we have a cut-free derivation of the same type, but also that the original derivation and the cut-free derivation are equal as arrows in a category (which is not a preorder: not all arrows of the same type are equal in this category).

As we indicated above, this paper is not self-contained. A more detailed and more self-contained investigation of star-autonomous categories and of their connection with the graphs of Kelly and Mac Lane, and with the proof nets of classical linear logic, is in the study [13].

Sections 2, 3 and 5 of this paper introduce gradually the notion of symmetric linearly distributive category with negation freely generated by a set of objects. Section 4 introduces a precise notion of graph of the kind of Kelly and Mac Lane, and states the previous coherence results on which we rely. Sections 6 and 7 contain the cut-elimination result, and Section 8 the coherence result, which we have announced.

All the categories considered in this paper are small. We have no need here for categories whose collections of objects or arrows are bigger than sets.

## 2. The category DS

The objects of the category **DS** are the formulae of the propositional language $\mathcal{L}_{\wedge,\vee}$, generated from a set $\mathcal{P}$ of propositional letters, which we call simply *letters*, with the binary connectives $\wedge$ and $\vee$. We use $p, q, r, \ldots$, sometimes with indices, for letters, and $A, B, C, \ldots$, sometimes with indices, for formulae. As usual, we omit the outermost parentheses of formulae and other expressions later on.

To define the arrows of **DS**, we define first inductively a set of expressions called the *arrow terms* of **DS**. Every arrow term of **DS** will have a *type*, which is an ordered pair of formulae of $\mathcal{L}_{\wedge,\vee}$. We write $f : A \vdash B$ when the arrow term $f$ is of type $(A, B)$. (We use the turnstile $\vdash$ instead of the more usual $\rightarrow$, which we reserve for a connective and a biendofunctor.) We use $f, g, h, \ldots$, sometimes with indices, for arrow terms.

For all formulae $A$, $B$ and $C$ of $\mathcal{L}_{\wedge,\vee}$ the following *primitive arrow terms*:

$$\mathbf{1}_A : A \vdash A,$$

$$\overset{\wedge}{b}{}^{\rightarrow}_{A,B,C} : A \wedge (B \wedge C) \vdash (A \wedge B) \wedge C, \quad \overset{\vee}{b}{}^{\rightarrow}_{A,B,C} : A \vee (B \vee C) \vdash (A \vee B) \vee C,$$

$$\overset{\wedge}{b}{}^{\leftarrow}_{A,B,C} : (A \wedge B) \wedge C \vdash A \wedge (B \wedge C), \quad \overset{\vee}{b}{}^{\leftarrow}_{A,B,C} : (A \vee B) \vee C \vdash A \vee (B \vee C),$$

$$\overset{\wedge}{c}_{A,B} : A \wedge B \vdash B \wedge A, \qquad \overset{\vee}{c}_{A,B} : B \vee A \vdash A \vee B,$$

$$d_{A,B,C} : A \wedge (B \vee C) \vdash (A \wedge B) \vee C$$

are arrow terms of **DS**. If $g : A \vdash B$ and $f : B \vdash C$ are arrow terms of **DS**, then $f \circ g : A \vdash C$ is an arrow term of **DS**; and if $f : A \vdash D$ and $g : B \vdash E$ are arrow terms of **DS**, then $f \xi g : A \xi B \vdash D \xi E$, for $\xi \in \{\wedge, \vee\}$, is an arrow term of **DS**. This concludes the definition of the arrow terms of **DS**.

Next we define inductively the set of *equations* of **DS**, which are expressions of the form $f = g$, where $f$ and $g$ are arrow terms of **DS** of the same type. We stipulate first that all instances of $f = f$ and of the following equations are equations of **DS**:

$(cat\ 1)$ $\quad f \circ \mathbf{1}_A = \mathbf{1}_B \circ f = f : A \vdash B,$

$(cat\ 2)$ $\quad h \circ (g \circ f) = (h \circ g) \circ f,$

for $\xi \in \{\wedge, \vee\}$,

$(\xi\ 1)$ $\quad \mathbf{1}_A \xi \mathbf{1}_B = \mathbf{1}_{A \xi B},$

$(\xi\ 2)$ $\quad (g_1 \circ f_1) \xi (g_2 \circ f_2) = (g_1 \xi g_2) \circ (f_1 \xi f_2),$

for $f : A \vdash D$, $g : B \vdash E$ and $h : C \vdash F$,

$(\overset{\xi}{b}{}^{\rightarrow}\ nat)$ $\quad ((f \xi g) \xi h) \circ \overset{\xi}{b}{}^{\rightarrow}_{A,B,C} = \overset{\xi}{b}{}^{\rightarrow}_{D,E,F} \circ (f \xi (g \xi h)),$

$(\overset{\wedge}{c}\ nat)$ $\quad (g \wedge f) \circ \overset{\wedge}{c}_{A,B} = \overset{\wedge}{c}_{D,E} \circ (f \wedge g),$

$(\overset{\vee}{c}\ nat)$ $\quad (g \vee f) \circ \overset{\vee}{c}_{B,A} = \overset{\vee}{c}_{E,D} \circ (f \vee g),$

$(d\ nat)\qquad ((f \wedge g) \vee h) \circ d_{A,B,C} = d_{D,E,F} \circ (f \wedge (g \vee h)),$

$(\overset{\xi}{b}\overset{\xi}{b})\quad \overset{\xi}{b}{}^{\leftarrow}_{A,B,C} \circ \overset{\xi}{b}{}^{\rightarrow}_{A,B,C} = \mathbf{1}_{A\xi(B\xi C)}, \qquad \overset{\xi}{b}{}^{\rightarrow}_{A,B,C} \circ \overset{\xi}{b}{}^{\leftarrow}_{A,B,C} = \mathbf{1}_{(A\xi B)\xi C},$

$(\overset{\xi}{b}5)\quad \overset{\xi}{b}{}^{\leftarrow}_{A,B,C\xi D} \circ \overset{\xi}{b}{}^{\leftarrow}_{A\xi B,C,D} = (\mathbf{1}_A\ \xi\ \overset{\xi}{b}{}^{\leftarrow}_{B,C,D}) \circ \overset{\xi}{b}{}^{\leftarrow}_{A,B\xi C,D} \circ (\overset{\xi}{b}{}^{\leftarrow}_{A,B,C}\ \xi\ \mathbf{1}_D),$

$(\hat{c}\hat{c})\quad \hat{c}_{B,A} \circ \hat{c}_{A,B} = \mathbf{1}_{A\wedge B},$

$(\check{c}\check{c})\quad \check{c}_{A,B} \circ \check{c}_{B,A} = \mathbf{1}_{A\vee B},$

$(\hat{b}\hat{c})\quad (\mathbf{1}_B \wedge \hat{c}_{C,A}) \circ \hat{b}^{\leftarrow}_{B,C,A} \circ \hat{c}_{A,B\wedge C} \circ \hat{b}^{\leftarrow}_{A,B,C} \circ (\hat{c}_{B,A} \wedge \mathbf{1}_C) = \hat{b}^{\leftarrow}_{B,A,C},$

$(\check{b}\check{c})\quad (\mathbf{1}_B \vee \check{c}_{A,C}) \circ \check{b}^{\leftarrow}_{B,C,A} \circ \check{c}_{B\vee C,A} \circ \check{b}^{\leftarrow}_{A,B,C} \circ (\check{c}_{A,B} \vee \mathbf{1}_C) = \check{b}^{\leftarrow}_{B,A,C},$

$(d\wedge)\quad (\hat{b}^{\leftarrow}_{A,B,C} \vee \mathbf{1}_D) \circ d_{A\wedge B,C,D} = d_{A,B\wedge C,D} \circ (\mathbf{1}_A \wedge d_{B,C,D}) \circ \hat{b}^{\leftarrow}_{A,B,C\vee D},$

$(d\vee)\quad d_{D,C,B\vee A} \circ (\mathbf{1}_D \wedge \check{b}^{\leftarrow}_{C,B,A}) = \check{b}^{\leftarrow}_{D\wedge C,B,A} \circ (d_{D,C,B} \vee \mathbf{1}_A) \circ d_{D,C\vee B,A},$

for $d^R_{C,B,A} =_{df} \check{c}_{C,B\wedge A} \circ (\hat{c}_{A,B} \vee \mathbf{1}_C) \circ d_{A,B,C} \circ (\mathbf{1}_A \wedge \check{c}_{B,C}) \circ \hat{c}_{C\vee B,A}$:

$$(C \vee B) \wedge A \vdash C \vee (B \wedge A),$$

$(d\hat{b})\quad d^R_{A\wedge B,C,D} \circ (d_{A,B,C} \wedge \mathbf{1}_D) = d_{A,B,C\wedge D} \circ (\mathbf{1}_A \wedge d^R_{B,C,D}) \circ \hat{b}^{\leftarrow}_{A,B\vee C,D},$

$(d\check{b})\quad (\mathbf{1}_D \vee d_{C,B,A}) \circ d^R_{D,C,B\vee A} = \check{b}^{\leftarrow}_{D,C\wedge B,A} \circ (d^R_{D,C,B} \vee \mathbf{1}_A) \circ d_{D\vee C,B,A}.$

The set of equations of **DS** is closed under symmetry and transitivity of equality and under the rules

$$(cong\ \xi)\quad \frac{f = f_1 \qquad g = g_1}{f\ \xi\ g = f_1\ \xi\ g_1}$$

where $\xi \in \{\circ, \wedge, \vee\}$, and if $\xi$ is $\circ$, then $f \circ g$ is defined (namely, $f$ and $g$ have appropriate, composable, types).

On the arrow terms of **DS** we impose the equations of **DS**. This means that an arrow of **DS** is an equivalence class of arrow terms of **DS** defined with respect to the smallest equivalence relation such that the equations of **DS** are satisfied (see [12], Section 2.3, for details).

The equations $(\xi\ 1)$ and $(\xi\ 2)$ say that $\wedge$ and $\vee$ are biendofunctors (i.e. 2-endofunctors in the terminology of [12], Section 2.4). Equations in the list above with "*nat*" in their names, and analogous derivable equations, will be called *naturality* equations. Such equations say that $\hat{b}^{\rightarrow}$, $\hat{b}^{\leftarrow}$, $\hat{c}$, etc. are natural transformations.

The equations $(d\wedge)$, $(d\vee)$, $(d\hat{b})$ and $(d\check{b})$ stem from [7] (Section 2.1; see [6], Section 2.1, for an announcement). The equation $(d\check{b})$ of [12] (Section 7.2) amounts with $(\check{b}\check{b})$ to the present one.

## 3. The category $\mathbf{PN}^{\neg}$

The category $\mathbf{PN}^{\neg}$ is defined as **DS** save that we make the following changes and additions. Instead of $\mathcal{L}_{\wedge,\vee}$, we have the propositional language $\mathcal{L}_{\neg,\wedge,\vee}$, which has in addition to what we have for $\mathcal{L}_{\wedge,\vee}$ the unary connective $\neg$.

To define the arrow terms of $\mathbf{PN}^{\neg}$, in the inductive definition that we had for the arrow terms of **DS** we assume in addition that for all formulae $A$ and $B$ of $\mathcal{L}_{\neg,\wedge,\vee}$ the following *primitive arrow terms*:

$\hat{\Delta}_{B,A}\colon A \vdash A \wedge (\neg B \vee B),$

$\check{\Sigma}_{B,A}\colon (B \wedge \neg B) \vee A \vdash A,$

are arrow terms of $\mathbf{PN}^{\neg}$.

To define the arrows of $\mathbf{PN}^{\neg}$, we assume in the inductive definition that we had for the equations of **DS** the following additional equations:

$(\hat{\Delta}\ nat)$ $\quad (f \wedge \mathbf{1}_{\neg B \vee B}) \circ \hat{\Delta}_{B,A} = \hat{\Delta}_{B,D} \circ f,$

$(\check{\Sigma}\ nat)$ $\quad f \circ \check{\Sigma}_{B,A} = \check{\Sigma}_{B,D} \circ (\mathbf{1}_{B \wedge \neg B} \vee f),$

$(\hat{b}\hat{\Delta})$ $\quad \hat{b}^{\leftarrow}_{A,B,\neg C \vee C} \circ \hat{\Delta}_{C,A \wedge B} = \mathbf{1}_A \wedge \hat{\Delta}_{C,B},$

$(\check{b}\check{\Sigma})$ $\quad \check{\Sigma}_{C,B \vee A} \circ \check{b}^{\leftarrow}_{C \wedge \neg C,B,A} = \check{\Sigma}_{C,B} \vee \mathbf{1}_A,$

for $\hat{\Sigma}_{B,A} =_{df} \hat{c}_{A,\neg B \vee B} \circ \hat{\Delta}_{B,A} : A \vdash (\neg B \vee B) \wedge A,$

$(d\hat{\Sigma})$ $\quad d_{\neg A \vee A,B,C} \circ \hat{\Sigma}_{A,B \vee C} = \hat{\Sigma}_{A,B} \vee \mathbf{1}_C,$

for $\check{\Delta}_{B,A} =_{df} \check{\Sigma}_{B,A} \circ \check{c}_{B \wedge \neg B,A} : A \vee (B \wedge \neg B) \vdash A,$

$(d\check{\Delta})$ $\quad \check{\Delta}_{A,C \wedge B} \circ d_{C,B,A \wedge \neg A} = \mathbf{1}_C \wedge \check{\Delta}_{A,B},$

$(\check{\Sigma}\hat{\Delta})$ $\quad \check{\Sigma}_{A,A} \circ d_{A,\neg A,A} \circ \hat{\Delta}_{A,A} = \mathbf{1}_A,$

for $\hat{\Delta}'_{B,A} =_{df} (\mathbf{1}_A \wedge \check{c}_{B,\neg B}) \circ \hat{\Delta}_{B,A} : A \vdash A \wedge (B \vee \neg B)$ and

$\check{\Sigma}'_{B,A} =_{df} \check{\Sigma}_{B,A} \circ (\hat{c}_{\neg B,B} \vee \mathbf{1}_A) : (\neg B \wedge B) \vee A \vdash A,$

$(\check{\Sigma}'\hat{\Delta}')$ $\quad \check{\Sigma}'_{A,\neg A} \circ d_{\neg A,A,\neg A} \circ \hat{\Delta}'_{A,\neg A} = \mathbf{1}_{\neg A}.$

The naturality equations $(\hat{\Delta}\ nat)$ and $(\check{\Sigma}\ nat)$ say that $\hat{\Delta}$ and $\check{\Sigma}$ are natural transformations in the second index. We have analogous naturality equations for $\hat{\Sigma}$, $\check{\Delta}$, $\hat{\Delta}'$ and $\check{\Sigma}'$.

The arrow $\hat{\Delta}_{B,A}$: $A \vdash A \wedge (\neg B \vee B)$ is analogous to the arrow of type $A \vdash A \wedge \top$ that one finds in monoidal categories. However, $\hat{\Delta}_{B,A}$ does not have an inverse in $\mathbf{PN}^{\neg}$. The equation $(\hat{b}\hat{\Delta})$ is analogous to an equation that holds in monoidal categories (see [20], Section VII.1, [12], Section 4.6, and Section 5 below).

A *proof-net* category is a category with two biendofunctors $\wedge$ and $\vee$, a unary operation $\neg$ on objects, and the natural transformations $\hat{b}^{\rightarrow}$, $\hat{b}^{\leftarrow}$, $\check{b}^{\rightarrow}$, $\check{b}^{\leftarrow}$, $\hat{c}$, $\check{c}$, $d$, $\hat{\Delta}$ and $\check{\Sigma}$ that satisfy the equations $(\overset{\xi}{b}5)$, $(\overset{\xi}{b}\overset{\xi}{b})$, $\dots$, $(\check{\Sigma}'\hat{\Delta}')$ of $\mathbf{PN}^{\neg}$.

It is clear how to define the notion of proof-net functor between proof-net categories, which preserves the proof-net structure of a category strictly (i.e. "on the nose"; cf. [12], Section 2.8). The functor $G$ from $\mathbf{PN}^{\neg}$ to *Br* defined in the next section is a proof-net functor in this sense. The other functors $G$ mentioned later in the paper also each preserve a certain categorial structure "on the nose".

The category $\mathbf{PN}^{\neg}$ is, up to isomorphism, the free proof-net category generated by the set of letters $\mathcal{P}$, thought of as a discrete category.

## 4. The category *Br*

We are now going to introduce a category called *Br*. This category serves to formulate a coherence result for proof-net categories, which says that there is a faithful functor from $\mathbf{PN}^{\neg}$ to *Br*. The name of the category *Br* comes from "Brauerian". The arrows of this category correspond to graphs, or diagrams, that were introduced in [5] in connection with Brauer algebras. Analogous graphs were investigated in [15], and in [17] Kelly and Mac Lane relied on them to prove their coherence result for symmetric monoidal closed categories.

Let $\mathcal{M}$ be a set whose subsets are denoted by $X, Y, Z, \dots$. For $i \in \{s,t\}$ (where $s$ stands for "source" and $t$ for "target"), let $\mathcal{M}^i$ be a set in one-to-one correspondence with $\mathcal{M}$, and let $i : \mathcal{M} \to \mathcal{M}^i$ be a bijection. Let $X^i$ be the subset of $\mathcal{M}^i$ that is the image of the subset $X$ of $\mathcal{M}$ under $i$. If $u \in \mathcal{M}$, then we use $u_i$ as an abbreviation for $i(u)$. We assume also that $\mathcal{M}$, $\mathcal{M}^s$ and $\mathcal{M}^t$ are mutually disjoint.

For $X, Y \subseteq \mathcal{M}$, let a *split relation* of $\mathcal{M}$ be a triple $\langle R, X, Y\rangle$ such that $R \subseteq (X^s \cup Y^t)^2$. The set $X^s \cup Y^t$ may be conceived as the disjoint union of $X$ and $Y$. We denote a split relation $\langle R, X, Y\rangle$ more suggestively by $R\colon X \vdash Y$.

A split relation $R\colon X \vdash Y$ is a *split equivalence* when $R$ is an equivalence relation. We denote by part$(R)$ the partition of $X_s \cup Y_t$ corresponding to the split equivalence $R\colon X \vdash Y$.

A split equivalence $R\colon X \vdash Y$ is *Brauerian* when every member of part$(R)$ is a two-element set. For $R\colon X \vdash Y$ a Brauerian split equivalence, every member of part$(R)$ is either of the form $\{u_s, v_t\}$, in which case it is called a *transversal*, or of the form $\{u_s, v_s\}$, in which case it is called a *cup*, or, finally, of the form $\{u_t, v_t\}$, in which case it is called a *cap*.

For $X, Y, Z \in \mathcal{M}$, we want to define the composition $P * R\colon X \vdash Z$ of the split relations $R\colon X \vdash Y$ and $P\colon Y \vdash Z$ of $\mathcal{M}$. For that we need some auxiliary notions.

For $X, Y \subseteq \mathcal{M}$, let the function $\varphi^s\colon X \cup Y^t \to X^s \cup Y^t$ be defined by

$$\varphi^s(u) = \begin{cases} u_s & \text{if } u \in X \\ u & \text{if } u \in Y^t, \end{cases}$$

and let the function $\varphi^t\colon X^s \cup Y \to X^s \cup Y^t$ be defined by

$$\varphi^t(u) = \begin{cases} u & \text{if } u \in X^s \\ u_t & \text{if } u \in Y. \end{cases}$$

For a split relation $R\colon X \vdash Y$, let the two relations $R^{-s} \subseteq (X \cup Y^t)^2$ and $R^{-t} \subseteq (X^s \cup Y)^2$ be defined by

$$(u, v) \in R^{-i} \quad \text{iff} \quad (\varphi^i(u), \varphi^i(v)) \in R$$

for $i \in \{s, t\}$. Finally, for an arbitrary binary relation $R$, let Tr$(R)$ be the transitive closure of $R$.

Then we define $P * R$ by

$$P * R =_{df} \mathrm{Tr}(R^{-t} \cup P^{-s}) \cap (X^s \cup Z^t)^2.$$

It is easy to conclude that $P * R\colon X \vdash Z$ is a split relation of $\mathcal{M}$, and that if $R\colon X \vdash Y$ and $P\colon Y \vdash Z$ are (Brauerian) split equivalences, then $P * R$ is a (Brauerian) split equivalence.

We now define the category *Br*. The set of objects of *Br* is $\boldsymbol{N}$, the set of finite ordinals. The arrows of *Br* are the Brauerian split equivalences $R\colon m \vdash n$ of $\boldsymbol{N}$. The identity arrow $\mathbf{1}_n\colon n \vdash n$ of *Br* is the Brauerian split equivalence such that

$$\mathrm{part}(\mathbf{1}_n) = \{\{m_s, m_t\} \mid m < n\}.$$

Composition in *Br* is the operation $*$ defined above.

That *Br* is indeed a category (i.e. that $*$ is associative and that $\mathbf{1}_n$ is an identity arrow) is proved in [10] and [11]. This proof is obtained via an isomorphic representation of *Br* in the category *Rel*, whose objects are the finite ordinals and whose arrows are all the relations between these objects. Composition in *Rel* is the ordinary composition of relations. A direct formal proof would be more involved, though what we have to prove is rather clear if we represent Brauerian split equivalences geometrically (as this is done in [5] and [15]).

For example, for $R \subseteq (3^s \cup 9^t)^2$ and $P \subseteq (9^s \cup 1^t)^2$ such that

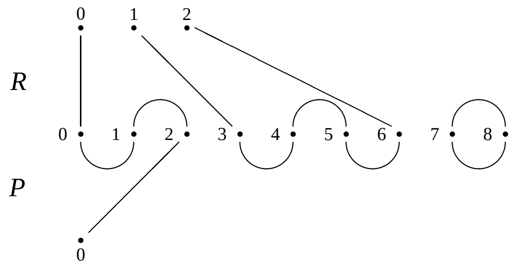
$$\mathrm{part}(R) = \{\{0_s, 0_t\}, \{1_s, 3_t\}, \{2_s, 6_t\}\} \cup \{\{n_t, (n+1)_t\} \mid n \in \{1, 4, 7\}\},$$

$$\mathrm{part}(P) = \{\{2_s, 0_t\}\} \cup \{\{n_s, (n+1)_s\} \mid n \in \{0, 3, 5, 7\}\},$$

the composition $P * R \subseteq (3^s \cup 1^t)^2$, for which we have

$$\mathrm{part}(P * R) = \{\{0_s, 0_t\}, \{1_s, 2_s\}\},$$

is obtained from the following diagram:



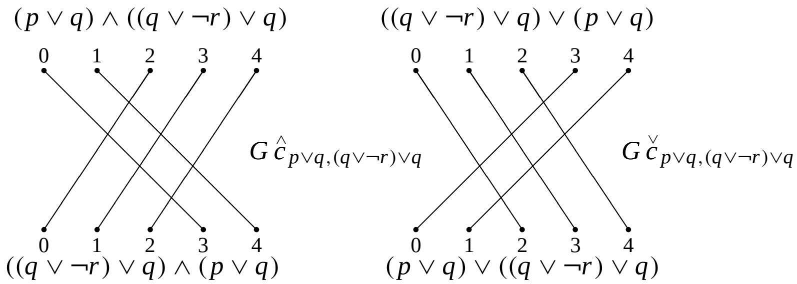

Every bijection $f$ from $X^s$ to $Y^t$ corresponds to a Brauerian split equivalence $R\colon X \vdash Y$ such that the members of part$(R)$ are of the form $\{u, f(u)\}$. The composition of such Brauerian split equivalences, which correspond to bijections, is then a simple matter: it amounts to composition of these bijections. If in *Br* we keep as arrows only such Brauerian split equivalences, then we obtain a subcategory of *Br* isomorphic to the category *Bij* whose objects are again the finite ordinals and whose arrows are the bijections between these objects. The category *Bij* is a subcategory of the category *Rel*, whose objects are the finite ordinals and whose arrows are all the relations between these objects. Composition in *Bij* and *Rel* is the ordinary composition of relations. The category *Rel* (which played an important role in [12]) is isomorphic to a subcategory of the category whose arrows are split relations of finite ordinals, of whom *Br* is also a subcategory.

We define a functor $G$ from $\mathbf{PN}^{\neg}$ to *Br* in the following way. On objects, we stipulate that $GA$ is the number of occurrences of letters in $A$. On arrows, we have first that $G\alpha$ is an identity arrow of *Br* for $\alpha$ being $\mathbf{1}_A$, $\overset{\xi}{b}{}^{\rightarrow}_{A,B,C}$, $\overset{\xi}{b}{}^{\leftarrow}_{A,B,C}$ and $d_{A,B,C}$, where $\xi \in \{\wedge, \vee\}$.

Next, for $i, j \in \{s, t\}$, we have that $\{m_i, n_j\}$ belongs to part$(G\,\hat{c}_{A,B})$ iff $\{n_i, m_j\}$ belongs to part$(G\,\check{c}_{A,B})$, iff $i$ is $s$ and $j$ is $t$, while $m, n < GA+GB$ and

$$(m-n-GA)(m-n+GB) = 0.$$

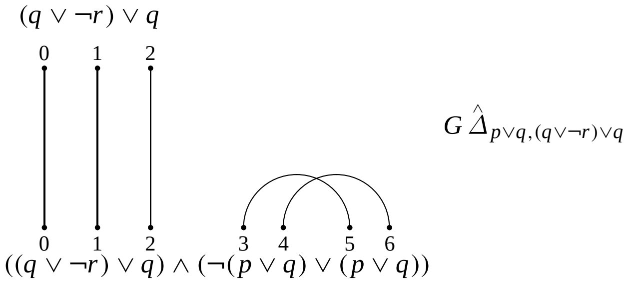
In the following example, we have $G(p \vee q) = 2 = \{0, 1\}$ and $G((q \vee \neg r) \vee q) = 3 = \{0, 1, 2\}$, and we have the diagrams




We have that $\{m_i, n_j\}$ belongs to part$(G\,\hat{\Delta}_{B,A})$ iff either

$i$ is $s$ and $j$ is $t$, while $m, n < GA$ and $m = n$, or
$i$ and $j$ are both $t$, while $m, n \in \{GA, \dots, GA+2GB-1\}$ and $|m-n| = GB$.

In the following example, for $A$ being $(q \vee \neg r) \vee q$ and $B$ being $p \vee q$, we have

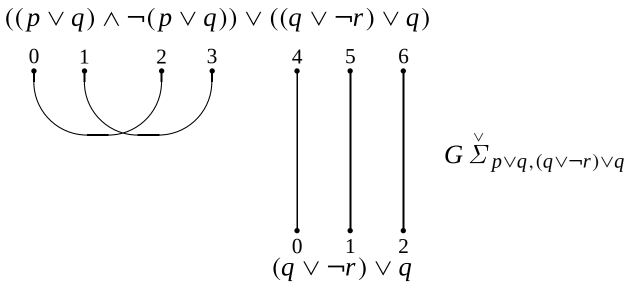



We have that $\{m_i, n_j\}$ belongs to part$(G \overset{\vee}{\Sigma}_{B,A})$ iff either

$i$ is $s$ and $j$ is $t$, while $m \in \{2GB, \ldots, 2GB+GA-1\}$, $n < GA$ and $m-2GB = n$, or

$i$ and $j$ are both $s$, while $m, n < 2GB$ and $|m-n| = GB$.

For $A$ and $B$ being as in the previous example, we have




Let $G(f \circ g) = Gf * Gg$. To define $G(f \;\xi\; g)$, for $\xi \in \{\wedge, \vee\}$, we need an auxiliary notion.

Suppose $b_X$ is a bijection from $X$ to $X_1$ and $b_Y$ a bijection from $Y$ to $Y_1$. Then for $R \subseteq (X^s \cup Y^t)^2$ we define $R^{b_X}_{b_Y} \subseteq (X_1^s \cup Y_1^t)^2$ by

$$(u_i, v_j) \in R^{b_X}_{b_Y} \quad \text{iff} \quad (i(b_U^{-1}(u)), j(b_V^{-1}(v))) \in R,$$

where $(i, U), (j, V) \in \{(s, X), (t, Y)\}$.

If $f : A \vdash D$ and $g : B \vdash E$, then for $\xi \in \{\wedge, \vee\}$ the set of ordered pairs $G(f \;\xi\; g)$ is

$$Gf \cup Gg^{+GA}_{+GD}$$

where $+GA$ is the bijection from $GB$ to $\{n+GA \mid n \in GB\}$ that assigns $n+GA$ to $n$, and $+GD$ is the bijection from $GE$ to $\{n+GD \mid n \in GE\}$ that assigns $n+GD$ to $n$.

It is not difficult to check that $G$ so defined is indeed a functor from **PN**$^\neg$ to *Br*. For that, we determine by induction on the length of derivation that for every equation $f = g$ of **PN**$^\neg$ we have $Gf = Gg$ in *Br*. We have shown by this induction that *Br* is a proof-net category, and the existence of a structure-preserving functor $G$ from **PN**$^\neg$ to *Br* follows from the freedom of **PN**$^\neg$.

We can define analogously to $G$ a functor, which we also call $G$, from the category **DS** to *Br*. We just omit from the definition of $G$ above the clauses involving $\overset{\wedge}{\Delta}_{B,A}$ and $\overset{\vee}{\Sigma}_{B,A}$. The image of **DS** by $G$ in *Br* is the subcategory of *Br* isomorphic to *Bij*, which we mentioned above. The following is proved in [12] (Section 7.6).

**DS Coherence.** *The functor $G$ from* **DS** *to Br is faithful.*

It follows immediately from this coherence result that **DS** is isomorphic to a subcategory of **PN**$^\neg$ (cf. [12], Section 14.4).

The following result is proved in [13] (Section 2.7) and [14].

**$\mathbf{PN}^{\neg}$ Coherence.** *The functor $G$ from $\mathbf{PN}^{\neg}$ to Br is faithful.*

## 5. The category S

The objects of the category **S** are the formulae of the propositional language $\mathcal{L}_{\top,\bot,\neg,\wedge,\vee}$ generated by $\mathcal{P}$, where $\neg$, $\wedge$ and $\vee$ are as before, and $\top$ and $\bot$ are nullary connectives, i.e. propositional constants. As primitive arrow terms we have $\mathbf{1}_A$, $\hat{b}^{\rightarrow}_{A,B,C}$, $\hat{b}^{\leftarrow}_{A,B,C}$, $\hat{c}_{A,B}$, $\check{b}^{\rightarrow}_{A,B,C}$, $\check{b}^{\leftarrow}_{A,B,C}$, $\check{c}_{A,B}$, $d_{A,B,C}$ (see Section 2), $\hat{\Delta}_{B,A}$, $\check{\Sigma}_{B,A}$ (see Section 3), plus

$$\hat{\delta}^{\rightarrow}_{A}\colon A\wedge\top\vdash A,\quad \hat{\delta}^{\leftarrow}_{A}\colon A\vdash A\wedge\top,$$

$$\check{\delta}^{\rightarrow}_{A}\colon A\vee\bot\vdash A,\quad \check{\delta}^{\leftarrow}_{A}\colon A\vdash A\vee\bot,$$

These primitive arrow terms together with the operations on arrow terms $\circ$, $\wedge$ and $\vee$ (the same as we had for **DS** and $\mathbf{PN}^{\neg}$ in Sections 2 and 3) define the arrow terms of **S**.

The equations of **S** are obtained by assuming all the equations we have assumed for $\mathbf{PN}^{\neg}$, plus

$$(\hat{\delta}^{\rightarrow}\ nat)\qquad f\circ\hat{\delta}^{\rightarrow}_{A}=\hat{\delta}^{\rightarrow}_{B}\circ(f\wedge\mathbf{1}_{\top}),$$

$$(\hat{\delta}\hat{\delta})\qquad \hat{\delta}^{\rightarrow}_{A}\circ\hat{\delta}^{\leftarrow}_{A}=\mathbf{1}_A,\qquad \hat{\delta}^{\leftarrow}_{A}\circ\hat{\delta}^{\rightarrow}_{A}=\mathbf{1}_{A\wedge\top},$$

$$(\hat{b}\hat{\delta})\qquad \hat{b}^{\leftarrow}_{A,B,\top}\circ\hat{\delta}^{\leftarrow}_{A\wedge B}=\mathbf{1}_A\wedge\hat{\delta}^{\leftarrow}_{B},$$

$$(\check{\delta}^{\rightarrow}\ nat)\qquad f\circ\check{\delta}^{\rightarrow}_{A}=\check{\delta}^{\rightarrow}_{B}\circ(f\vee\mathbf{1}_{\bot}),$$

$$(\check{\delta}\check{\delta})\qquad \check{\delta}^{\rightarrow}_{A}\circ\check{\delta}^{\leftarrow}_{A}=\mathbf{1}_A,\qquad \check{\delta}^{\leftarrow}_{A}\circ\check{\delta}^{\rightarrow}_{A}=\mathbf{1}_{A\vee\bot},$$

$$(\check{b}\check{\delta})\qquad \check{b}^{\leftarrow}_{A,B,\bot}\circ\check{\delta}^{\leftarrow}_{A\vee B}=\mathbf{1}_A\vee\check{\delta}^{\leftarrow}_{B},$$

for $\hat{\sigma}^{\leftarrow}_{A}=_{df}\hat{c}_{A,\top}\circ\hat{\delta}^{\leftarrow}_{A}$,

$$(d\hat{\sigma})\qquad d_{\top,B,C}\circ\hat{\sigma}^{\leftarrow}_{B\vee C}=\hat{\sigma}^{\leftarrow}_{B}\vee\mathbf{1}_C,$$

$$(d\check{\delta})\qquad \check{\delta}^{\rightarrow}_{C\wedge B}\circ d_{C,B,\bot}=\mathbf{1}_C\wedge\check{\delta}^{\rightarrow}_{B}.$$

The set of equations of **S** is closed under symmetry and transitivity of equality and under the rules $(cong\ \xi)$ for $\xi\in\{\circ,\wedge,\vee\}$ (see Section 2). This defines the equations of **S**.

We have the following definitions:

$$\hat{\sigma}^{\rightarrow}_{A}=_{df}\hat{\delta}^{\rightarrow}_{A}\circ\hat{c}_{\top,A},\qquad \check{\sigma}^{\rightarrow}_{A}=_{df}\check{\delta}^{\rightarrow}_{A}\circ\check{c}_{A,\bot},\qquad \check{\sigma}^{\leftarrow}_{A}=_{df}\check{c}_{\bot,A}\circ\check{\delta}^{\leftarrow}_{A},$$

which give isomorphisms in **S**. Note that $\check{\sigma}^{\rightarrow}_{A}\colon\bot\vee A\vdash A$ is analogous to $\check{\Sigma}_{B,A}\colon(B\wedge\neg B)\vee A\vdash A$, though $\check{\Sigma}_{B,A}$ is not an isomorphism. The equation $(\hat{b}\check{\Sigma})$ of Section 3 is analogous to the following equation of **S** (an equation of monoidal categories):

$$\check{\sigma}^{\rightarrow}_{B\vee A}\circ\check{b}^{\leftarrow}_{\bot,B,A}=\check{\sigma}^{\rightarrow}_{B}\vee\mathbf{1}_A.$$

The equations $(d\hat{\sigma})$ and $(d\check{\delta})$, which amount to the equations $(\hat{\sigma}\,d^L)$ and $(\check{\delta}\,d^L)$ of Section 7.9 of [12] (these equations stem from [7], Section 2.1), are analogous to the equations $(d\hat{\Sigma})$ and $(d\check{\Delta})$ of Section 3.

With the definitions

$$\tau^L_B=_{df}\hat{\sigma}^{\rightarrow}_{\neg B\vee B}\circ\hat{\Delta}_{B,\top}\colon\top\vdash\neg B\vee B,$$

$$\gamma^R_B=_{df}\check{\Sigma}_{B,\bot}\circ\check{\delta}^{\leftarrow}_{B\wedge\neg B}\colon B\wedge\neg B\vdash\bot,$$

in **S**, on the one hand, and

$$\hat{\Delta}_{B,A} =_{df} (\mathbf{1}_A \wedge \tau^L_B) \circ \hat{\delta}^{\leftarrow}_A : A \vdash A \wedge (\neg B \vee B),$$

$$\check{\Sigma}_{B,A} =_{df} \check{\sigma}^{\rightarrow}_A \circ (\gamma^R_B \vee \mathbf{1}_A) \colon (B \wedge \neg B) \vee A \vdash A,$$

on the other hand, it can easily be established that **S** is isomorphic to the free *symmetric linearly* (alias *weakly*) *distributive category with negation* in the sense of [7] (Section 4, Definition 4.3) generated by $\mathcal{P}$.

## 6. The gentzenization of S

We will now define a new language of arrow terms to denote the arrows of the category **S**. We call these arrow terms *Gentzen terms*, and we prove for Gentzen terms a result analogous to Gentzen's cut-elimination theorem, which we will use to prove that the category $\mathbf{PN}^{\neg}$ is isomorphic to a full subcategory of **S**.

As the arrow terms of **S**, Gentzen terms will be defined inductively starting from primitive Gentzen terms. As *primitive* Gentzen terms we have $\mathbf{1}_A : A \vdash A$, for $A$ being a letter, or $\top$, or $\bot$. To define the operations on Gentzen terms, called *Gentzen operations*, which are mostly partial operations, we need some preparation.

We define inductively a notion that for $\xi \in \{\wedge, \vee\}$ we call a $\xi$*-context*:

$\Box$ is a $\xi$-context;
if $Z$ is a $\xi$-context and $A$ an object of **S**, then $Z \,\xi\, A$ and $A \,\xi\, Z$ are $\xi$-contexts.

A $\xi$-context is called *proper* when it is not $\Box$.

Next we define inductively what it means for a $\xi$-context $Z$ to be applied to an object $B$ of **S**, which we write $Z(B)$, or to an arrow term $f$ of **S**, which we write $Z(f)$:

$$\begin{aligned} \Box(B) &= B, & \qquad \Box(f) &= f, \\ (Z \,\xi\, A)(B) &= Z(B) \,\xi\, A, & (Z \,\xi\, A)(f) &= Z(f) \,\xi\, \mathbf{1}_A, \\ (A \,\xi\, Z)(B) &= A \,\xi\, Z(B); & (A \,\xi\, Z)(f) &= \mathbf{1}_A \,\xi\, Z(f). \end{aligned}$$

We use $X$, perhaps with indices, as a variable for $\wedge$-contexts, and $Y$, perhaps with indices, as a variable for $\vee$-contexts.

Then we have the Gentzen operation $\hat{B}^{\leftarrow}_X$, which involves types specified by

$$\frac{f : X(A \wedge (B \wedge C)) \vdash D}{\hat{B}^{\leftarrow}_X f : X((A \wedge B) \wedge C) \vdash D}$$

This is read "if $f$ is a Gentzen term, then $\hat{B}^{\leftarrow}_X f$ is a Gentzen term", all that of the required types. We use this rule notation for operations also in the future. The Gentzen term $\hat{B}^{\leftarrow}_X f$ denotes the arrow of **S** named on the right-hand side of the $=_{dn}$ sign below:

$$\hat{B}^{\leftarrow}_X f =_{dn} f \circ X(\hat{b}^{\leftarrow}_{A,B,C}).$$

We also have the following Gentzen operation:

$$\frac{f : D \vdash Y(A \vee (B \vee C))}{\check{B}^{\rightarrow}_Y f =_{dn} Y(\check{b}^{\rightarrow}_{A,B,C}) \circ f : D \vdash Y((A \vee B) \vee C)}$$

and the following four analogous Gentzen operations, where the types can be easily guessed:

$$\begin{aligned} \hat{B}^{\rightarrow}_X f &=_{dn} f \circ X(\hat{b}^{\rightarrow}_{A,B,C}), & \qquad \check{B}^{\leftarrow}_Y f &=_{dn} Y(\check{b}^{\leftarrow}_{A,B,C}) \circ f, \\ \hat{C}_X f &=_{dn} f \circ X(\hat{c}_{A,B}), & \check{C}_Y f &=_{dn} Y(\check{c}_{A,B}) \circ f. \end{aligned}$$

We also have the Gentzen operations in the following list:

$$\frac{f : A \vdash B}{\top^{\rightarrow} f =_{dn} f \circ \hat{\sigma}^{\rightarrow}_{A} : \top \wedge A \vdash B} \qquad \frac{f : B \vdash A}{\bot^{\leftarrow} f =_{dn} \check{\delta}^{\leftarrow}_{A} \circ f : B \vdash A \vee \bot}$$

$$\frac{g : \top \wedge A \vdash B}{\top^{\leftarrow} g =_{dn} g \circ \hat{\sigma}^{\leftarrow}_{A} : A \vdash B} \qquad \frac{g : B \vdash A \vee \bot}{\bot^{\rightarrow} g =_{dn} \check{\delta}^{\rightarrow}_{A} \circ g : B \vdash A}$$

for $\check{e}'_{D,C,B,A} =_{df} (\hat{c}_{C,D} \vee \mathbf{1}_{B\vee A}) \circ \check{b}^{\leftarrow}_{C\wedge D,B,A} \circ ((d_{C,D,B} \circ \hat{c}_{D\vee B,C}) \vee \mathbf{1}_{A}) \circ$
$\circ\, d_{D\vee B,C,A} : (D \vee B) \wedge (C \vee A) \vdash (D \wedge C) \vee (B \vee A)$,

$$\frac{f_1 : B_1 \vdash A_1 \vee C_1 \qquad\qquad f_2 : B_2 \vdash A_2 \vee C_2}{\wedge(f_1, f_2) =_{dn} \check{e}'_{A_1,A_2,C_1,C_2} \circ (f_1 \wedge f_2) : B_1 \wedge B_2 \vdash (A_1 \wedge A_2) \vee (C_1 \vee C_2)}$$

for $\hat{e}'_{A,B,C,D} =_{df} d_{A,C,B\wedge D} \circ (\mathbf{1}_A \wedge (\check{c}_{C,B\wedge D} \circ d_{B,D,C})) \circ \hat{b}^{\leftarrow}_{A,B,D\vee C} \circ$
$\circ (\mathbf{1}_{A\wedge B} \wedge \check{c}_{D,C}) : (A \wedge B) \wedge (C \vee D) \vdash (A \wedge C) \vee (B \wedge D)$,

$$\frac{f_1 : C_1 \wedge A_1 \vdash B_1 \qquad\qquad f_2 : C_2 \wedge A_2 \vdash B_2}{\vee(f_1, f_2) =_{dn} (f_1 \vee f_2) \circ \hat{e}'_{C_1,C_2,A_1,A_2} : (C_1 \wedge C_2) \wedge (A_1 \vee A_2) \vdash B_1 \vee B_2}$$

(see [12], Section 7.6, for $\check{e}'$ and $\hat{e}'$),

$$\frac{f : B \vdash A \vee C}{\neg^{\mathrm{L}} f =_{dn} \check{\Sigma}'_{A,C} \circ d_{\neg A,A,C} \circ \hat{c}_{A\vee C,\neg A} \circ (f \wedge \mathbf{1}_{\neg A}) : B \wedge \neg A \vdash C}$$

$$\frac{f : C \wedge A \vdash B}{\neg^{\mathrm{R}} f =_{dn} (\mathbf{1}_{\neg A} \vee f) \circ \check{c}_{\neg A,C\wedge A} \circ d_{C,A,\neg A} \circ \hat{\Delta}'_{A,C} : C \vdash \neg A \vee B}$$

To define the remaining Gentzen operations, we need some preparation. For every proper $\wedge$-context $X$ we define inductively as follows an object $E_X$ of **S**:

$E_{\Box\wedge B} = E_{B\wedge\Box} = B$,

$E_{X\wedge B} = E_X \wedge B$, for $X$ proper,

$E_{B\wedge X} = B \wedge E_X$, for $X$ proper.

For every proper $\wedge$-context $X$ and every object $A$ of **S** we define inductively as follows an arrow term $\hat{\tau}_{X,A}$: $E_X \wedge A \vdash X(A)$ **S**:

$\hat{\tau}_{B\wedge\Box,A} =_{df} \mathbf{1}_{B\wedge A} : B \wedge A \vdash B \wedge A$,

$\hat{\tau}_{B\wedge X,A} =_{df} (\mathbf{1}_B \wedge \hat{\tau}_{X,A}) \circ \hat{b}^{\leftarrow}_{B,E_X,A} : (B \wedge E_X) \wedge A \vdash B \wedge X(A)$,
for $X$ proper,

$\hat{\tau}_{\Box\wedge B,A} =_{df} \hat{c}_{B,A} : B \wedge A \vdash A \wedge B$,

$\hat{\tau}_{X\wedge B,A} =_{df} (\hat{\tau}_{X,A} \wedge \mathbf{1}_B) \circ \hat{b}^{\rightarrow}_{E_X,A,B} \circ (\mathbf{1}_{E_X} \wedge \hat{c}_{B,A}) \circ \hat{b}^{\leftarrow}_{E_X,B,A}$:
$(E_X \wedge B) \wedge A \vdash X(A) \wedge B$, for $X$ proper.

For every proper $\vee$-context $Y$ we define inductively as follows an object $D_Y$ of **S**:

$$D_{\Box\vee B} = D_{B\vee\Box} = B,$$
$$D_{Y\vee B} = D_Y \vee B, \quad \text{for } Y \text{ proper},$$
$$D_{B\vee Y} = B \vee D_Y, \quad \text{for } Y \text{ proper}.$$

For every proper $\vee$-context $Y$ and every object $A$ of **S** we define inductively as follows an arrow term $\overset{\vee}{\tau}_{Y,A}\colon Y(A) \vdash A \vee D_Y$ of **S**:

$$\overset{\vee}{\tau}_{\Box\vee B,A} =_{df} \mathbf{1}_{A\vee B} : A \vee B \vdash A \vee B,$$

$$\overset{\vee}{\tau}_{Y\vee B,A} =_{df} \overset{\vee}{b}{}^{\leftarrow}_{A,D_Y,B} \circ (\overset{\vee}{\tau}_{Y,A} \vee \mathbf{1}_B) : Y(A) \vee B \vdash A \vee (D_Y \vee B), \quad \text{for } Y \text{ proper},$$

$$\overset{\vee}{\tau}_{B\vee\Box,A} =_{df} \overset{\vee}{c}_{A,B} : B \vee A \vdash A \vee B,$$

$$\overset{\vee}{\tau}_{B\vee Y,A} =_{df} \overset{\vee}{b}{}^{\leftarrow}_{A,B,D_Y} \circ (\overset{\vee}{c}_{A,B} \vee \mathbf{1}_{D_Y}) \circ \overset{\vee}{b}{}^{\rightarrow}_{B,A,D_Y} \circ (\mathbf{1}_B \vee \overset{\vee}{\tau}_{Y,A}) : B \vee Y(A) \vdash A \vee (B \vee D_Y), \quad \text{for } Y \text{ proper}.$$

For $f : A \vdash B$, the following equations hold in **S**:

$$(\overset{\wedge}{\tau}\ nat) \qquad X(f) \circ \overset{\wedge}{\tau}_{X,A} = \overset{\wedge}{\tau}_{X,B} \circ (\mathbf{1}_{E_X} \wedge f),$$

$$(\overset{\vee}{\tau}\ nat) \qquad (f \vee \mathbf{1}_{D_Y}) \circ \overset{\vee}{\tau}_{Y,A} = \overset{\vee}{\tau}_{Y,B} \circ Y(f);$$

they are proved by applying naturality equations.

It is clear that for $\xi \in \{\wedge, \vee\}$ and $\overset{\xi}{\tau}_{X,A}\colon A_1 \vdash A_2$ there is an arrow term $\overset{\xi}{\tau}{}^{-1}_{X,A}\colon A_2 \vdash A_1$ of **S**, which is a "mirror image" of $\overset{\xi}{\tau}_{X,A}$, such that in **S** we have

$$\overset{\xi}{\tau}{}^{-1}_{X,A} \circ \overset{\xi}{\tau}_{X,A} = \mathbf{1}_{A_1}, \qquad \overset{\xi}{\tau}_{X,A} \circ \overset{\xi}{\tau}{}^{-1}_{X,A} = \mathbf{1}_{A_2}.$$

For example, with

$$\overset{\wedge}{\tau}_{F\wedge((C\wedge\Box)\wedge B),A} = (\mathbf{1}_F \wedge (\overset{\wedge}{b}{}^{\rightarrow}_{C,A,B} \circ (\mathbf{1}_C \wedge \overset{\wedge}{c}_{B,A}) \circ \overset{\wedge}{b}{}^{\leftarrow}_{C,B,A})) \circ \overset{\wedge}{b}{}^{\leftarrow}_{F,C\wedge B,A}$$

we have

$$\overset{\wedge}{\tau}{}^{-1}_{F\wedge((C\wedge\Box)\wedge B),A} = \overset{\wedge}{b}{}^{\rightarrow}_{F,C\wedge B,A} \circ (\mathbf{1}_F \wedge (\overset{\wedge}{b}{}^{\rightarrow}_{C,B,A} \circ (\mathbf{1}_C \wedge \overset{\wedge}{c}_{A,B}) \circ \overset{\wedge}{b}{}^{\leftarrow}_{C,A,B})).$$

Officially, $\overset{\xi}{\tau}{}^{-1}_{X,A}$ is defined inductively as $\overset{\xi}{\tau}_{X,A}$, in a dual manner.

Next, we introduce the following abbreviation:

$$d_{X,A,Y} =_{df} \overset{\vee}{\tau}{}^{-1}_{Y,X(A)} \circ (\overset{\wedge}{\tau}_{X,A} \vee \mathbf{1}_{D_Y}) \circ d_{E_X,A,D_Y} \circ (\mathbf{1}_{E_X} \wedge \overset{\vee}{\tau}_{Y,A}) \circ \overset{\wedge}{\tau}{}^{-1}_{X,Y(A)} : X(Y(A)) \vdash Y(X(A)).$$

When $X$ or $Y$ is $\Box$, then we assume that $d_{X,A,Y}$ stands for $\mathbf{1}_{X(Y(A))}$, which is of type $X(Y(A)) \vdash Y(X(A))$, i.e. $Y(A) \vdash Y(A)$ or $X(A) \vdash X(A)$.

We can finally define the remaining Gentzen operations, which are all of the following form:

$$\frac{g : B \vdash Y(A) \qquad\qquad f : X(A) \vdash C}{cut_{X,Y}(f, g) =_{dn} Y(f) \circ d_{X,A,Y} \circ X(g) : X(B) \vdash Y(C)}$$

This concludes the definition of Gentzen operations. The set of Gentzen terms is the smallest set containing primitive Gentzen terms and closed under the Gentzen operations above.

It is easy to infer from **DS** Coherence of Section 4 that the following equations hold in **S**:

$(d{\wedge}X)$ $\quad d_{A\wedge X,C,Y} = d_{A\wedge\Box,X(C),Y} \circ (\mathbf{1}_A \wedge d_{X,C,Y})$,

$(dX{\wedge})$ $\quad d_{X\wedge A,C,Y} = d_{\Box\wedge A,X(C),Y} \circ (d_{X,C,Y} \wedge \mathbf{1}_A)$,

$(d{\vee}Y)$ $\quad d_{X,C,A\vee Y} = (\mathbf{1}_A \vee d_{X,C,Y}) \circ d_{X,Y(C),A\vee\Box}$,

$(dY{\vee})$ $\quad d_{X,C,Y\vee A} = (d_{X,C,Y} \vee \mathbf{1}_A) \circ d_{X,Y(C),\Box\vee A}$.

The equation $(d{\wedge}X)$ is analogous to the equation $(d{\wedge})$ of Section 2, while $(d{\vee}Y)$ is analogous to $(d{\vee})$ of Section 2.

We can then prove the following.

**Gentzenization Lemma.** *Every arrow of* **S** *is denoted by a Gentzen term.*

**Proof.** We first show by induction on the complexity of $A$ that for every $A$ the arrow $\mathbf{1}_A : A \vdash A$ is denoted by a Gentzen term. For $A$ being a letter, or $\top$, or $\bot$, this is trivial. For the induction step we use the following equations of **S**:

$$(\wedge) \qquad \bot^{\rightarrow}\bot^{\rightarrow}\check{B}^{\rightarrow}_{\Box}\wedge(\bot^{\leftarrow} f_1, \bot^{\leftarrow} f_2) = f_1 \wedge f_2,$$

$$(\vee) \qquad \top^{\leftarrow}\top^{\leftarrow}\hat{B}^{\rightarrow}_{\Box}\vee(\top^{\rightarrow} f_1, \top^{\rightarrow} f_2) = f_1 \vee f_2.$$

For $(\wedge)$ we use

$$\check{e}'_{A_1,A_2,\bot,\bot} = (\mathbf{1}_{A_1\wedge A_2} \vee \check{\delta}^{\leftarrow}_{\bot}) \circ \check{\delta}^{\leftarrow}_{A_1\wedge A_2} \circ (\check{\delta}^{\rightarrow}_{A_1} \wedge \check{\delta}^{\rightarrow}_{A_2}),$$

which follows essentially from $(\check{b}\check{\delta})$ and $(d\,\check{\delta})$ of Section 5 (we may apply here the Symmetric Bimonoidal Coherence of [12], Section 6.4, which reduces to Mac Lane's symmetric monoidal coherence of [19]; see [20], Section VII.7, and [12], Section 5.3). We proceed analogously for $(\vee)$.

We also have for the induction step the following equations of **S**:

$$\bot^{\rightarrow}\neg^{\mathrm{R}}\hat{C}_{\Box}\neg^{\mathrm{L}}\bot^{\leftarrow}\mathbf{1}_A = \top^{\leftarrow}\neg^{\mathrm{L}}\check{C}_{\Box}\neg^{\mathrm{R}}\top^{\rightarrow}\mathbf{1}_A = \mathbf{1}_{\neg A},$$

for which we use $(d\,\check{\delta})$ and $(\check{\Sigma}'\hat{\Delta}')$, among other equations. The Gentzen term that denotes $\mathbf{1}_A$ is written $\mathbf{1}_A$.

Next we have the following in **S**:

$$\hat{B}^{\rightarrow}_{\Box}\mathbf{1}_{(A\wedge B)\wedge C} =_{dn} \hat{b}^{\rightarrow}_{A,B,C}, \qquad \check{B}^{\rightarrow}_{\Box}\mathbf{1}_{A\vee(B\vee C)} =_{dn} \check{b}^{\rightarrow}_{A,B,C},$$

$$\hat{B}^{\leftarrow}_{\Box}\mathbf{1}_{A\wedge(B\wedge C)} =_{dn} \hat{b}^{\leftarrow}_{A,B,C}, \qquad \check{B}^{\leftarrow}_{\Box}\mathbf{1}_{(A\vee B)\vee C} =_{dn} \check{b}^{\leftarrow}_{A,B,C},$$

$$\hat{C}_{\Box}\mathbf{1}_{B\wedge A} =_{dn} \hat{c}_{A,B}, \qquad \check{C}_{\Box}\mathbf{1}_{B\vee A} =_{dn} \check{c}_{A,B},$$

$$cut_{A\wedge\Box,\Box\vee C}(\mathbf{1}_{A\wedge B}, \mathbf{1}_{B\vee C}) =_{dn} d_{A,B,C};$$

by using abbreviations according to $(\wedge)$ and $(\vee)$ above,

$$\top^{\leftarrow}\hat{C}_{\Box}(\mathbf{1}_A \wedge \neg^{\mathrm{R}}\top^{\rightarrow}\mathbf{1}_B) =_{dn} \hat{\Delta}_{B,A},$$

$$\bot^{\rightarrow}\check{C}_{\Box}(\neg^{\mathrm{L}}\bot^{\leftarrow}\mathbf{1}_B \vee \mathbf{1}_A) =_{dn} \check{\Sigma}_{B,A},$$

$$\hat{C}_{\Box}\top^{\rightarrow}\mathbf{1}_A =_{dn} \hat{\delta}^{\rightarrow}_{A}, \qquad \bot^{\rightarrow}\mathbf{1}_{A\vee\bot} =_{dn} \check{\delta}^{\rightarrow}_{A},$$

$$\top^{\leftarrow}\hat{C}_{\Box}\mathbf{1}_{A\wedge\top} =_{dn} \hat{\delta}^{\leftarrow}_{A}, \qquad \bot^{\leftarrow}\mathbf{1}_A =_{dn} \check{\delta}^{\leftarrow}_{A}.$$

For the equations involving $\hat{\Delta}_{B,A}$ and $\check{\Sigma}_{B,A}$ we rely on $(d\,\hat{\sigma})$ and $(d\,\check{\delta})$ of Section 5, and on other equations, called stem-increasing equations in [13] (Section 2.5) and [14] (Section 6).

For composition we have the following equation of **S**:

$$cut_{\Box,\Box}(f, g) = f \circ g,$$

and for the operations $\wedge$ and $\vee$ on arrows we have the equations $(\wedge)$ and $(\vee)$ above. $\dashv$

## 7. Cut elimination in S

For the proof of the Cut-Elimination Theorem below we will introduce analogues of Gentzen's notions of rank and degree. We need some preliminary definitions to define these notions.

For $\xi \in \{\wedge, \vee\}$, we define first by induction the notion of $\xi$-*superficial* subformula of a formula of $\mathcal{L}_{\top,\bot,\neg,\wedge,\vee}$:

$A$ of the form $p$, $\bot$, $A_1 \vee A_2$, or $\neg A'$, is a $\wedge$-superficial subformula of $A$;
$A$ of the form $p$, $\top$, $A_1 \wedge A_2$, or $\neg A'$, is a $\vee$-superficial subformula of $A$;
if $A$ is a $\xi$-superficial subformula of $B$, then $A$ is a $\xi$-superficial subformula of $B\,\xi\,C$ and $C\,\xi\,B$.

Consider a Gentzen term $f$ of the form

$$\wedge(f_1, f_2) \colon B_1 \wedge B_2 \vdash (A_1 \wedge A_2) \vee (C_1 \vee C_2).$$

The $\vee$-superficial subformula $A_1 \wedge A_2$ that is the left disjunct of the target of $f$ is called the *leaf* of $f$. All the other $\vee$-superficial subformulae of the target of $f$, which are subformulae of $C_1$ or $C_2$, and all the $\wedge$-superficial subformulae of the source of $f$, which are subformulae of $B_1$ or $B_2$, are called *lower parameters* of $f$.

To every lower parameter $x$ of $f$, there corresponds unambiguously a subformula $y$ in the target or the source of either $f_1 \colon B_1 \vdash A_1 \vee C_1$ or $f_2 \colon B_2 \vdash A_2 \vee C_2$, which we call the *upper parameter of f corresponding to x*. The lower parameter $x$ is a $\wedge$-superficial subformula of the source of $f$ iff the corresponding upper parameter $y$ is a $\wedge$-superficial subformula of the source of either $f_1$ or $f_2$ (it cannot be in both), and analogously for parameters that are $\vee$-superficial subformulae of targets. If $y$ is in the type of $f_1$, then $f_1$ is called the *subterm of f for the upper parameter y*, and analogously for $f_2$.

For example, if $f$ is

$$\wedge(\mathbf{1}_{p\vee q}, \bot^{\leftarrow}\mathbf{1}_r) \colon (p \vee q) \wedge r \vdash (p \wedge r) \vee (q \vee \bot),$$

then $p\wedge r$ in the target is the leaf of $f$, while $q$ in the target of $f$ and $p \vee q$ and $r$ in the source of $f$ are lower parameters of $f$. To the lower parameter $q$ of $f$ corresponds the upper parameter of $f$ that is the occurrence of $q$ in the target of the subterm $\mathbf{1}_{p\vee q} \colon p \vee q \vdash p \vee q$ for this upper parameter; to the lower parameter $p \vee q$ of $f$ corresponds the upper parameter of $f$ that is the source of the subterm $\mathbf{1}_{p\vee q}$ for this upper parameter; and to the lower parameter $r$ of $f$ corresponds the upper parameter of $f$ that is the source of the subterm $\bot^{\leftarrow}\mathbf{1}_r \colon r \vdash r \vee \bot$ for this upper parameter. Note that the subformula $\bot$ in the target of $f$ is not a $\vee$-superficial subformula of this target, and hence is not a lower parameter of $f$.

If the Gentzen term $f$ is of the form

$$\vee(f_1, f_2) \colon (C_1 \wedge C_2) \wedge (A_1 \vee A_2) \vdash B_1 \vee B_2,$$

then the $\wedge$-superficial subformula $A_1 \vee A_2$ that is the right conjunct of the source of $f$ is the leaf of $f$, while all the other $\wedge$-superficial subformulae of the source of $f$ and the $\vee$-superficial subformulae of the target of $f$ are the lower parameters of $f$. The upper parameters of $f$ corresponding to these lower parameters, and the subterms of $f$ for these upper parameters, are defined analogously to what we had in the previous case.

The leaf of $\neg^{\mathrm{L}} f \colon B \wedge \neg A \vdash C$ is the $\wedge$-superficial subformula $\neg A$ that is the right conjunct of its source, while the leaf of $\neg^{\mathrm{R}} f \colon C \vdash \neg A \vee B$ is the $\vee$-superficial subformula $\neg A$ that is the left disjunct of its target. In both cases, the remaining $\wedge$-superficial subformulae of the source or the remaining $\vee$-superficial subformulae of the target are lower parameters, to whom correspond, analogously to what we had before, upper parameters in the source or target of the subterm $f$ for these upper parameters.

If our Gentzen term is of the form

$$\hat{B}^{\leftarrow}_X f,\ \hat{B}^{\rightarrow}_X f,\ \check{B}^{\rightarrow}_Y f,\ \check{B}^{\leftarrow}_Y f,\ \hat{C}_X f,\ \check{C}_Y f,\ \top^{\rightarrow} f,\ \top^{\leftarrow} f,\ \bot^{\leftarrow} f,\ \bot^{\rightarrow} f,\ \text{ or } cut_{X,Y}(f, g),$$

then it has no leaves, and all the $\wedge$-superficial subformulae of its source and all the $\vee$-superficial subformulae of its target are lower parameters, to which upper parameters correspond in an obvious manner.

Finally, the Gentzen term $\mathbf{1}_p \colon p \vdash p$ has two leaves, which are its source $p$ and its target $p$. There are no parameters of $\mathbf{1}_p$, either lower or upper. The Gentzen term $\mathbf{1}_\top \colon \top \vdash \top$ has as its leaf the target $\top$, and no parameters (the source

$\top$ of $\mathbf{1}_\top$ is not a $\wedge$-superficial subformula of itself). The Gentzen term $\mathbf{1}_\bot : \bot \vdash \bot$ has as its leaf the source $\bot$, and no parameters (the target $\bot$ of $\mathbf{1}_\bot$ is not a $\vee$-superficial subformula of itself).

Let $x$ be a $\wedge$-superficial subformula of the source of a Gentzen term $f$ or a $\vee$-superficial subformula of the target of $f$. Then the *cluster* of $x$ in $f$ is a sequence of occurrences of formulae defined inductively as follows:

if $x$ is a leaf of $f$, then the cluster of $x$ in $f$ is $x$,
if $x$ is not a leaf of $f$, then $x$ is a lower parameter of $f$, and for $y_1$ being the upper parameter of $f$ corresponding to $x$, take the cluster $y_1 \dots y_n$, where $n \geq 1$, of $y_1$ in the proper subterm $f'$ of $f$ that is the subterm of $f$ for the upper parameter $y_1$ (the sequence $y_1 \dots y_n$ is already defined, by the induction hypothesis); the cluster of $x$ in $f$ is the sequence $xy_1 \dots y_n$.

All occurrences of formulae in a cluster are $\xi$-superficial subformulae for $\xi$ being one of $\wedge$ and $\vee$. If $\xi$ is $\wedge$, then the cluster is a *source* cluster, and if $\xi$ is $\vee$, then it is a *target* cluster.

A *cut* is a Gentzen term of the form $cut_{X,Y}(f, g)$. For $g : B \vdash Y(A)$ and $f : X(A) \vdash C$ let the formula $A$ be called the *cut formula* of the cut $cut_{X,Y}(f, g)$. Let $x$ be the displayed occurrence of $A$ in the source $X(A)$ of $f$, and let $s$ be the length of the cluster of $x$ in $f$ (we write $s$ because we have here a source cluster). Let $y$ be the displayed occurrence of $A$ in the target $Y(A)$ of $g$, and let $t$ be the length of the cluster of $y$ in $g$ (we write $t$ because we have here a target cluster).

Depending on the form of $A$, we define a number $r$, which we call the *rank* of the cut $cut_{X,Y}(f, g)$. If the cut formula $A$ is of the form $p$ or $\neg A'$, then

$$r = \min(s,t) - 1, \quad \text{if } A \text{ is } p,$$
$$r = s + t - 2, \quad \text{if } A \text{ is } \neg A'.$$

(As a matter of fact, when $A$ is $p$, we could stipulate that $r$ is either $s+t-2$, as when it is $\neg A'$, or $s-1$, or $t-1$, but the computation of rank that we have introduced makes the cut-elimination procedure run faster, and does not complicate the proof.)

If the cut formula $A$ is of the form $\top$ or $A_1 \wedge A_2$, then $r = t - 1$. If, finally, the cut formula $A$ is of the form $\bot$ or $A_1 \vee A_2$, then $r = s - 1$.

We define the *degree* $d$ of a cut as the number of occurrences of $\wedge$, $\vee$ and $\neg$ in its cut formula. The *complexity* of a cut is the ordered pair $(d, r)$, where $d$ is its degree and $r$ its rank. The complexities of cuts are lexicographically ordered (i.e., $(d_1, r_1) < (d_2, r_2)$ iff $d_1 < d_2$, or $d_1 = d_2$ and $r_1 < r_2$).

A Gentzen term is called *cut-free* when no subterm of it is a cut. A cut $cut_{X,Y}(f, g)$ is *topmost* when $f$ and $g$ are cut-free. (Since in the proof below, we compute the rank only for topmost cuts, our definition of cluster can be shortened a little bit by not considering the parameters of cuts; but this is not a substantial shortening.)

We can then prove the following.

**Cut-Elimination Theorem.** *For every Gentzen term $h$ there is a cut-free Gentzen term $h'$ such that $h = h'$ in* **S**.

**Proof.** It suffices to prove the theorem when $h$ is a topmost cut. We proceed by induction on the complexity $(d, r)$ of this topmost cut.

Suppose $r = 0$ and $d = 0$. Then $h$ can be of one of the following forms:

$$cut_{X,\Box}(f, \mathbf{1}_A) \quad \text{for } A \text{ being } p \text{ or } \top,$$
$$cut_{\Box,Y}(\mathbf{1}_A, g) \quad \text{for } A \text{ being } p \text{ or } \bot,$$

and we have in **S**

$$cut_{X,\Box}(f, \mathbf{1}_A) = f,$$
$$cut_{\Box,Y}(\mathbf{1}_A, g) = g.$$

This settles the basis of the induction.

Suppose $r = 0$ and $d > 0$. Then the cut formula must be of the form $A_1 \wedge A_2$ or $A_1 \vee A_2$ or $\neg A'$. In the first case, for $f : X(A_1 \wedge A_2) \vdash D$, $g_1 : B_1 \vdash A_1 \vee C_1$ and $g_2 : B_2 \vdash A_2 \vee C_2$ we have the equation

$$cut_{X,\Box\vee(C_1\vee C_2)}(f, \wedge(g_1, g_2)) = \overset{\vee}{B}{}^{\leftarrow}_{\Box} cut_{X'',\Box\vee C_2}(cut_{X',\Box\vee C_1}(f, g_1), g_2)$$

where $X'(C)$ is $X(C \wedge A_2)$ and $X''(C)$ is $X(B_1 \wedge C)$. To prove this equation we apply naturality equations and **DS** Coherence of Section 4.

The complexity of the topmost cut $cut_{X',\Box\vee C_1}(f, g_1)$ is $(d', r')$ with $d' < d$, and we can apply the induction hypothesis to obtain a cut-free Gentzen term $f'$ equal to it in **S**. The complexity of the topmost cut $cut_{X'',\Box\vee C_2}(f', g_2)$ is $(d'', r'')$ with $d'' < d$, and we can again apply the induction hypothesis.

In the case where the cut formula is $A_1 \vee A_2$, we have an analogous equation, for which we use again **DS** Coherence, and we reason analogously, applying the induction hypothesis twice.

In the case where the cut formula is $\neg A'$, for $f : D \wedge A' \vdash E$ and $g : B \vdash A' \vee C$ we have the equation

$$cut_{B\wedge\Box,\Box\vee E}(\neg^{\mathrm{L}} g, \neg^{\mathrm{R}} f) = \check{C}_\Box \hat{C}_\Box\, cut_{D\wedge\Box,\Box\vee C}(f, g),$$

which holds by naturality equations and $\mathbf{PN}^\neg$ Coherence of Section 4. Then we apply the induction hypothesis to the topmost cut on the right-hand side, which has a smaller degree.

Suppose now $r > 0$. If $r$ was computed as $s-1$, or as $s+t-2$, where $s > 1$, then we may apply equations of **S** of the following form

$$(*) \quad cut_{X,Y}(\gamma f', g) = \gamma_1 \ldots \gamma_n cut_{X',Y}(f', g)$$

for $\gamma, \gamma_1, \ldots, \gamma_n$ unary Gentzen operations. If $(d, r)$ is the complexity of the topmost cut $cut_{X,Y}(\gamma f', g)$, then the complexity of the topmost cut $cut_{X',Y}(f', g)$ is $(d, r-1)$, and so we may apply to it the induction hypothesis.

If $\gamma$ is a unary Gentzen operation different from $\top^{\rightarrow}$, $\top^{\leftarrow}$, $\bot^{\leftarrow}$ and $\bot^{\rightarrow}$, then so are $\gamma_1, \ldots, \gamma_n$, and to prove $(*)$ we apply naturality equations and $\mathbf{PN}^\neg$ Coherence (sometimes **DS** Coherence suffices, depending on $\gamma$). We have analogous equations involving binary Gentzen operations, which are proved analogously, relying on **DS** Coherence (cf. [12], Section 11.2, Case (6), where on p. 251, in the second line $\wedge^R(f, cut(g, h))$ should be replaced by $\wedge^R(g, (f, h))$, and in the third line $cut(g, h)$ should be replaced by $cut(f, h)$).

If $\gamma$ in $(*)$ is $\top^{\rightarrow}$, then $n = 1$ and $\gamma_1$ is $\top^{\rightarrow}$. To prove $(*)$, we then apply essentially the equation

$$Y(\hat{\sigma}^{\rightarrow}_{X(A)}) \circ d_{\top\wedge X,A,Y} = d_{X,A,Y} \circ \hat{\sigma}^{\rightarrow}_{X(Y(A))},$$

which we obtain with the help of $(d\wedge X)$ of the preceding section, $(d\,\hat{\sigma})$ of Section 3.3, and $(\check{\tau}\ nat)$ of the preceding section (as a matter of fact, we may apply here the Symmetric Bimonoidal Coherence of [12], Section 6.4). We proceed analogously if $\gamma$ is $\top^{\leftarrow}$.

If $\gamma$ in $(*)$ is $\bot^{\leftarrow}$ or $\bot^{\rightarrow}$, then we apply essentially Mac Lane's symmetric monoidal coherence of [19] (see also [20], Section VII.7, and [12], Section 5.3).

If $r$ was computed as $t-1$, or as $s+t-2$, where $t > 1$, then we proceed in a dual manner. Instead of $(*)$, we have equations of **S** of the following form:

$$cut_{X,Y}(f, \gamma g') = \gamma_1 \ldots \gamma_n cut_{X,Y'}(f, g').$$

This concludes the proof of the theorem. ⊣

## 8. $\mathbf{S}^c$ coherence

There is a functor $G$ from the category **S** to *Br*, which is defined as the functor $G$ from $\mathbf{PN}^\neg$ to *Br* (see Section 4) with the additional clauses that say that $G\alpha$ is an identity arrow of *Br* for $\alpha$ being $\overset{\xi}{\delta}{}^{\rightarrow}_{A}$ and $\overset{\xi}{\delta}{}^{\leftarrow}_{A}$, where $\xi \in \{\wedge, \vee\}$. It follows from the existence of these functors and $\mathbf{PN}^\neg$ Coherence of Section 4 that $\mathbf{PN}^\neg$ is isomorphic to a subcategory of **S** (cf. [12], Section 14.4).

The following theorem can be proved with the help of the Cut-Elimination Theorem of the preceding section.

**Conservativeness Theorem.** *If $A$ and $B$ are objects of $\mathbf{PN}^\neg$, then for every arrow $f : A \vdash B$ of* **S** *there is an arrow term $f' : A \vdash B$ of $\mathbf{PN}^\neg$ such that $f = f'$ in* **S**.

This theorem implies that $\mathbf{PN}^\neg$ is isomorphic to a full subcategory of **S**. In these isomorphisms every object of $\mathbf{PN}^\neg$ is mapped to itself, and so every object of $\mathbf{PN}^\neg$ in **S** is in the image of $\mathbf{PN}^\neg$.

Let $\mathbf{S}'$ be the full subcategory of $\mathbf{S}$ whose objects are all the objects $A$ of $\mathbf{S}$ such that there is an isomorphism of type $A \vdash A'$ of $\mathbf{S}$ for $A'$ an object of $\mathbf{PN}^{\neg}$. Then we can restrict the functor $G$ from $\mathbf{S}$ to *Br* to a functor $G$ from $\mathbf{S}'$ to *Br*, for which we can prove the following, relying on the Conservativeness Theorem.

**$\mathbf{S}'$ Coherence.** *The functor $G$ from $\mathbf{S}'$ to Br is faithful.*

**Proof.** Suppose $A$ and $B$ are objects of $\mathbf{S}'$, and let $j_A\colon A \vdash A'$ and $j_B\colon B \vdash B'$ be isomorphisms of $\mathbf{S}$ for $A'$ and $B'$ objects of $\mathbf{PN}^{\neg}$. Suppose that $f_1, f_2\colon A \vdash B$ are arrows of $\mathbf{S}$, i.e. of $\mathbf{S}'$, such that $Gf_1 = Gf_2$.

Since $\mathbf{PN}^{\neg}$ is isomorphic to a full subcategory of $\mathbf{S}$ such that every object of $\mathbf{PN}^{\neg}$ in $\mathbf{S}$ is in the image of $\mathbf{PN}^{\neg}$, we have in $\mathbf{S}$ that

$$j_B \circ f_i \circ j_A^{-1} = f_i'$$

for $i \in \{1,2\}$ and $f_i'$ an arrow term of $\mathbf{PN}^{\neg}$. It follows that $Gf_1' = Gf_2'$, and, according to what we said immediately after the definition of the functor $G$ from $\mathbf{S}$ to *Br*, by $\mathbf{PN}^{\neg}$ Coherence we have that $f_1' = f_2'$ in $\mathbf{PN}^{\neg}$, and hence also in $\mathbf{S}$. So $f_1 = f_2$ in $\mathbf{S}$. ⊣

The category $\mathbf{S}'$ is a category equivalent to $\mathbf{PN}^{\neg}$, and its coherence is a consequence of $\mathbf{PN}^{\neg}$ Coherence. We can find full subcategories of $\mathbf{S}'$ that are not only equivalent, but also isomorphic, to $\mathbf{PN}^{\neg}$.

Let $\mathbf{S}^c$ be the full subcategory of $\mathbf{S}$ whose objects are all the objects $A$ of $\mathbf{S}$ such that there is an isomorphism of type $A \vdash A'$ of $\mathbf{S}$ for $A'$ being either an object of $\mathbf{PN}^{\neg}$, or $\top$, or $\bot$. Then we can restrict the functor $G$ from $\mathbf{S}$ to *Br* to a functor $G$ from $\mathbf{S}^c$ to *Br*, for which we can prove the following, relying on the Conservativeness Theorem and on $\mathbf{S}'$ Coherence.

**$\mathbf{S}^c$ Coherence.** *The functor $G$ from $\mathbf{S}^c$ to Br is faithful.*

**Proof.** There is no arrow of type $\top \vdash \bot$ in $\mathbf{S}$. (Otherwise, classical propositional logic would be inconsistent.) There is also no arrow of type $\bot \vdash \top$ in $\mathbf{S}$. If $f : \bot \vdash \top$ were such an arrow, then we would have in $\mathbf{S}$ the arrow

$$((\hat{\delta}^{\rightarrow}_{p} \circ (\mathbf{1}_p \wedge f)) \vee \mathbf{1}_q) \circ d_{p,\bot,q} \circ (\mathbf{1}_p \wedge \check{\sigma}^{\leftarrow}_{q})\colon p \wedge q \vdash p \vee q.$$

Hence, by the Conservativeness Theorem, there would be an arrow term $f'\colon p \wedge q \vdash p \vee q$ of $\mathbf{PN}^{\neg}$, and that such an $f'$ does not exist can be shown by appealing to the connectedness condition of proof nets (see [8]).

Suppose $A$ and $B$ are objects of $\mathbf{S}^c$; so $A$ and $B$ are isomorphic in $\mathbf{S}$ to respectively $A'$ and $B'$ each of which is either an object of $\mathbf{PN}^{\neg}$, or $\top$, or $\bot$. Suppose that $f_1, f_2\colon A \vdash B$ are arrows of $\mathbf{S}$, i.e. of $\mathbf{S}^c$, such that $Gf_1 = Gf_2$.

As we have seen above, it is excluded that one of $A'$ and $B'$ is $\top$ while the other is $\bot$. If $A'$ and $B'$ are objects of $\mathbf{PN}^{\neg}$, then we apply $\mathbf{S}'$ Coherence.

Let $\mathbf{S}_{+p}$ be $\mathbf{S}$ generated by $\mathcal{P} \cup \{p\}$ for a letter $p$ foreign to $\mathcal{P}$, and hence also to $A$ and $B$. Let $\mathbf{S}'_{+p}$ be the $\mathbf{S}'$ subcategory of $\mathbf{S}_{+p}$. In the remaining cases, if either $A'$ or $B'$ is $\top$, then $G(f_1 \wedge \mathbf{1}_p) = G(f_2 \wedge \mathbf{1}_p)$. It is easy to see that $f_1 \wedge \mathbf{1}_p, f_2 \wedge \mathbf{1}_p\colon A \wedge p \vdash B \wedge p$ are arrows of $\mathbf{S}'_{+p}$, and so $f_1 \wedge \mathbf{1}_p = f_2 \wedge \mathbf{1}_p$ in $\mathbf{S}_{+p}$ by $\mathbf{S}'$ Coherence applied to $\mathbf{S}'_{+p}$. Then in $\mathbf{S}$ generated by $\mathcal{P}$ we have $f_1 \wedge \mathbf{1}_\top = f_2 \wedge \mathbf{1}_\top$ (we just substitute $\top$ for $p$ in the derivation of $f_1 \wedge \mathbf{1}_p = f_2 \wedge \mathbf{1}_p$ in $\mathbf{S}_{+p}$), and so we have in $\mathbf{S}$

$$\begin{aligned} f_1 &= f_1 \circ \hat{\delta}^{\rightarrow}_{A} \circ \hat{\delta}^{\leftarrow}_{A}, \quad \text{by } (\hat{\delta}\hat{\delta}),\\ &= \hat{\delta}^{\rightarrow}_{B} \circ (f_1 \wedge \mathbf{1}_\top) \circ \hat{\delta}^{\leftarrow}_{A}, \quad \text{by } (\hat{\delta}^{\rightarrow}\ nat),\\ &= \hat{\delta}^{\rightarrow}_{B} \circ (f_2 \wedge \mathbf{1}_\top) \circ \hat{\delta}^{\leftarrow}_{A}\\ &= f_2. \end{aligned}$$

If either $A'$ or $B'$ in the remaining cases is $\bot$, then $G(f_1 \vee \mathbf{1}_p) = G(f_2 \vee \mathbf{1}_p)$, and we proceed analogously. ⊣

Let $\mathcal{L}_{\top,\wedge,\rightarrow}$ be the propositional language generated by $\mathcal{P}$ with the nullary connective $\top$ and the binary connectives $\wedge$ and $\rightarrow$. The formulae of $\mathcal{L}_{\top,\wedge,\rightarrow}$ are the objects of the free symmetric monoidal closed category **SMC** generated by $\mathcal{P}$ (see [20], Section VII.7, and [13], Section 3.1).

We call a formula $A$ of $\mathcal{L}_{\top,\wedge,\rightarrow}$ *consequential* when for every subformula $B \rightarrow C$ of $A$ we have that either $B$ is letterless or $C$ has letters occurring in it. An alternative way to characterize consequential formulae is to say that these are formulae $A$ of $\mathcal{L}_{\top,\wedge,\rightarrow}$ for which there is an isomorphism of type $A \vdash A'$ of **SMC** such that either $\top$ does not occur in $A'$ or $A'$ is $\top$. (To establish the equivalence of these two characterizations, one may rely on the results of [9].)

Let $\mathbf{SMC}^c$ be the full subcategory of **SMC** whose objects are consequential formulae. With an appropriate definition of the functor $G$ from $\mathbf{SMC}^c$ to *Br*, Kelly and Mac Lane's coherence theorem for symmetric monoidal closed categories of [17] amounts to the assertion that the functor $G$ from $\mathbf{SMC}^c$ to *Br* is faithful. Both $\mathbf{S}'$ Coherence and $\mathbf{S}^c$ Coherence are analogous to this result of Kelly and Mac Lane. For $\mathbf{S}^c$ Coherence the analogy is complete.

The proof of the Conservativeness Theorem is accomplished with the help of a technical lemma, for whose formulation we introduce the following terminology.

An object of **S**, i.e. a formula of $\mathcal{L}_{\top,\bot,\neg,\wedge,\vee}$, is *constant-free* when neither $\top$ nor $\bot$ occurs in it. In other words, the constant-free objects of **S** are the objects of $\mathbf{PN}^{\neg}$.

An object of **S** is called *literate* when at least one letter occurs in it; otherwise, it is *letterless*. Every constant-free formula is literate (but not conversely).

For $\xi \in \{\wedge, \vee\}$, we define inductively when a formula of $\mathcal{L}_{\top,\bot,\neg,\wedge,\vee}$ is $\xi$*-nice*:

$\top$ is $\wedge$-nice and $\bot$ is $\vee$-nice;

constant-free objects of **S** are $\xi$-nice;

if $A$ and $B$ are $\xi$-nice, then $A\,\xi\,B$ is $\xi$-nice.

For a $\xi$-nice formula $A$ we define inductively an arrow term $\overset{\xi}{\rho}_A\colon A \vdash A^r$ of **S** such that $A^r$ is constant-free if $A$ is literate, $A^r$ is $\top$ if $A$ is letterless and $\wedge$-nice, and $A^r$ is $\bot$ if $A$ is letterless and $\vee$-nice:

$$\overset{\wedge}{\rho}_\top = \mathbf{1}_\top, \qquad \overset{\vee}{\rho}_\bot = \mathbf{1}_\bot, \qquad \overset{\xi}{\rho}_A = \mathbf{1}_A, \qquad \text{for } A \text{ constant-free,}$$

$$\overset{\xi}{\rho}_{A\xi B} = \overset{\xi}{\rho}_A\ \xi\ \overset{\xi}{\rho}_B, \qquad \text{for } A \text{ and } B \text{ literate,}$$

$$\overset{\xi}{\rho}_{A\xi B} = \overset{\xi}{\delta}{}^{\rightarrow}_A \circ (\overset{\xi}{\rho}_A\ \xi\ \overset{\xi}{\rho}_B), \qquad \text{for } B \text{ letterless,}$$

$$\overset{\xi}{\rho}_{A\xi B} = \overset{\xi}{\sigma}{}^{\rightarrow}_B \circ (\overset{\xi}{\rho}_A\ \xi\ \overset{\xi}{\rho}_B), \qquad \text{for } A \text{ letterless.}$$

It is clear that $\overset{\xi}{\rho}_A$ is an isomorphism of **S**, with inverse $\overset{\xi}{\rho}{}^{-1}_A\colon A^r \vdash A$.

The Conservativeness Theorem is a corollary of the following lemma (we just instantiate statement (1) of this lemma).

**Lemma.** *Let $f : A \vdash B$ be an arrow of* **S** *such that $A$ is $\wedge$-nice and $B$ is $\vee$-nice.*

(1) *If both $A$ and $B$ are literate, then there is an arrow term $f^r : A^r \vdash B^r$ of $\mathbf{PN}^{\neg}$ such that in* **S** *we have*

$$\overset{\vee}{\rho}_B \circ f \circ \overset{\wedge}{\rho}{}^{-1}_A = f^r.$$

(2) *If $A$ is letterless and $B$ is literate, then for every constant-free $C$ there is an arrow term $f^r : C \vdash C \wedge B^r$ of $\mathbf{PN}^{\neg}$ such that in* **S** *we have*

$$(\mathbf{1}_C \wedge (\overset{\vee}{\rho}_B \circ f \circ \overset{\wedge}{\rho}{}^{-1}_A)) \circ \overset{\wedge}{\delta}{}^{\leftarrow}_C = f^r.$$

(3) *If $A$ is literate and $B$ is letterless, then for every constant-free $C$ there is an arrow term $f^r : A^r \vee C \vdash C$ of $\mathbf{PN}^{\neg}$ such that in* **S** *we have*

$$\overset{\vee}{\sigma}{}^{\rightarrow}_C \circ ((\overset{\vee}{\rho}_B \circ f \circ \overset{\wedge}{\rho}{}^{-1}_A) \vee \mathbf{1}_C) = f^r.$$

The proof of this lemma, which may be found in [13] (Section 4.3), is based on the Gentzenization Lemma and the Cut-Elimination Theorem of the preceding two sections. We take that $f$ in the lemma is a cut-free Gentzen term, and we proceed by induction on the complexity of $f$.

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