Advances in Mathematics **133**, 1–32 (1998)
Article No. AI971680

The Tree Property

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Received July 1, 1996; accepted May 29, 1997

We construct a model in which there are no $\aleph_n$-Aronszajn trees for any finite $n \geqslant 2$, starting from a model with infinitely many supercompact cardinals. We also construct a model in which there is no κ^{++}-Aronszajn tree for κ a strong limit cardinal of cofinality ω, starting from a model with a supercompact cardinal and a weakly compact cardinal above it.

Key words: forcing, large cardinals, Aronszajn trees.

1. INTRODUCTION

We will prove the following theorems.

THEOREM 1. *If "ZFC + there exist infinitely many supercompact cardinals" is consistent, then "ZFC + there are no $\aleph_n$-Aronszajn trees for $2 \leqslant n < \omega$" is also consistent.*

THEOREM 2. *If "ZFC + there exists a supercompact cardinal with a weakly compact cardinal above it" is consistent then "ZFC + there exists κ a strong limit cardinal of cofinality ω such that there are no κ^{++}-Aronszajn trees" is also consistent.*

We start by recalling the definition of "κ-Aronszajn tree" and some related concepts.

* Partially supported by a postdoctoral fellowship at the Mathematics Institute, Hebrew University of Jerusalem.
† Partially supported by DMS 9500484.

0001-8708/98 $25.00

DEFINITION 1.1. Let κ be regular:

1. A κ-*tree* is a tree of height κ whose every level has size less than κ.
2. A κ-*Aronszajn* tree is a κ-tree with no cofinal branch.
3. κ has the *tree property* iff there is no κ-Aronszajn tree.
4. A λ^+-tree T is *special* iff there is a function $F: T \to \lambda$ such that $x <_T y$ implies $F(x) \neq F(y)$.

Aronszajn trees arise naturally in infinitary combinatorics, and the tree property has been investigated by many set theorists. We list a few classical results:

- (König [6]) $\aleph_0$ has the tree property.
- (Aronszajn, see [9]) $\aleph_1$ does not have the tree property.
- (Specker [15]) If $\tau^{<\tau} = \tau$ then there is a special τ^+-tree. In particular CH implies that $\aleph_2$ fails to have the tree property.
- If κ is strongly inaccessible, κ has the tree property if and only if κ is weakly compact.
- (Mitchell [14]) The theories "$ZFC +$ there exists a weakly compact cardinal" and "$ZFC + \aleph_2$ has the tree property" are equiconsistent.

The construction from [14] is quite general. Let $\delta < \kappa$ with δ regular and κ weakly compact; there is a forcing which preserves cardinals up to δ^+, makes $2^\delta = \delta^{++} = \kappa$, and preserves the tree property of κ in the generic extension. In the other direction, any cardinal with the tree property in V will be weakly compact in L.

It is natural to ask whether two small cardinals can simultaneously have the tree property. It turns out that starting from two weakly compact cardinals it is fairly easy to make a model where (for example) $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$, $2^{\aleph_2} = 2^{\aleph_3} = \aleph_4$, and both $\aleph_2$ and $\aleph_4$ have the tree property. However a naive approach will fail if one starts with two weak compacts and tries to make them into *successive* cardinals with the tree property.

This is more than just a technical problem; Magidor (see [1]) showed that getting a pair of successive cardinals to have the tree property requires at least a measurable cardinal. This result was improved by Foreman and Magidor to show that a strong cardinal is required. Abraham [1] showed that using large enough cardinals this situation is consistent; he started with $\delta < \kappa < \lambda$ wich are respectively regular, supercompact, and weakly compact and forced to make $2^\delta = \delta^{++} = \kappa$, $2^{\delta^+} = \delta^{+++} = \lambda$, preserving the tree property at κ and λ.

In the first part of this paper we will start with ω supercompact cardinals $\langle \kappa_n : n < \omega \rangle$ and will collapse so that κ_n becomes $\aleph_{n+2}$ and still enjoys the tree property. The idea is essentially to iterate Abraham's forcing from [1], although there are several new technical problems that have to be coped with. The authors discovered the construction independently in slightly different versions; Cummings' version is the one given there.

We will also discuss the tree property for κ^{++}, where κ is singular strong limit. Since $2^\kappa = \kappa^+ \Rightarrow (\kappa^+)^{<\kappa^+} = \kappa^+$, there will be a special κ^{++}-tree unless the Singular Cardinals Hypothesis fails at κ. Foreman showed that the tree property can hold in this situation, and in the second part of the paper we will give the proof. The method can be used to show that $\aleph_{\omega+2}$ can have the tree property.

The tree property at successors of singulars is a problem with a different flavour, since here there are connections with the existence of weak squares and much larger cardinals are used in the consistency proofs. It is proved in [11] that $\aleph_{\omega+1}$ can have the tree property. The consistency of $\aleph_{\omega+1}$ and $\aleph_{\omega+2}$ having the tree property simultaneously (or even of $\aleph_{\omega+1}$ having the tree property with $2^{\aleph_\omega} > \aleph_{\omega+1}$) is open. We also do not know whether "the tree property holds for all $\aleph_n$ with $2 \leqslant n < \omega$ and for $\aleph_{\omega+1}$" is consistent.

We would like to thank Menachem Magidor for a very helpful discussion of the material of Section 7.

2. PRELIMINARIES

In this section we collect some technical definitions and facts for later use. The impatient reader is advised to skip ahead to the next section and refer back if necessary.

2.1. *Forcing Conventions*

We mostly follow the notation and forcing conventions of [7]. In particular a forcing $\mathbb{P}$ is a preordering with a distinguished maximum element $1_{\mathbb{P}}$, "$p \leqslant q$" means that p is stronger than q, and "κ-closed" means that descending sequences of length less than κ have lower bounds. We will write "$p \leqslant_{\mathbb{P}} q$" when there is a possibility of confusion about which ordering is meant. For iterated forcing we mostly follow the conventions of [2].

Recall that a forcing is *separative* iff $p \leqslant q \Leftrightarrow p \Vdash \hat{q} \in \dot{G}$. Not all of the forcings in this paper will have this property.

We will use $V^{\mathbb{P}}$ to denote the class of $\mathbb{P}$-names and will use $V[\mathbb{P}]$ to denote a generic extension by some unspecified $\mathbb{P}$-generic filter. When we have a particular generic filter G in mind we will use $V[G]$ to denote the extension by G.

We will need several refinements and variations of the standard chain condition and closure properties.

Definition 2.1. Let κ be regular; let $\mathbb{P}$ be some forcing:

1. $\mathbb{P}$ is *κ-Knaster* if and only if for all sequences $\langle p_\alpha : \alpha < \kappa \rangle$ from $\mathbb{P}$ there is X unbounded in κ such that $\langle p_\alpha : \alpha \in X \rangle$ consists of pairwise compatible elements.

2. $\mathbb{P}$ is *κ-directed closed* if and only if every directed subset of $\mathbb{P}$ of size less than κ has a lower bound.

3. $\mathbb{P}$ is *canonically κ-directed closed* if and only if every directed subset of $\mathbb{P}$ of size less than κ has a greatest lower bound.

4. $\mathbb{P}$ is *$<\kappa$-distributive* if and only if every family of fewer than κ many dense open sets has a nonempty intersection. This is equivalent to the property that no sequence of ordinals of length less than κ is added by $\mathbb{P}$.

The following easy fact will be useful later.

Lemma 2.2. *Let $\mathbb{P}$ be canonically κ-directed closed, let G be a $\mathbb{P}$-generic filter over V, and let $A \subseteq G$, where $V[G] \models |A| < \kappa$. Then $A \in V$, A is directed, and the greatest lower bound of A is a member of G.*

2.2. *Trees and Forcing*

It will be important in what follows to know that certain kinds of forcing cannot add branches to a tree.

Lemma 2.3 (Kunen and Tall [8]). *Let κ be regular, let $\mathbb{P}$ be κ-Knaster, and suppose that T is a tree of height κ with no cofinal branch. Then T has no cofinal branch in $V[\mathbb{P}]$.*

Proof. Suppose not, and let $\dot{b} \in V^{\mathbb{P}}$ be a name for a cofinal branch of the tree T. Choose for each α a condition p_α such that p_α forces the element of b on level α to be a particular point x_α. Now using the Knaster property we may find $X \subseteq \kappa$ of cardinality κ such that $\{p_\alpha : \alpha \in X\}$ is pairwise compatible. But then $\{x_\alpha : \alpha \in X\}$ is a cofinal linearly ordered subset of T in V, which can easily be extended to a cofinal branch. Contradiction. ∎

It is crucial to the intended application of Lemma 2.3 that it applies to all trees of height κ, not just to κ-trees. By contrast, the next lemma is specific to κ-trees.

Lemma 2.4 (Silver). *Let τ, κ be regular and suppose $\tau < \kappa \leqslant 2^\tau$. Let $\mathbb{P}$ be τ^+-closed, let T be a κ-tree. Then every branch of T in $V[\mathbb{P}]$ is in fact a member of V.*

Proof. We may assume that τ is minimal with $2^\tau \geqslant \kappa$. Let $\dot{b} \in V^{\mathbb{P}}$ name a new branch. Now we may build by induction for each $s \in {}^{\leqslant \tau+1}2$ conditions p_s and points x_s of T with the following properties:

1. If s properly extends t then $p_s \leqslant p_t$, $x_s >_T x_t$.
2. p_s forces x_s is on the branch $\dot{b}$.
3. For each α the points $\{x_s : s \in {}^{\alpha}2\}$ are all on the same level of T, say level η_α.
4. For each $s \in {}^{<\tau}2$ the points $x_{s \frown 0}$ and $x_{s \frown 1}$ are incompatible.

The minimal choice of τ ensures that for $\alpha < \tau$ the set $\{x_s : s \in {}^{\alpha}2\}$ has size less than κ, so that we can choose $\eta_{\alpha+1}$. The closure of $\mathbb{P}$ guarantees that the construction works at limit stages.

This leads to a contradiction, because the level η_τ of T must have fewer than κ elements, yet we have constructed 2^τ many distinct ones. ∎

2.3. *Cohen Forcing*

We need a careful analysis of the properties of Cohen forcing, as this forcing will be the key building-block in the main construction.

Definition 2.5. Let κ be regular; let X be a set of ordinals. Then $\mathrm{Add}(\kappa, X)$ is the forcing whose conditions are functions p such that $\mathrm{dom}(p) \subseteq X$, $\mathrm{rge}(p) \subseteq 2$, $|\mathrm{dom}(p)| < \kappa$. The ordering is by extension.

When we have defined $\mathbb{P} = \mathrm{Add}(\kappa, \lambda)$ and $\eta < \lambda$, we will often use "$\mathbb{P} \upharpoonright \eta$" as a convenient shorthand for $\mathrm{Add}(\kappa, \eta)$. Notice that $\mathrm{Add}(\kappa, \eta)$ is a complete subordering of $\mathrm{Add}(\kappa, \lambda)$.

It is easy to see that $\mathrm{Add}(\kappa, \lambda)$ is canonically κ-directed closed, and a standard Δ-system argument (see [7]) shows that $\mathrm{Add}(\kappa, \lambda)$ is $(2^{<\kappa})^+$-Knaster. It turns out that if we want to add subsets to κ in a model where $2^{<\kappa}$ is big, via a forcing which has some reasonable chain condition, then it can be helpful to look at Cohen forcing defined in some inner model where $2^{<\kappa}$ is small. This is the motivation behind the following lemma, which is implicit in [1].

Lemma 2.6 (Abraham). *Let $\tau < \kappa$ and assume that $V \models$ "τ is regular" and $V \models$ "κ is inaccessible." Let $\mathbb{P} = \mathrm{Add}(\tau, \eta)$. Let $W \supseteq V$ be a model of ZFC such that*

1. *κ and τ are still cardinals in W.*
2. *If $X \in W$ is a set of ordinals such that $W \models |X| < \kappa$, then there is $Y \supseteq X$ such that $Y \in V$ and $V \models |Y| < \kappa$.*

Then $\mathbb{P}$ has the κ-Knaster property in W.

Proof (Sketch). Working in W, mimic the usual Δ-system argument. Use the inaccessibility of κ in V and the "covering" property to show that the argument goes through. ∎

2.4. *Projections*

Another key idea in [1] is to use the concept of a *projection* from one forcing to another. We collect some more or less standard information about projections in a series of definitions and lemmas. The proofs are routine.

DEFINITION 2.7. Let $\mathbb{P}$ and $\mathbb{Q}$ be forcings. Then $\pi\colon \mathbb{P} \to \mathbb{Q}$ is a *projection* if and only if

1. $p_1 \leqslant p_0 \Rightarrow \pi(p_1) \leqslant \pi(p_0)$.
2. $\pi(1_{\mathbb{P}}) = 1_{\mathbb{Q}}$.
3. $\forall p \in \mathbb{P}\ \forall q \leqslant \pi(p)\ \exists p_1 \leqslant p\ \pi(p_1) \leqslant q$.

Notice that π "$\mathbb{P}$ is dense in $\mathbb{Q}$, by setting $p = 1_{\mathbb{P}}$ in clause 3.

LEMMA 2.8. *Let $\pi\colon \mathbb{P} \to \mathbb{Q}$ be a projection. Then*

1. *If G is $\mathbb{P}$-generic over V, then $H = \{q : \exists p \in G\ \pi(p) \leqslant q\}$ is $\mathbb{Q}$-generic over V.*

2. *Let H be $\mathbb{Q}$-generic over V. Let $\bar{\mathbb{P}} = \{p : \pi(p) \in H\}$, ordered as a suborder of $\mathbb{P}$. Then $\bar{\mathbb{P}}$ is nonempty, and if G is $\bar{\mathbb{P}}$-generic over $V[H]$ then G is $\mathbb{P}$-generic over V. Moreover, π"G generates H.*

3. *Let G be $\mathbb{P}$-generic, and define H as in* 1 *and then $\bar{\mathbb{P}}$ as in* 2. *Then $G \subseteq \bar{\mathbb{P}}$, and G is $\bar{\mathbb{P}}$-generic over $V[H]$. That is, we can factor forcing with $\mathbb{P}$ as forcing with $\mathbb{Q}$ followed by forcing with $\bar{\mathbb{P}}$ over $V[\mathbb{Q}]$.*

Some projections have a stronger property, which enables us to give a slightly different factor analysis.

DEFINITION 2.9. Let $\mathbb{P}$ and $\mathbb{Q}$ be forcings. Then $\pi\colon \mathbb{P} \to \mathbb{Q}$ is a *good projection* if and only if

1. $p_1 \leqslant p_0 \Rightarrow \pi(p_1) \leqslant \pi(p_0)$.
2. $\pi(1_{\mathbb{P}}) = 1_{\mathbb{Q}}$.
3. For all $p \in \mathbb{P}$ and $q \leqslant \pi(p)$ there is $p_1 \leqslant p$ such that
   (a) $\pi(p_1) = q$.
   (b) For all $r \leqslant p$, if $\pi(r) \leqslant q$ then $r \leqslant p_1$.

Notice that the p_1 in part 3 of the definition is essentially unique, because if p_1' also has these properties then it is easy to see that $p_1 \leqslant p_1' \leqslant p_1$. We let $\mathrm{Ext}(p, q)$ denote some extension of p with these properties.

LEMMA 2.10. *Let $\pi\colon \mathbb{P} \to \mathbb{Q}$ be a good projection and suppose that H is $\mathbb{Q}$-generic. Let $\bar{\mathbb{P}} = \{p \in \mathbb{P} : \pi(p) \in H\}$ and define an ordering $\leqslant^*$ on $\bar{\mathbb{P}}$ by*

$$p \leqslant^* q \Leftrightarrow \exists r \leqslant \pi(p)(r \in H \wedge \mathrm{Ext}(p, r) \leqslant q).$$

Then forcing over $V[H]$ with $\bar{\mathbb{P}}$ ordered as a subset of $\mathbb{P}$ is equivalent to forcing over $V[H]$ with $\bar{\mathbb{P}}$ ordered by $\leqslant^*$.

In the case which we will be using later, $\mathbb{P}$ is an iteration, $\mathbb{Q} = \mathbb{P} \upharpoonright \beta$ is an initial segment of $\mathbb{P}$, and $\pi\colon p \mapsto p \upharpoonright \beta$. In this case $\mathrm{Ext}(p, q) = q \frown (p \upharpoonright \mathrm{dom}(p) \setminus \beta)$.

Suppose that $\mathbb{P}$ and $\mathbb{Q}$ are two forcings with the same underlying set but different orderings, and suppose that the identity function projects $\mathbb{P}$ to $\mathbb{Q}$. If G is $\mathbb{P}$-generic, G is directed as a subset of $\mathbb{Q}$ and generates a generic filter H on $\mathbb{Q}$. If H is $\mathbb{Q}$-generic then forcing with H considered as a suborder of $\mathbb{P}$ produces a $\mathbb{P}$-generic G such that $G \subseteq H$, and G generates H.

It is important to notice that the constructions of this section can produce nonseparative posets.

2.5. *Easton's Lemma*

As we mentioned in Section 2.3 we will be interested in analysing the properties of forcings defined in inner models of the universe. The following lemmas will be used for this purpose.

LEMMA 2.11 (Easton's lemma). *Let κ be regular. If $\mathbb{P}$ has the κ-chain condition and $\mathbb{Q}$ is κ-closed, then*

1. $\Vdash_{\mathbb{Q}} \mathbb{P}$ *has the κ-chain condition.*

2. $\Vdash_{\mathbb{P}} \mathbb{Q}$ *is $<\kappa$-distributive.*

3. *If G is $\mathbb{P}$-generic over V and H is $\mathbb{Q}$-generic over V, then G and H are mutually generic.*

4. *With the same assumptions as the last claim, if $X \in V[G][H]$ is a set of ordinals and $V[G][H] \models |X| < \kappa$, then there is $Y \supseteq X$ in V such that $V \models |Y| < \kappa$.*

5. *If $\mathbb{R}$ is κ-closed, then $\Vdash_{\mathbb{P} \times \mathbb{Q}} \mathbb{R}$ is $<\kappa$-distributive.*

Proof (Sketch). We take each claim in turn:

1. If $\dot{A}$ is a $\mathbb{Q}$-name for an antichain of size κ in $\mathbb{P}$, we can build a decreasing sequence of length κ in $\mathbb{Q}$ such that element α of the chain decides element α in the antichain. This gives us an antichain of size κ in V, contradiction.

2. If X is a sequence of ordinals of length less than κ in $V[\mathbb{Q} \times \mathbb{P}]$ then the last claim implies that X has a name $\dot{t}$ of size less than κ in $V[\mathbb{Q}]^{\mathbb{P}}$. $\mathbb{Q}$ is κ-closed so $\dot{t} \in V$ and $X \in V[\mathbb{P}]$.

3. Every maximal antichain of $\mathbb{P}$ in $V[H]$ has size less than κ, so is in V.

4. $X \in V[G]$ and $\mathbb{P}$ is κ-c.c.

5. $\mathbb{R} \times \mathbb{Q}$ is $<\kappa$-distributive in $V[\mathbb{P}]$. ∎

We make the remark that claims 4 and 5 will actually be true for any extension intermediate between V and $V[\mathbb{P} \times \mathbb{Q}]$. The next lemma is an easy result with the same flavour as Easton's lemma.

LEMMA 2.12. *Let $\mathbb{P}$ be κ-closed and let $\mathbb{Q}$ be $<\kappa$-distributive, then*

1. $\Vdash_{\mathbb{Q}} \mathbb{P}$ *is κ-closed.*
2. $\Vdash_{\mathbb{P}} \mathbb{Q}$ *is $<\kappa$-distributive.*

The final lemma is rather ad hoc and will be used to propagate some inductive claims in the main construction.

LEMMA 2.13. *Let τ be regular, and let $\mathbb{A} = \mathrm{Add}(\tau, \eta)$ for some η. Let κ be inaccessible with $\tau < \kappa$. Then*

1. *If $\mathbb{Q}$ is κ-c.c. and $\mathbb{Q}$ is a projection of $\mathbb{P} \times \mathbb{U}$, where $\mathbb{P}$ is τ-c.c. and $\mathbb{U}$ is τ-closed, then $V[\mathbb{Q}] \models$ "$\mathbb{A}$ is κ-Knaster and $<\tau$-distributive."*

2. *Suppose that $V[\mathbb{Q}] \models$ "$\mathbb{Q}^*$ is a projection of $\mathrm{Add}(\tau, \zeta)_V \times \mathbb{U}^*$" and also that $V[\mathbb{Q}] =$ "$\mathbb{U}^*$ is κ-closed." Then $V[\mathbb{Q} * \dot{\mathbb{Q}}^*] \models$ "$\mathbb{A}$ is κ-Knaster."*

Proof. A series of routine applications of Easton's lemma and Lemma 2.6. ∎

2.6. *Elementary Embeddings*

We will assume familiarity with the theory of large cardinals and elementary embeddings, as developed, for example, in [5]. The following fact was proved by Laver in [10] and used by him there as a kind of prediction principle in a forcing iteration. Our application will be similar; we will use it to build some degree of "look-ahead" into the main construction.

LEMMA 2.14 (Laver). *If κ is supercompact then there exists $f: \kappa \to V_\kappa$ with the following property: for all λ, for all $x \in H_{\lambda^+}$, there is $j: V \to M$ such that $j(\kappa) > \lambda$, ${}^\lambda M \subseteq M$, and $j(f)(\kappa) = x$.*

We will frequently be interested in lifting elementary embeddings onto some generic extension. The following lemma gives a necessary and sufficient condition for this to be possible.

LEMMA 2.15 (Silver). *Let $k: M \to N$ be an elementary embedding between inner models of ZFC. Let $\mathbb{P} \in M$ be a forcing and suppose that G is $\mathbb{P}$-generic over M, H is $k(\mathbb{P})$-generic over N, and*

$$k``G \subseteq H.$$

Then there is a unique $k^+: M[G] \to N[H]$ such that $k^+ \restriction M = k$ and $k^+(G) = H$.

Proof. It is clear that if k^+ exists, it must be given by the formula

$$k^+(\dot{t}^G) = k(\dot{t})^H$$

for $\dot{t} \in M^{\mathbb{P}}$. It is fairly routine to use the truth lemma and elementarity to check that this definition works. ▮

It is important to notice that in this result it need not be the case that $M = V$, that $N \subseteq M$, or that $H \in M[G]$. We also note a connection with the theory [12] of proper forcing; if $\mathbb{P} \in N \prec H_\theta$ and $\pi: N \to \bar{N}$ is the transitive collapse map, then $p \in \mathbb{P}$ is $(N, \mathbb{P})$-generic precisely when it forces that $G^* =_{\text{def}} \pi``G_{\mathbb{P}} \cap N$ is $\pi(\mathbb{P})$-generic over $\bar{N}$, and in this case we can lift π^{-1} to get an embedding from $\bar{N}[G^*]$ to $N[G]$.

PART 1. THE TREE PROPERTY FOR ω SUCCESSIVE CARDINALS

In this part of the paper we will prove Theorem 1.

3. THE MAIN FORCING

3.1. *Defining $\mathbb{R}$*

Before we define the main forcing, a few words of motivation may be in order. The principal aim of the main forcing is to take cardinals $\tau < \kappa$ with τ regular and κ supercompact, and (while preserving all cardinals up to τ^+) to force that $2^\tau = \kappa = \tau^{++}$ and that κ retains the tree property. We aim to iterate the main forcing in a certain way, and stage $n+1$ of the iteration will be defined using some forcing computed at stage n; this explains why an inner model is one of the parameters in the definition of the main forcing. The main forcing is also designed to make κ have the tree property in a very "indestructible" way, so the last parameter in the definition is a function from κ to V_κ whose role is to guess some information about subsequent forcing extensions.

Definition 3.1. Let $V \subseteq W$ be two models of set theory and suppose that we have τ, κ such that $W \models$ "$\tau = \mathrm{cf}(\tau) < \kappa$ and κ is inaccessible." Let $\mathbb{P} = \mathrm{Add}(\tau, \kappa)_V$ and suppose also that $W \models$ "$\mathbb{P}$ is τ^+-c.c. and $<\tau$-distributive." Let $F \in W$ be a function with $F: \kappa \to (V_\kappa)_W$. Define in W a forcing

$$\mathbb{R} = \mathbb{R}(\tau, \kappa, V, W, F)$$

as follows. The definition is by induction; for each $\beta \leqslant \kappa$ we will define a forcing $\mathbb{R} \upharpoonright \beta$, and we will finally set $\mathbb{R} = \mathbb{R} \upharpoonright \kappa$. To start the induction, $\mathbb{R} \upharpoonright 0$ is the trivial forcing. (p, q, f) is a condition in $\mathbb{R} \upharpoonright \beta$ iff

1. $p \in \mathbb{P} \upharpoonright \beta = \mathrm{Add}(\tau, \beta)_V$.
2. q is a partial function on β, $|q| \leqslant \tau$, $\mathrm{dom}(q)$ consists of successor ordinals, and if $\alpha \in \mathrm{dom}(q)$ then $q(\alpha) \in W^{\mathbb{P} \upharpoonright \alpha}$ and $\Vdash^W_{\mathbb{P} \upharpoonright \alpha} q(\alpha) \in \mathrm{Add}(\tau^+, 1)_{W[\mathbb{P} \upharpoonright \alpha]}$.
3. f is a partial function on β, $|f| \leqslant \tau$, $\mathrm{dom}(f)$ consists of limit ordinals, and $\mathrm{dom}(f)$ is a subset of

$$\{\alpha : \Vdash^W_{\mathbb{R} \upharpoonright \alpha} F(\alpha) \text{ is a canonically } \tau^+\text{-directed-closed forcing}\}.$$

4. If $\alpha \in \mathrm{dom}(f)$ then $f(\alpha) \in W^{\mathbb{R} \upharpoonright \alpha}$ and $\Vdash^W_{\mathbb{R} \upharpoonright \alpha} f(\alpha) \in F(\alpha)$.

The conditions in $\mathbb{R} \upharpoonright \beta$ are ordered

$$(p_1, q_1, f_1) \leqslant (p, q, f)$$

if and only if

1. $p_1 \leqslant p$ in $\mathbb{P} \upharpoonright \beta$.
2. For all $\alpha \in \mathrm{dom}(f)$, $p_1 \upharpoonright \alpha \Vdash^W_{\mathbb{P} \upharpoonright \alpha} q_1(\alpha) \leqslant q(\alpha)$.
3. For all $\alpha \in \mathrm{dom}(q)$, $(p_1, q_1, f_1) \upharpoonright \alpha \Vdash^W_{\mathbb{R} \upharpoonright \alpha} f_1(\alpha) \leqslant f(\alpha)$.

It is important to notice that if $(p, q, f) \in \mathbb{R}$ and $\alpha < \kappa$ then the term $q(\alpha)$ depends only on $\mathbb{P} \upharpoonright \alpha$, while $f(\alpha)$ depends on $\mathbb{R} \upharpoonright \alpha$. It is also important that the definition is made in the model W, and the only appearance of V is as the model where the Cohen forcing $\mathbb{P}$ is to be defined.

One should think of $\mathbb{R}$ as aiming to add several objects to W:

1. A $\mathbb{P}$-generic object.
2. For each successor α, an $\mathrm{Add}(\tau^+, 1)_{W[\mathbb{P} \upharpoonright \alpha]}$-generic object over $W[\mathbb{P} \upharpoonright \alpha]$.
3. For each appropriate limit α, an $F(\alpha)$-generic object over $W[\mathbb{R} \upharpoonright \alpha]$.

We need to analyze the properties of $\mathbb{R}$, which we will do in a series of lemmas.

3.2. *Easy Properties of* $\mathbb{R}$

LEMMA 3.2. *$|\mathbb{R}| = \kappa$, and $\mathbb{R}$ has the κ-Knaster property.*

Proof. An easy argument counting terms shows that $|\mathbb{R}| = \kappa$, the key point being that at each α there are fewer than κ possibilities for $q(\alpha)$ or $f(\alpha)$. A standard Δ-system argument then shows that $\mathbb{R}$ has the κ-Knaster property. ∎

LEMMA 3.3. *$\mathbb{R}$ can be projected to $\mathbb{P}$, $\mathbb{R} \upharpoonright \alpha * F(\alpha)$, and $\mathbb{P} \upharpoonright \alpha * \mathrm{Add}(\tau^+, 1)_{W[\mathbb{P} \upharpoonright \alpha]}$.*

Proof. Define projection maps

$$\pi_0 \colon (p, q, f) \mapsto p$$
$$\pi_1 \colon (p, q, f) \mapsto ((p, q, f) \upharpoonright \alpha, f(\alpha))$$
$$\pi_2 \colon (p, q, f) \mapsto (p \upharpoonright \alpha, q(\alpha)).$$

It is routine to check that these are projections (this was, in fact, one motivation for the definition of $\mathbb{R}$). ∎

It follows that if G is $\mathbb{R}$-generic then G induces a $\mathbb{P}$-generic object over V, g say. G also induces an $\mathrm{Add}(\tau^+, 1)_{V[g \upharpoonright \alpha]}$-generic object over $V[g \upharpoonright \alpha]$ and a $F(\alpha)^{G \upharpoonright \alpha}$-generic object over $V[G \upharpoonright \alpha]$.

LEMMA 3.4. *$\mathbb{R}$ adds at least κ subsets to τ.*

Proof. $\mathbb{R}$ projects to $\mathbb{P}$, $\mathbb{P}$ adds at least κ subsets to τ. ∎

LEMMA 3.5. *$\mathbb{R}$ collapses every cardinal between τ^+ and κ to τ^+.*

Proof. Let α be such a cardinal. $\mathbb{R}$ projects to a forcing which makes $2^\tau \geqslant \alpha$ and then adds a Cohen subset of τ^+. This forcing will collapse α to τ^+. ∎

LEMMA 3.6. *Let $\eta \leqslant \tau$. If $\mathbb{P}$ is η-closed in W then $\mathbb{R}$ is η-closed in W.*

Proof. Let $\mu < \eta$ and suppose that $\langle (p_\alpha, a_\alpha, f_\alpha) : \alpha < \mu \rangle$ is a decreasing sequence in $\mathbb{R}$. By hypothesis we can find $p \in \mathbb{P}$ such that $p \leqslant p_\alpha$ for all $\alpha < \mu$. It is easy to see that for each β the condition $p \restriction \beta$ forces the sequence $\langle q_\alpha(\beta) : \alpha < \mu \rangle$ to be decreasing in $\mathrm{Add}(\tau^+, 1)_{W[\mathbb{P} \restriction \beta]}$; $\mathrm{Add}(\tau^+, 1)_{W[\mathbb{P} \restriction \beta]}$ is τ^+-closed in $W[\mathbb{P} \restriction \beta]$ so we may choose a $\mathbb{P} \restriction \beta$-term $q(\beta)$ which is forced by $p \restriction \beta$ to be a lower bound. The domain of q has cardinality at most τ.

Now we will define $f(\beta)$ by induction on β in such a way that $(p, q, f) \restriction \beta$ is a lower bound for $\langle (p_\alpha, q_\alpha, f_\alpha) \restriction \beta : \alpha < \mu \rangle$. If we have this up to stage β then $(p, q, f) \restriction \beta$ forces that $\langle f_\alpha(\beta) : \alpha < \mu \rangle$ is decreasing in $F(\beta)$; $F(\beta)$ is τ^+-closed in $W[\mathbb{R} \restriction \beta]$, so we may choose a $\mathbb{R} \restriction \beta$-term $f(\beta)$ which is forced by $(p, q, f) \restriction \beta$ to be a lower bound. The domain of f will also have cardinality at most τ, so the induction can proceed and at the end (p, q, f) is the lower bound that we require. ∎

To sum up, we have seen that $\mathbb{R}$ preserves cardinals greater than or equal to κ, collapses all cardinals strictly between τ^+ and κ, and adds κ subsets to τ. To understand what is happening in $W[\mathbb{R}]$ at τ^+ and below, we will analyze $\mathbb{R}$ as the projection of $\mathbb{P} \times \mathbb{U}$ for some τ^+-closed forcing U.

3.3. *Factoring* $\mathbb{R}$

DEFINITION 3.7. With the same data as for $\mathbb{R}$, $\mathbb{U} = \mathbb{U} = \mathbb{U}(\tau, \kappa, V, W, F)$ is the ordering with underlying set $\{(0, q, f) : (0, q, f) \in \mathbb{R}\}$ and the partial ordering inherited from $\mathbb{R}$.

LEMMA 3.8. *$\mathbb{U}$ has the κ-chain condition.*

Proof. A standard Δ-system argument. ∎

LEMMA 3.9. *In W, $\mathbb{U}$ is a canonically τ^+-directed closed poset.*

Proof. We write out explicitly the definition of the ordering on $\mathbb{U}$. $(0, q_1, f_1) \leqslant (0, q_0, f_0)$ if and only if

1. $\mathrm{dom}(q_0) \subseteq \mathrm{dom}(q_1)$, and $\models^W_{\mathbb{P} \restriction \alpha} q_1(\alpha) \leqslant q_0(\alpha)$ for all $\alpha \in \mathrm{dom}(q_0)$.
2. $\mathrm{dom}(f_0) \subseteq \mathrm{dom}(f_1)$, and $(0, q_1, f_1) \restriction \alpha \models^W_{\mathbb{R} \restriction \alpha} f_1(\alpha) \leqslant f_0(\alpha)$ for all $\alpha \in \mathrm{dom}(f_0)$.

Let $\{(0, q_\eta, f_\eta) : \eta < \tau\}$ be a directed set of conditions. Let us define $A_1 =_{\mathrm{def}} \bigcup_{\eta < \tau} \mathrm{dom}(q_\eta)$, and observe that $|A_1| \leqslant \tau$. We will define a function q with domain A_1. For $\alpha \in A_1$, consider $\{q_\eta(\alpha) : \eta < \tau\}$. If $\eta, \zeta < \tau$ then for some $\mu < \tau$ we have that $(0, q_\mu, f_\mu)$ is a common refinement of $(0, q_\eta, f_\eta)$ and $(0, q_\zeta, f_\zeta)$. In particular, $\Vdash q_\mu(\alpha) \leqslant q_\eta(\alpha), q_\zeta(\alpha)$. So we can look at

$\{q_\eta(\alpha) : \eta < \tau\}$ as a name in $W^{\mathbb{P} \upharpoonright \alpha}$ for a directed set of size τ in $\mathrm{Add}(\tau^+, 1)_{W[\mathbb{P} \upharpoonright \alpha]}$, and find $r(\alpha)$ which is forced to be the greatest lower bound. In particular, $\Vdash r(\alpha) \leqslant q_\eta(\alpha)$ for all $\eta < \tau$.

Now let $A_2 =_{\mathrm{def}} \bigcup_{\eta < \tau} \mathrm{dom}(f_\eta)$, and observe that $|A_2| \leqslant \tau$. We will define by induction on α a function g with domain A_2 such that $(0, r, g) \upharpoonright \alpha \Vdash g(\alpha) \leqslant f_\eta(\alpha)$ for all α, η. Fix α. As we remarked already, if $\eta, \zeta < \tau$ then for some $\mu < \tau$ we have that $(0, q_\mu, f_\mu)$ is a common refinement of $(0, q_\eta, f_\eta)$ and $(0, q_\zeta, f_\zeta)$; in particular $(0, q_\mu, f_\mu) \upharpoonright \alpha \Vdash f_\mu(\alpha) \leqslant f_\eta(\alpha), f_\zeta(\alpha)$. By induction $(0, r, g) \upharpoonright \alpha \leqslant (0, q_\mu, f_\mu) \upharpoonright \alpha$, so $((0, r, g) \upharpoonright \alpha \Vdash f_\mu(\alpha) \leqslant f_\eta(\alpha), f_\zeta(\alpha)$. Now $(0, r, g) \upharpoonright \alpha$ forces that $\{f_\eta(\alpha) : \eta < \tau\}$ is directed. We define $g(\alpha)$ rather carefully to be a name which denotes the greatest lower bound of $\{f_\eta(\alpha) : \eta < \tau\}$ if that set is directed, and the trivial condition otherwise. In particular $(0, r, g) \upharpoonright \alpha \Vdash g(\alpha) \leqslant f_\eta(\alpha)$ for all η. At the end we have constructed a condition $(0, r, g)$ which is a lower bound for the directed set $\{(0, q_\eta, f_\eta) : \eta < \tau\}$.

It remains to be seen that $(0, r, g)$ is a greatest lower bound. Let $(0, s, h)$ be any condition such that for all η $(0, s, h) \leqslant (0, q_\eta, f_\eta)$. Clearly $A_1 \subseteq \mathrm{dom}(s)$ and $A_2 \subseteq \mathrm{dom}(h)$. For each $\alpha \in \mathrm{dom}(s)$ we have $\Vdash s(\alpha) \leqslant f_\eta(\alpha)$ for all η and, since $r(\alpha)$ is forced to be a greatest lower bound, $\Vdash s(\alpha) \leqslant r(\alpha)$. We attempt to show by induction that $(0, s, h) \upharpoonright \alpha \Vdash h(\alpha) \leqslant g(\alpha)$. If it is true below α then $(0, s, h) \upharpoonright \alpha \leqslant (0, r, g) \upharpoonright \alpha$, so that $(0, s, h) \upharpoonright \alpha$ forces that $\{f_\eta(\alpha) : \eta < \tau\}$ is directed. It also follows from the hypothesis that $(0, s, h) \upharpoonright \alpha \Vdash h(\alpha) \leqslant f_\eta(\alpha)$ for all η, so that by our choice of $g(\alpha)$, $(0, s, h) \upharpoonright \alpha \Vdash h(\alpha) \leqslant g(\alpha)$. So the induction goes through, and at the end we have shown $(0, s, h) \leqslant (0, r, g)$. Hence, $(0, r, g)$ is the greatest lower bound. ▮

LEMMA 3.10. *In $\mathbb{R}$, (p, q, f) is the greatest lower bound for $(p, 0, 0)$ and $(0, q, f)$.*

Proof. (p, q, f) is clearly a lower bound. Suppose that (p_1, q_1, f_1) is also a lower bound. Then by definition $p_1 \leqslant p$, $p_1 \upharpoonright \alpha \Vdash q_1(\alpha) \leqslant q(\alpha)$, and $(p_1, q_1, f_1) \upharpoonright \alpha \Vdash f_1(\alpha) \leqslant f(\alpha)$. That is to say, $(p_1, q_1, f_1) \leqslant (p, q, f)$. ▮

LEMMA 3.11. *With $\mathbb{P}$ and $\mathbb{U}$ as above*

1. *$\mathbb{P} \times \mathbb{U}$ is κ-c.c.*
2. *All τ-sequences of ordinals in $W[\mathbb{P} \times \mathbb{U}]$ are in $W[\mathbb{P}]$.*

Proof. By Easton's lemma, $\mathbb{P}$ is τ^+-c.c. in $W[\mathbb{U}]$. Since $\mathbb{U}$ is κ-c.c. in W, $\mathbb{P} \times \mathbb{U}$ is κ-c.c. Easton's lemma also shows that all τ-sequences of ordinals from $W[\mathbb{P} \times \mathbb{U}]$ are in $W[\mathbb{P}]$. ▮

DEFINITION 3.12. $\pi\colon \mathbb{P}\times\mathbb{U}\to\mathbb{R}$ is the function given by

$$\pi\colon (p, (0, q, f))\mapsto(p, q, f).$$

LEMMA 3.13. *π is a projection.*

Proof. It is clear that π preserves the identity and respects the ordering relation.

Let us have $(p_1, q_1, f_1)\leqslant(p_0, q_0, f_0)$ in $\mathbb{R}$. Observe that for all α we have $p_1\upharpoonright\alpha\Vdash q_1(\alpha)\leqslant q_0(\alpha)$. Define $\bar{q}(\alpha)$ as a name with the following property; for G any $\mathbb{P}\upharpoonright\alpha$-generic object, $\bar{q}(\alpha)$ interprets as $q_1(\alpha)^G$ if $p_1\upharpoonright\alpha\in G$, and interprets as $q_0(\alpha)^G$ if $p_1\upharpoonright\alpha\notin G$. By construction, $\Vdash\bar{q}(\alpha)\leqslant q_0(\alpha)$ and $p_1\upharpoonright\alpha\Vdash\bar{q}(\alpha)=q_1(\alpha)$.

Now we attempt to define by induction a term $\bar{f}(\alpha)$ such that $(p_1, \bar{q}, \bar{f})\upharpoonright\alpha\Vdash\bar{f}(\alpha)=f_1(\alpha)$ and $(0, \bar{q}, \bar{f})\upharpoonright\alpha\Vdash\bar{f}(\alpha)\leqslant f_0(\alpha)$. If we have done this for stages below α, then the conditions $(p_1, \bar{q}, \bar{f})\upharpoonright\alpha$ and $(p_1, q_1, f_1)\upharpoonright\alpha$ are equivalent in $\mathbb{R}\upharpoonright\alpha$.

By hypothesis, $(p_1, q_1, f_1)\upharpoonright\alpha\Vdash f_1(\alpha)\leqslant f_0(\alpha)$. Define $\bar{f}(\alpha)$ as follows: for any $\mathbb{R}\upharpoonright\alpha$-generic object G, the interpretation of $\bar{f}(\alpha)$ is $f_1(\alpha)^G$ if $(p_1, \bar{q}, \bar{f})\upharpoonright\alpha\in G$ and $f_0(\alpha)^G$ otherwise. Now $(p_1, \bar{q}, \bar{f})\upharpoonright\alpha\Vdash\bar{f}(\alpha)=f_1(\alpha)$, and $\Vdash\bar{f}(\alpha)\leqslant f_0(\alpha)$, so we are done.

At the end of this construction we have shown that the conditions $(p_1, \bar{q}, \bar{f})$ and (p_1, q_1, f_1) are equivalent in $\mathbb{R}$ and $(0, \bar{q}, \bar{f})\leqslant(0, q_0, f_0)$, which is what is needed. ∎

Recall that we also have projections $\rho\colon\mathbb{R}\to\mathbb{P}$ and $\sigma\colon\mathbb{P}\times\mathbb{U}\to\mathbb{P}$ given by $\rho\colon(p, q, f)\to p$ and $\sigma\colon(p, (0, q, f))\mapsto p$. These projections commute as in Fig. 1. Consequently, we may take it that $W\subseteq W[\mathbb{R}]\subseteq W[\mathbb{P}\times\mathbb{U}]$.

COROLLARY 3.14. *Let G be $\mathbb{R}$-generic over W, let g be the $\mathbb{P}$-generic object added by G. If $X\in W[G]$ is a set of ordinals of size τ, then $X\in W[g]$.*

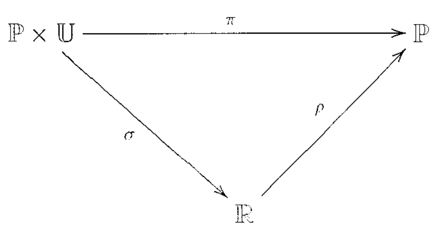

FIG. 1. Projections between $\mathbb{P}\times\mathbb{U}$, $\mathbb{R}$, and $\mathbb{P}$.

Proof. By Lemma 3.13 we can embed $W[G]$ in a larger extension $W[g][H]$, where H is $\mathbb{U}$-generic over $W[g]$. As $\mathbb{P}$ is τ^+-c.c. and $\mathbb{U}$ is τ^+-closed, Easton's lemma implies that $X \in W[g]$. ∎

Corollary 3.15. *$\mathbb{R}$ preserves τ^+ and forces that $2^\tau = \kappa = \tau^{++}$.*

Proof. If $f \in W[G]$ and $f\colon \tau \to \tau^+_W$ then by Corollary 3.14 $f \in W[g]$ and g is generic for τ^+-c.c. forcing so that f is bounded. Similarly, $\mathscr{P}\tau \cap W[G] = \mathscr{P}\tau \cap W[g]$, and standard arguments show that $2^\tau = \kappa$ in $W[g]$. Finally, we saw in Lemma 3.5 that $\mathbb{R}$ collapses everything in the interval $[\tau^+, \kappa)$ to τ^+. ∎

Corollary 3.16. *$\mathbb{R}$ is $<\tau$-distributive in W, so it preserves cardinals less than or equal to τ.*

Proof. If $X \in W[G]$ is a set of ordinals of size less than τ, then $X \in W[g]$ by Corollary 3.14. But $\mathbb{P}$ is $<\tau$–distributive in W, so $X \in W$. ∎

Corollary 3.17. *$\mathbb{U}$ is κ-c.c. and τ^+-closed. In $W[\mathbb{U}]$ we have $\tau^{++} = \kappa$.*

Proof. We already saw the closure and chain condition results. $\mathbb{U}$ must collapse κ to τ^{++} because $W[\mathbb{U}] \Vdash$ "$\mathbb{P}$ is τ^+-c.c. and $<\tau$-distributive," and κ is collapsed to τ^{++} in $W[\mathbb{U} \times \mathbb{P}]$. ∎

Corollary 3.18. *If $X \in W[G]$ is a set of ordinals of size τ, then X is covered by a set of size τ in W.*

Proof. $X \in W[g]$ and $\mathbb{P}$ is τ^+-c.c. in W, so the claim follows by a standard chain condition argument. ∎

3.4. *The Forcing $\mathbb{S}$*

$W[\mathbb{P} \times \mathbb{U}]$ is a generic extension of $W[\mathbb{R}]$, and we need some information about this extension.

Definition 3.19. Let G be $\mathbb{R}$-generic over W. Then define in $W[G]$ a forcing $\mathbb{S} = \mathbb{S}(\tau, \kappa, V, W, F, G)$ whose conditions are $\{(p, (0, q, f)) : (p, q, f) \in G\}$, ordered as a suborder of $\mathbb{P} \times \mathbb{U}$.

$\mathbb{S}$ is a version of the forcing to expand G to an $\mathbb{P} \times \mathbb{U}$-generic object which projects to G. We need to analyze the properties of $\mathbb{S}$. For a careful analysis of this sort of forcing, in a slightly different setting, we refer the reader to Foreman's paper [3].

LEMMA 3.20. *Let G be generic for $\mathbb{R}$ and let $\mathbb{S} =_{\mathrm{def}} \mathbb{S}(\tau, \kappa, V, W, F, G)$. Then $W[G] \Vdash$ "$\mathbb{S}$ is $<\tau^+$-distributive, τ-closed and κ-c.c." In particular forcing with $\mathbb{S}$ over $W[G]$ does not collapse cardinals.*

Proof. The κ-chain condition and $<\tau^+$-distributivity follow at once from our earlier remarks about $\mathbb{P} \times \mathbb{U}$. It remains to show τ-closure.

In $W[G]$, let $\langle (p_\zeta, (0, q_\zeta, f_\zeta)) : \zeta < \mu \rangle$ be a decreasing μ-sequence of conditions from $\mathbb{S}$ for some $\mu < \tau$. By Corollary 3.16, this sequence is a member of W. Let $p = \bigcup_\zeta p_\zeta$, then (by Lemma 2.2 or elementary facts about Cohen forcing) $p \in g$, where g is the $\mathbb{P}$-generic added by $\mathbb{R}$.

Since the sequence $\langle (0, q_\zeta, f_\zeta) : \zeta < \mu \rangle$ is decreasing in $\mathbb{U}$, we may perform the construction of Lemma 3.9 to get $(0, \bar{q}, \bar{f})$ which is a greatest lower bound in $\mathbb{U}$ for this sequence. We now claim that $(p, \bar{q}, \bar{f}) \in G$.

We already know that $p \in g$. Fix a successor $\alpha < \kappa$, then by the definition of $\bar{q}(\alpha)$ we see that $\bar{q}^{g \restriction \alpha}$ is the greatest lower bound for the sequence $\langle q_\zeta(\alpha)^{g \restriction \alpha} : \zeta < \mu \rangle$ in the forcing $\mathrm{Add}(\tau^+, 1)_{W[g \restriction \alpha]}$. If we let G^0_α be the $\mathrm{Add}(\tau^+, 1)_{W[g \restriction \alpha]}$-generic filter added by G then we know that $q_\zeta(\alpha)^{g \restriction \alpha} \in G^0_\alpha$ for all ζ, and so by another application of Lemma 2.2, $\bar{q}(\alpha)^{g \restriction \alpha} \in G^0_\alpha$.

For each relevant limit $\alpha < \kappa$ let G^1_α denote the $F(\alpha)^{G \restriction \alpha}$-generic filter added by G. We will prove by induction on α that $\bar{f}(\alpha)^{G \restriction \alpha} \in G^1_\alpha$. Suppose that we have done this up to stage α, so that in particular $(0, \bar{q}, \bar{f}) \restriction \alpha \in G \restriction \alpha$. Since $(0, \bar{q}, \bar{f})$ is a lower bound for $\langle (0, q_\zeta, f_\zeta) : \zeta < \mu \rangle$ the condition $(0, \bar{q}, \bar{f}) \restriction \alpha$ forces that $\langle f_\zeta(\alpha) : \zeta < \mu \rangle$ is decreasing, so that $\langle f_\zeta(\alpha)^{G \restriction \alpha} \rangle$ is a decreasing sequence of members of G^1_α. What is more $\bar{q}(\alpha)^{G \restriction \alpha}$ is the greatest lower bound of this sequence, so that applying Lemma 3.9 one more time $\bar{q}(\alpha)^{G \restriction \alpha} \in G^1_\alpha$ and we are done. ∎

3.5. $\mathbb{R}^*$ *and* $\mathbb{U}^*$

To conclude the analysis of $\mathbb{R}$, we need to look at the forcing obtained when we factor $\mathbb{R}$ over one of its initial segments. Fix $\beta < \kappa$ and consider the projection $\pi_\beta : \mathbb{R} \to \mathbb{R} \restriction \beta$ given by restriction. It is easy to see that this is a good projection, in particular if G_β is $\mathbb{R} \restriction \beta$-generic then we may consider the forcing to prolong G_β to an $\mathbb{R}$-generic object as given by the following definition.

DEFINITION 3.21. Given $\beta < \kappa$ and G_β generic over W for $\mathbb{R}$, define

$$\mathbb{R}^* = \mathbb{R}^*(\tau, \kappa, V, W, F, G_\beta) = \{ r \in \mathbb{R} : r \restriction \beta \in G_\beta \},$$

with the ordering on $\mathbb{R}^*$ given by $r_1 \leqslant r_0 \Leftrightarrow \exists s \in G_\beta\, \mathrm{Ext}(r_1, s) \leqslant r_0$. We observe that here $\mathrm{Ext}(r_1, s)$ is just the extension of r_1 in which $r_1 \restriction \beta$ is replaced by s.

DEFINITION 3.22. Given β and G_β as above, define

$$\mathbb{U}^* = \mathbb{U}^*(\tau, \kappa, V, W, F, G_\beta) = \{(0, q, f) : (0, q, f) \in \mathbb{R}^*\},$$

ordered as a suborder of $\mathbb{R}^*$.

DEFINITION 3.23. Given β and G_β as above, let $\mathbb{P}^*$ be $\{p \in \mathbb{P} : (p, 0, 0) \in \mathbb{R}^*\}$, ordered as a suborder of $\mathbb{P}$.

It is easy to see that $\mathbb{P}^*$ is essentially $\mathbb{P} \upharpoonright \kappa - \beta = \mathrm{Add}(\tau, \kappa - \beta)_V$.

LEMMA 3.24. *If we define π on domain $\mathbb{P}^* \times \mathbb{U}^*$ by $\pi : (p, (0, q, f)) \mapsto (p, q, f)$, then π is a projection from $\mathbb{P}^* \times \mathbb{U}^*$ to $\mathbb{R}^*$.*

Proof. It follows from Lemma 3.10 that $(p, q, f) \in \mathbb{R}^*$. The rest is also fairly routine. ▮

LEMMA 3.25. *$\mathbb{U}^*$ is τ^+-closed in $W[G_\beta]$.*

Proof. The proof is similar to that of Lemma 2.18 in [1]. Let $\dot{\tau}$ name a descending τ-sequence in $\mathbb{U}^*$. Let g_β be the $\mathbb{P} \upharpoonright \beta$-generic object added by G_β. We may assume that $\dot{\tau} \in W^{\mathbb{P} \upharpoonright \beta}$, because all τ-sequences in $W[G_\beta]$ come from $W[g_\beta]$. $\dot{\tau}_\eta$ will denote the canonical term for entry η in the sequence named by $\dot{\tau}$. We adopt from [1] the convention that

$$L(0, q, f) = q$$
$$R(0, q, f) = f.$$

Let G_γ be $\mathbb{R} \upharpoonright \gamma$-generic, let g_γ be the associated $\mathbb{P} \upharpoonright \gamma$-generic object. Then $\dot{\tau}_\eta^{g_\gamma \upharpoonright \beta}$ is a condition in $\mathbb{U}^*$, so that $[R(\dot{\tau}_\eta^{g_\gamma \upharpoonright \beta})]^{G_\gamma} \in F(\gamma)^{G_\gamma}$. Similarly, $[L(\dot{\tau}_\eta^{g_\gamma \upharpoonright \beta})]^{g_\gamma} \in \mathrm{Add}(\tau^+, 1)_{W[g_\gamma]}$.

We will define in W a condition $(0, q^*, f^*) \in \mathbb{R}$ such that

- $\mathrm{dom}(q^*) \subseteq \kappa - \beta$, $\mathrm{dom}(f^*) \subseteq \kappa - \beta$.
- $\mathrm{dom}(q^*)$ is the set of $\gamma \geqslant \beta$ such that for some $\eta < \tau$, γ is a potential member of the domain of $L(\dot{\tau}_\eta)$.
- $\mathrm{dom}(f^*)$ is the set of $\gamma \geqslant \beta$ such that for some $\eta < \tau$, γ is potential member of the domain of $R(\dot{\tau}_\eta)$.
- For all $\gamma \geqslant \beta$, if G_γ is $\mathbb{R} \upharpoonright \gamma$-generic and $(0, q^*, f^*) \upharpoonright \gamma \in G_\gamma$, then $\forall \eta < \tau\ \dot{\tau}_\eta^{g_\gamma \upharpoonright \beta} \upharpoonright \gamma \in G_\gamma$.

We observe that by τ^+-c.c. for $\mathbb{P}$ in W, the domains are not too big. We will start by setting $(0, q^*, f^*) \upharpoonright \beta = (0, 0, 0)$. Suppose we have defined $(0, q^*, f^*) \upharpoonright \gamma$ successfully. We will now define $f^*(\gamma)$. Let G_γ be $\mathbb{R} \upharpoonright \gamma$-generic, and assume that $(0, q^*, f^*) \upharpoonright \gamma \in G_\gamma$. By our induction hypothesis, $\forall \eta < \tau\ \dot{\tau}_\eta^{g_\gamma \upharpoonright \beta} \upharpoonright \gamma \in G_\gamma$. We will work in $W[G_\gamma]$.

Claim. *Define a τ-sequence of conditions in $F(\gamma)^{G_\gamma}$ by $p(\eta) = [R(\dot{\tau}_\eta^{g_\gamma \upharpoonright \beta})]^{G_\gamma}$. Then this is a decreasing sequence.*

Proof. Let $\zeta < \eta < \tau$, and suppose that

$$\dot{\tau}_\zeta^{g_\gamma \upharpoonright \beta} = (0, q, f)$$

$$\dot{\tau}_\eta^{g_\gamma \upharpoonright \beta} = (0, \bar{q}, \bar{f}).$$

Notice that $p(\zeta) = f(\gamma)^{G_\gamma}$, $p(\eta) = \bar{f}(\gamma)^{G_\gamma}$. What is more $(0, q, f) \upharpoonright \gamma$ and $(0, \bar{q}, \bar{f}) \upharpoonright \gamma$ are in G_γ. We may choose $s \in G_\gamma \upharpoonright \beta$ such that $s \Vdash \dot{\tau}_\zeta = (0, q, f)$, $s \Vdash \dot{\tau}_\eta = (0, \bar{q}, \bar{f})$, and $\mathrm{Ext}(s, (0, \bar{q}, \bar{f}) \leqslant (0, q, f)$ in $\mathbb{R}$. Now $\mathrm{Ext}(s, (0, \bar{q}, \bar{f})) \upharpoonright \gamma \in G_\gamma$, so that (by the definition of extension in $\mathbb{R}$) $p(\eta) = \bar{f}(\gamma)^{G_\gamma} \leqslant f(\gamma)^{G_\gamma} = p(\zeta)$. This proves the claim. ▮

Now we choose $f^*(\gamma)$ to be a name, forced by $(0, q^*, f^*) \upharpoonright \gamma$ to be a lower bound for that sequence **p**. We observe for the record that if we assume $(0, q^*, f^*) \upharpoonright \gamma \in G_\gamma$, then $f^*(\gamma)^{G_\gamma} \leqslant [R(\dot{\tau}_\eta^{g_\gamma \upharpoonright \beta})]^{G_\gamma}$. The choice of $q^*(\gamma)$ is similar. Let G_γ, g_γ be as usual, where we assume that $(0, q^*, f^*) \upharpoonright \gamma \in G_\gamma$. Working in $W[g_\gamma]$ define a sequence q in $\mathrm{Add}(\tau^+, 1)$ by $q(\eta) = [L(\dot{\tau}_\eta^{g_\gamma \upharpoonright \beta})]^{g_\gamma}$. Working much as before we can show that q is decreasing. Now choose $q^*(\gamma)$ to be a $\mathbb{P} \upharpoonright \gamma$-name for a lower bound.

We check that the induction hypothesis goes through. Suppose that $(0, q^*, f^*) \upharpoonright \gamma + 1 \in G_{\gamma+1}$, and let $\eta < \tau$. Suppose that $\dot{\tau}_\eta^{g_{\gamma+1} \upharpoonright \beta} = t_\eta = (0, q_\eta, f_\eta)$. Then $t_\eta \upharpoonright \gamma \in G_\gamma$, and by construction we know that $q^*(\gamma)^{g_\gamma} \leqslant q_\eta(\gamma)^{g_\gamma}$ and $f^*(\gamma)^{G_\gamma} \leqslant q_\eta(\gamma)^{G_\gamma}$. So $t_\eta \upharpoonright \gamma + 1 \in G_{\gamma+1}$.

Limits do not present a problem, so that the construction of $(0, q^*, f^*)$ can proceed. We finish by showing that we have constructed a lower bound.

Claim. *Let G_β be $\mathbb{R} \upharpoonright \beta$-generic. Then $(0, q^*, f^*)$ is a lower bound in $\mathbb{U}^*$ for the sequence $\dot{\tau}^{g_\beta}$.*

Proof. Let $\zeta < \tau$, and suppose $\dot{\tau}_\zeta^{g_\beta} = (0, q, f)$. Choose $r \in G_\beta$ such that $r \leqslant (0, q, f) \upharpoonright \beta$ and $r \Vdash \dot{\tau}_\zeta = (0, q, f)$. Now by construction

$$r \upharpoonright \gamma \Vdash q^*(\gamma) \leqslant q(\gamma)$$

$$\mathrm{Ext}((0, q^*, f^*) \upharpoonright \gamma, r) \Vdash f^*(\gamma) \leqslant f(\gamma)$$

for each γ, so that $\mathrm{Ext}((0, q^*, f^*), r) \leqslant (0, q, f)$. ▮

With the last claim, the proof is done. ▮

4. THE FINAL MODEL

4.1. *Building the Model*

Let $\langle \kappa_n : n < \omega \rangle$ be an increasing sequence of supercompact cardinals. We will build a model by iterating the forcing described in the last section, in such a way that κ_n becomes $\aleph_{n+2}$ in the final model.

DEFINITION 4.1. Fix $\langle F_n : n < \omega \rangle$ such that $F_n : \kappa_n \to V_{\kappa_n}$ has the "diamond" property described in Lemma 2.14. We will define a forcing iteration $\mathbb{R}_\omega$ of length ω:

1. The first stage is $\mathbb{Q}_0 =_{\text{def}} \mathbb{R}(\aleph_0, \kappa_0, V, V, F_0)$, and $\mathbb{R}_1 =_{\text{def}} \mathbb{Q}_0$.

2. Define a name $\dot{F}_1$ in $V^{\mathbb{Q}_0}$ for a function from κ_1 to $(V_{\kappa_1})_{V[\mathbb{Q}_0]}$, by setting $\dot{F}_1^{G_0}(\alpha) = F_1(\alpha)^{G_0}$ if $F_1(\alpha)$ is a $\mathbb{Q}_0$-name and $\dot{F}_1^{G_0}(\alpha) = 0$ otherwise. Then define $\dot{\mathbb{Q}}_1$ to be the canonical name for $\mathbb{R}(\aleph_1^V, \kappa_1, V, V[\mathbb{Q}_0], F_1^*)$, where F_1^* is the realisation of $\dot{F}_1$ in $V[\mathbb{Q}_0]$ and $\mathbb{R}_2 =_{\text{def}} \mathbb{Q}_0 * \dot{\mathbb{Q}}_1$.

3. Suppose that we have defined the iteration up to stage n, where $n \geqslant 2$. Let $\mathbb{R}_n =_{\text{def}} \mathbb{Q}_0 * \cdots * \dot{\mathbb{Q}}_{n-1}$. As at stage one, define a $\mathbb{R}_n$-name $\dot{F}_n$ by $\dot{F}_n^{G_n}(\alpha) = F_n(\alpha)^{G_n}$ if $F_n(\alpha)$ is a $\mathbb{R}_n$-name and $\dot{F}_n^{G_n}(\alpha) = 0$ otherwise. Then define $\dot{\mathbb{Q}}_n$ to name $\mathbb{R}(\kappa_{n-2}, \kappa_n, V[\mathbb{R}_{n-1}], V[\mathbb{R}_n], F_n^*)$, where F_n^* is the interpretation of F_n in $V[\mathbb{R}_n]$.

4. $\mathbb{R}_\omega$ is the inverse limit of $\langle \mathbb{R}_n : n < \omega \rangle$.

It is not yet clear that this definition is legitimate, because we can only define $\mathbb{R}(\tau, \kappa, V, W, F)$ when we know that certain things are true in W; namely τ must be regular, κ must be inaccessible, and $\text{Add}(\tau, \kappa)_V$ must be τ^+-c.c. and $<\tau$-distributive. We will now do an inductive analysis of $\mathbb{P}_n$ and $\dot{\mathbb{Q}}_n$, which will show among other things that the definition is a valid one. The case $n = 0$ needs special treatment; this case is essentially Abraham's analysis of the main forcing from [1].

LEMMA 4.2. *Let* $\mathbb{P}_0 =_{\text{def}} \text{Add}(\aleph_0, \kappa_0)$ *and* $\mathbb{U}_0 =_{\text{def}} \mathbb{U}(\aleph_0, \kappa_0, V, V, F_0)$.

1. $|\mathbb{Q}_0| = \kappa_0$, *and* $\mathbb{Q}_0$ *is* κ_0*-Knaster. In particular,*

(a) *All* V*-cardinals greater than or equal to* κ_0 *are preserved in* $V[\mathbb{Q}_0]$.

(b) *All* V*-inaccessibles greater than* κ_0 *remain inaccessible in* $V[\mathbb{Q}_0]$.

(c) *All sets of size less than* κ_0 *in* $V[\mathbb{Q}_0]$ *are covered by sets of size less than* κ_0 *in* V.

2. *$\mathbb{Q}_0$ is a projection of $\mathbb{P}_0 \times \mathbb{U}_0$, and $V[\mathbb{P}_0] \subseteq V[\mathbb{Q}_0] \subseteq V[\mathbb{P}_0 \times \mathbb{U}_0]$.*
3. *All ω-sequences of ordinals from $V[\mathbb{Q}_0]$ are in $V[\mathbb{P}_0]$.*
4. *$\mathbb{Q}_0$ preserves $\aleph_1$, and $2^{\aleph_0} = \kappa_0 = \aleph_2$ in $V[\mathbb{Q}_0]$.*
5. *$\mathrm{Add}(\aleph_0, \eta)_V$ has the $\aleph_1$-Knaster property in $V[\mathbb{Q}_0]$.*
6. *For all η, $V[\mathbb{Q}_0] \models$ "$\mathrm{Add}(\aleph_1, \eta)_V$ is $< \aleph_1$-distributive and κ_0-Knaster."*

Proof. We take the claims in turn:

1. This follows from Lemma 3.2 and standard facts about forcing.
2. This is exactly the content of Lemma 3.13.
3. This follows from 3.14.
4. Immediate by Corollary 3.15.
5. $\mathrm{Add}(\aleph_0, \eta)_V = \mathrm{Add}(\aleph_0, \eta)_{V[\mathbb{Q}_0]}$.
6. Use clause 1 from Lemma 2.13, with $\tau = \aleph_1$, $\kappa = \kappa_0$, $\mathbb{Q} = \mathbb{Q}_0$, $\mathbb{P} = \mathbb{P}_0$, and $\mathbb{U} = \mathbb{U}_0$. ∎

The definition of $\mathbb{Q}_1$ is now seen to be legitimate, because in $V[\mathbb{Q}_0]$ we know that $\aleph_1^V$ is regular, κ_1 is inaccessible, and $\mathrm{Add}(\aleph_1, \kappa_1)_V$ is $< \aleph_1$-distributive and $\aleph_2$-Knaster. We now turn to the general case. In the statement of the following lemma, when we refer to "$\aleph_i$" we mean $\aleph_i$ in the sense of $V[\mathbb{R}_n]$; a key point is that the values of the cardinals $\aleph_i$ for $i \leqslant n+1$ are already fixed in $V[\mathbb{R}_n]$; namely, $\aleph_0$ and $\aleph_1$ are as in the ground model and $\aleph_i = \kappa_{i-2}$ for $2 \leqslant i \leqslant n+1$.

LEMMA 4.3. *Let $n \geqslant 1$. Let $\mathbb{R}_n = \mathbb{Q}_0 * \cdots \mathbb{Q}_{n-1}$, $\mathbb{R}_{n-1} = \mathbb{Q}_0 * \cdots \mathbb{Q}_{n-2}$, $\mathbb{P}_n = \mathrm{Add}(\aleph_n, \kappa_n)_{V[\mathbb{R}_{n-1}]}$ and $\dot{\mathbb{U}}_n = \mathbb{U}(\aleph_n, \kappa_n, V[\mathbb{R}_{n-1}], V[\mathbb{R}_n], F_n^*)$:*

1. *In $V[\mathbb{R}_n]$, cardinal arithmetic follows the pattern $2^{\aleph_i} = \aleph_{i+2} = \kappa_i$ for $i < n$. κ_i is inaccessible for $i \geqslant n$.*

2. *$V[\mathbb{R}_n] \models$ "$\mathbb{Q}_n$ is $< \aleph_n$-distributive, κ_n-Knaster and $\aleph_{n-1}$-closed, and also $V[\mathbb{R}_n] \models$ "$|\mathbb{Q}_n| = \kappa_n$." In particular,*

   (a) *All V-cardinals greater than or equal to κ_n are preserved in $V[\mathbb{R}_n * \dot{\mathbb{Q}}_n]$.*

   (b) *All V-inaccessibles greater than κ_n remain inaccessible in $V[\mathbb{R}_n * \dot{\mathbb{Q}}_n]$.*

   (c) *All sets of ordinals of size less than κ_n in $V[\mathbb{R}_n * \dot{\mathbb{Q}}_n]$ are covered by sets of size less than κ_n in $V[\mathbb{R}_n]$.*

3. *All $\aleph_{n-1}$-sequences of ordinals from $V[\mathbb{R}_n * \dot{\mathbb{Q}}_n]$ are in $V[\mathbb{R}_{n-1} * \dot{\mathbb{P}}_{n-1}]$.*

4. *All cardinals up to* $\aleph_n$ *are preserved in* $V[\mathbb{R}_n * \dot{\mathbb{Q}}_n]$.

5. $V[\mathbb{R}_n] \models$ "$\mathbb{Q}_n$ *is a projection of* $\mathbb{P}_n \times \mathbb{U}_n$," *and* $V[\mathbb{R}_n * \dot{\mathbb{P}}_n] \subseteq V[\mathbb{R}_n * \dot{\mathbb{Q}}_n] \subseteq V[\mathbb{R}_n * (\dot{\mathbb{P}}_n \times \dot{\mathbb{U}}_n)]$.

6. *All* $\aleph_n$*-sequences of ordinals from* $V[\mathbb{R}_n * \dot{\mathbb{Q}}_n]$ *are in* $V[\mathbb{R}_n * \dot{\mathbb{P}}_n]$.

7. $\aleph_{n+1}$ (*which is* κ_{n-1}) *is preserved in* $V[\mathbb{R}_n * \dot{\mathbb{Q}}_n]$. *Cardinal arithmetic in* $V[\mathbb{R}_n * \dot{\mathbb{Q}}_n]$ *follows the pattern* $2^{\aleph_i} = \aleph_{i+2} = \kappa_i$ *for* $i \leqslant n$.

8. $\mathrm{Add}(\aleph_n, \eta)_{V[\mathbb{R}_{n-1}]}$ *is* $\aleph_{n+1}$*-Knaster in* $V[\mathbb{R}_n * \dot{\mathbb{Q}}_n]$, *for any ordinal* η.

9. $V[\mathbb{R}_n * \dot{\mathbb{Q}}_n] \models$ "$\mathrm{Add}(\aleph_{n+1}, \eta)_{V[\mathbb{R}_n]}$ *is* $<\aleph_{n+1}$*-distributive and* κ_n*-Knaster*," *for any ordinal* η.

Proof. We prove the lemma by induction on $n \geqslant 1$, using Lemma 4.2 to get some information in the case $n = 1$. Notice that by induction the forcing $\mathbb{P}_n$ has the right distributivity and chain condition in $V[\mathbb{R}_n]$, and that $\aleph_n$ and κ_n are respectively regular and inaccessible in $V[\mathbb{R}_n]$; it is therefore legitimate to define $\dot{\mathbb{Q}}_n$:

1. This is immediate by induction.

2. By Lemma 3.2, $|\mathbb{Q}_n| = \kappa_n$ and $\mathbb{Q}_n$ is κ_n-Knaster in $V[\mathbb{R}_n]$. By Lemma 3.16, $\mathbb{Q}_n$ is $<\aleph_n$-distributive in $V[\mathbb{R}_n]$. For the closure, observe that $\mathbb{P}_n$ is $\aleph_n$-closed in $V[\mathbb{R}_{n-1}]$ and that (by induction) $\mathbb{Q}_{n-1}$ is $<\aleph_{n-1}$-distributive in $V[\mathbb{R}_{n-1}]$, so that $\mathbb{P}_n$ is $\aleph_{n-1}$-closed in $V[\mathbb{R}_n]$. By Lemma 3.6, $\mathbb{Q}_n$ is $\aleph_{n-1}$-closed in $V[\mathbb{R}_n]$.

3. Since $\mathbb{Q}_n$ is $<\aleph_n$-distributive, every $\aleph_{n-1}$-sequence of ordinals from $V[\mathbb{R}_n * \dot{\mathbb{Q}}_n]$ is in $V[\mathbb{R}_n] = V[\mathbb{R}_{n-1} * \dot{\mathbb{Q}}_{n-1}]$. By induction, every $\aleph_{n-1}$-sequence of ordinals from $V[\mathbb{R}_{n-1} * \dot{\mathbb{Q}}_{n-1}]$ is in $V[\mathbb{R}_{n-1} * \dot{\mathbb{P}}_{n-1}]$.

4. This follows immediately from the last claim.

5. Apply Lemma 3.13 in $V[\mathbb{R}_n]$.

6. Apply Corollary 3.14 in $V[\mathbb{R}_n]$.

7. By Corollary 3.15, $\aleph_{n+1}$ is preserved in $V[\mathbb{R}_n * \dot{\mathbb{Q}}_n]$. Since $\mathbb{Q}_n$ is $<\aleph_n$-distributive in $V[\mathbb{R}_n]$, it follows that all cardinals up to $\aleph_n$ are preserved and that we still have $2^{\aleph_i} = \kappa_i = \aleph_{i+2}$ for $i < n$ in $V[\mathbb{R}_n * \dot{\mathbb{Q}}_n]$. By Corollary 3.15 again, $2^{\aleph_n} = \kappa_n = \aleph_{n+2}$ in $V[\mathbb{R}_n * \dot{\mathbb{Q}}_n]$.

8. Apply clause 2 of Lemma 2.13 in $V[\mathbb{R}_{n-1}]$ with $\tau = \aleph_n$, $\mathbb{Q} = \mathbb{Q}_{n-1}$, $\mathbb{Q}^* = \mathbb{Q}_n$.

9. Apply clause 1 of Lemma 2.13 in $V[\mathbb{R}_n]$ with $\tau = \aleph_{n+1}$, $\mathbb{Q} = \mathbb{Q}_n$. ∎

The fact that the closure of $\mathbb{Q}_n$ in $V[\mathbb{R}_n]$ increases with n enables us to see that the inverse limit $\mathbb{R}_\omega$ is well behaved. We make this precise in the following lemma.

LEMMA 4.4. *Let G_ω be $\mathbb{R}_\omega$-generic. Let $X \in V[G_\omega]$ be a κ_n-sequence of ordinals. Then $X \in V[G_0][\cdots][G_n][G_{n+1}][g_{n+2}]$, where $G_0 * \cdots G_{n+1} * g_{n+2}$ is the initial segment of G_ω generic for $\mathbb{Q}_0 * \cdots \mathbb{Q}_{n+1} * \mathbb{P}_{n+2}$.*

Proof. It is easy to see that $\mathbb{R}_\omega/\mathbb{R}_m$ is κ_{m-3}-closed, so that $X \in V[G_0][\cdots][G_{n+3}]$. Since $\mathbb{Q}_{n+3}$ is $<\kappa_{n+1}$-distributive in $V[G_0][\cdots][G_{n+2}]$, $X \in V[G_0][\cdots][G_{n+2}]$. Finally, all κ_n-sequences of ordinals in $V[G_0][\cdots][G_{n+2}]$ are in $V[G_0][\cdots][G_{n+1}][g_{n+2}]$, so we are done. ▮

In the interests of brevity, we will denote $V[G_0][\cdots][G_n]$ by V_n. Notice that g_{n+2} is generic over $V_{n+1} = V_n[G_{n+1}]$ for $\mathbb{P}_{n+2} \in V_n$, so that we can consider G_{n+1} and g_{n+2} as mutually generic over V_n.

COROLLARY 4.5. $2^{\aleph_n} = \kappa_n = \aleph_{n+2}$ *in* $V[G_\omega]$.

Proof. Immediate from Lemma 4.4 and Lemma 4.3. ▮

4.2. *Why the Model Works*

We are now ready to begin in the argument that all of the cardinals κ_n have the tree property in $V[G_\omega]$. By Lemma 4.4, if $T \in V[G_\omega]$ is a κ_n-tree then $T \in V_{n+1}[g_{n+2}]$. We will show that there are no κ_n-Aronszajn trees in $V_{n+1}[g_{n+2}]$.

We begin our work in V. Let λ be some cardinal greater than $\sup\{\kappa_n : n \in \omega\}$. Using the "diamond" property of F_n, choose $j: V \to M$ such that

1. $\operatorname{crit}(j) = \kappa_n$, $j(\kappa_n) > \lambda$, ${}^{\lambda}M \subseteq M$.

2. $j(F_n)(\kappa_n)$ is the canonical $\mathbb{R}_n$-name for the canonical $\mathbb{Q}_n$-name for $\operatorname{Add}(\kappa_n, \kappa_{n+2})_{V[\mathbb{R}_n * \dot{\mathbb{Q}}_n]} \times \mathbb{U}(\aleph_{n+1}, \kappa_{n+1}, V[\mathbb{R}_n], V[\mathbb{R}_n * \dot{\mathbb{Q}}_n], F^*_{n+1})$.

Notice that $j(F_n)(\kappa_n)$ is a name for a κ_n-directed closed forcing in $V[\mathbb{R}_n * \dot{\mathbb{Q}}_n]$.

Our aim is now to force over $V_{n+1}[g_{n+2}]$ to get a new model $V_{n+1}[g_{n+2}][X]$, in such a way that in $V_{n+1}[g_{n+2}][X]$ we may define an extension of j to a (generic) elementary embedding $j: V_{n+1}[g_{n+2}] \to N \subseteq V_{n+1}[g_{n+2}][X]$. We will break up the construction into a series of stages. To keep the notation simple, we will denote all the extensions of the original $j: V \to M$ by "j" also.

4.2.1. *Stage one.* Since $j(\mathbb{R}_n) = \mathbb{R}_n$, it is easy to lift the embedding j onto the extension by $\mathbb{R}_n$. We will then have an embedding $j\colon V_{n-1} \to V_{n-1}$ extending $j\colon V \to M$.

4.2.2. *Stage two.* We force over $V_{n+1}[g_{n+2}]$ with

$$\mathbb{S} =_{\text{def}} \mathbb{S}(\aleph_{n+1}, \kappa_{n+1}, V_{n-1}, V_n, F^*_{n+1}, G_{n+1}).$$

Since G_{n+1} and g_{n+2} are mutually generic, we can look at the resulting extension as being obtained from V_{n+1} by forcing first with $\mathbb{S}$ and then with $\mathbb{P}_{n+2}$.

$\mathbb{S}$ (by design) will expand V_{n+1} to a new model $V_n[g_{n+1} \times u_{n+1}]$, where $g_{n+1} \times u_{n+1}$ is generic for $\mathbb{P}_{n+1} \times \mathbb{U}_{n+1}$. So forcing with $\mathbb{S}$ over the model $V_{n+1}[g_{n+2}]$ gives us a model $V_n[g_{n+1} \times u_{n+1} \times g_{n+2}]$, with $g_{n+1} \times u_{n+1} \times g_{n+2}$ generic for $\mathbb{P}_{n+1} \times \mathbb{U}_{n+1} \times \mathbb{P}_{n+2}$ over V_n.

4.2.3. *Stage three.* It is at this point that the careful choice of j pays off. If we consider the construction of $j(\mathbb{Q}_n)$ in $M[G_0][\cdots][G_{n-1}]$ then it is clear that

- $j(\mathbb{Q}_n) \restriction \kappa_n = \mathbb{Q}_n$.
- At stage κ_n, the forcing at the third coordinate will be

$$\text{Add}(\kappa_n, \kappa_{n+2})_{V_n} \times \mathbb{U}(\aleph_{n+1}, \kappa_{n+1}, V_{n-1}, V_n, F^*_{n+1}),$$

- If we let $M_i =_{\text{def}} M[G_0][\cdots][G_i]$ then the agreement between V and M implies that this forcing equals

$$\text{Add}(\kappa_n, \kappa_{n+2})_{M_n} \times \mathbb{U}(\aleph_{n+1}, \kappa_{n+1}, M_{n-1}, M_n, F^*_{n+1}).$$

This means that we can look at the model $M_n[g_{n+2} \times u_{n+1}]$ as a generic extension of M_{n-1} by $j(\mathbb{Q}_n) \restriction \kappa_n + 1$. We will now force over $V_n[g_{n+1} \times u_{n+1} \times g_{n+2}]$ to get H_n a $j(\mathbb{Q}_n)$-generic object, with $H_n \restriction \kappa_n + 1 = G_n * (g_{n+2} \times u_{n+1})$. Notice that g_{n+1} is generic over $M_{n-1}[H_n]$ for $\mathbb{P}_{n+1}$.

Since $\mathbb{Q}_n$ is κ_n-c.c it is easy to see that $j \restriction \mathbb{Q}_n$ is a complete embedding from $\mathbb{Q}_n$ into $j(\mathbb{Q}_n)$, so we may lift $j\colon V_{n-1} \to V_{n-1}$ to get $j\colon V_n \to M_{n-1}[H_n]$.

4.2.4. *Stage four.* Recall that $\mathbb{P}_{n+1} = \text{Add}(\aleph_{n+1}, \kappa_{n+1})_{V_{n-1}}$. $\mathbb{P}_{n+1}$ is $\aleph_{n+2}$-Knaster (that is, κ_n-Knaster) in V_n, so that $j \restriction \mathbb{P}_n$ is a complete embedding of $\mathbb{P}_n$ into $j(\mathbb{P}_n)$. We can be more precise, $\mathbb{P}_n$ is isomorphic via $j \restriction \mathbb{P}_n$ to $\text{add}(\aleph_{n+1}, j''\kappa_{n+1})_{M_{n-1}}$, which is equal to $\text{Add}(\aleph_{n+1}, j''\kappa_{n+1})_{V_{n-1}}$ by the agreement between V and M.

We force over $V_n[H_n][g_{n+1}]$ with $\text{Add}(\aleph_{n+1}, j(\kappa_{n+1}) - j''\kappa_{n+1})_{V_{n-1}}$ to get a generic object h_{n+1} for $j(\mathbb{P}_n)$ such that $j''g_{n+1} \subseteq h_n$. We can now lift $j\colon V_n \to M_{n-1}[H_n]$ to get $j\colon V_n[g_{n+1}] \to M_{n-1}[H_n][h_{n+1}]$.

4.2.5. *Stage five.* By construction and the closure of M, we know that $j''u_{n+1} \in M_{n-1}[H_n]$. H_n collapses κ_{n+1} to have cardinality $\aleph_{n+1}$, and $j(\mathbb{U}_n)$ is $j(\kappa_n)$-directed closed where $j(\kappa_n) = (\aleph_{n+2})_{M_{n-1}[H_n]}$. Therefore we may find t such that $t \leqslant j(q)$ for all $q \in u_{n+1}$.

We now force over $V_n[H_n][h_n]$ with $j(U_n)$, below the condition t, to get a generic object x_{n+1} such that $j''u_{n+1} \subseteq x_{n+1}$. h_{n+1} and x_{n+1} are mutually generic over $M_{n-1}[H_n]$ by Easton's lemma, and $h_{n+1} \times x_{n+1}$ generates a filter H_{n+1} generic for $j(\mathbb{R}_{n+1})$ over $M_{n-1}[H_n]$.

We claim that $j''G_{n+1} \subseteq H_{n+1}$. To see this recall that G_{n+1} is generated by $g_{n+1} \times u_{n+1}$, so that if $(p, q, f) \in G_{n+1}$ we may find $\bar{p} \in g_{n+1}$ and $(0, \bar{q}, \bar{f}) \in u_{n+1}$ such that $(\bar{p}, \bar{q}, \bar{f}) \leqslant (p, q, f)$ in $\mathbb{R}_{n+1}$. Now $j(\bar{p}) \in h_{n+1}$, $(0, j(\bar{q}), j(\bar{f})) \in x_{n+1}$, so that $(j(\bar{p}), 0, 0)$ and $(0, j(\bar{q}), j(\bar{f}))$ are both in H_{n+1}. Their greatest lower bound is $j((\bar{p}, \bar{q}, \bar{f}))$, so this condition must be in H_{n+1}; moreover, $j((\bar{p}, \bar{q}, \bar{f})) \leqslant j((p, q, f))$ in $j(\mathbb{R}_{n+1})$, so that $j((p, q, f)) \in H_{n+1}$.

We may now lift $j: V_n[g_{n+1}] \to M_{n-1}[H_n][h_{n+1}]$, to get an embedding $j: V_{n+1} \to M_{n-1}[H_n][H_{n+1}]$.

4.2.6. *Stage six.* $j''g_{n+2} \in M_{n-1}[H_n]$ by construction and the closure of M, so if we let $s =_{\text{def}} \bigcup j''g_{n+2}$ then s is a condition in $j(\mathbb{P}_{n+2}) = \text{Add}(j(\kappa_n), j(\kappa_{n+2}))_{M_{n-1}[H_n]}$. So we can force over $V_n[H_n][h_n][x_{n+1}]$ with $j(\mathbb{P}_{n+2})$ below s and get a generic object h_{n+2} such that $j''g_{n+2} \subseteq h_{n+2}$.

We finish the construction by lifting $j: V_{n+1} \to M_{n-1}[H_n][H_{n+1}]$, to get $j: V_{n+1}[g_{n+2}] \to M_{n-1}[H_n][H_{n+1}][h_{n+2}]$.

Now that we have lifted j, we can return to the proof that κ_n has the tree property in $V_{n+1}[g_{n+2}]$. Suppose that T is a κ_n-Aronszajn tree in $V_{n+1}[g_{n+2}]$. If we apply j to T we get a tree $j(T)$ of height $j(\kappa_n) > \kappa_n$, which has an initial segment $j(T) \upharpoonright \kappa_n$ that is isomorphic to T. It follows that $T \in M_{n-1}[H_n][H_{n+1}][h_{n+2}]$, and T has a branch b in that model.

Now we will locate the tree T and its branch more precisely. Because of the resemblance between V and M, $T \in M_{n+1}[g_{n+2}]$. κ_{n+1} is collapsed to $\aleph_{n+1}$ in $M_{n-1}[H_n][H_{n+1}][h_{n+2}]$, so applying some familiar arguments $b \in M_{n-1}[H_n][h_{n+1}]$.

Notice that $M_{n+1}[g_{n+2}] = M_{n-1}[G_n][G_{n+1}][g_{n+2}]$ and

$$M_{n-1}[H_n][h_{n+1}] = M_{n-1}[G_n][g_{n+2} \times u_{n+1}][H_n^*][g_{n+1} \times h_{n+1}^*],$$

decomposing H_n as $G_n * (g_{n+2} \times u_{n+1}) * H_n^*$ and h_{n+1} as $g_{n+1} \times h_{n+1}^*$. We will analyze the forcing that takes us from $M_{n+1}[g_{n+2}]$ to $M_{n-1}[H_n][h_{n+1}]$, and show that this forcing cannot add a cofinal branch to a κ_n-Aronszajn tree. This contradiction will finish the proof that κ_n has the tree property in $V_{n+1}[g_{n+2}]$.

In the analysis that follows, we will use repeatedly and without comment the resemblance between V and M.

We begin by considering what happens when we force over $M_{n+1}[g_{n+2}]$ with $\mathbb{S}$ to get the model $M_n[g_{n+1}\times u_{n+1}\times g_{n+2}]$ (see stage two above). By Lemma 3.20, $M_{n+1}\models$ "$\mathbb{S}$ is $\aleph_{n+1}$-closed, $<\aleph_{n+2}$-distributive and $\aleph_{n+3}$-c.c." Since g_{n+2} is generic over M_{n+1} for $<\aleph_{n+2}$-distributive forcing, $\mathbb{S}$ is still $\aleph_{n+1}$-closed in $M_{n+1}[g_{n+2}]$. It follows that forcing with $\mathbb{S}$ over $M_{n+1}[g_{n+2}]$ preserves cardinals up to $\aleph_{n+1}$; since $2^{\aleph_n}=\aleph_{n+2}=\kappa_n$ in $M_{n+1}[g_{n+2}]$, it follows from Lemma 2.4 that T has no cofinal branch in $M_n[g_{n+1}\times u_{n+1}\times g_{n+2}]$.

$g_{n+2}\times u_{n+1}$ is generic over M_n for $\aleph_{n+2}$-directed closed forcing and g_{n+1} is generic over M_n for $\aleph_{n+2}$-Knaster forcing, so by Easton's lemma it follows that all $\aleph_{n+1}$-sequences of ordinals from $M_n[g_{n+1}\times u_{n+1}\times g_{n+2}]$ are in $M_{n+1}[g_{n+2}]$. In particular κ_n is still a cardinal (namely $\aleph_{n+2}$) in $M_n[g_{n+1}\times u_{n+1}\times g_{n+2}]$, and so T is still a κ_n-Aronszajn tree in that model.

Now we will force over $M_n[g_{n+1}\times u_{n+1}\times g_{n+2}]$ to add h^*_{n+1}; h^*_{n+1} is generic for $\mathbb{P}^*_{n+1}=_{\text{def}}\text{Add}(\aleph_{n+1}, j(\kappa_{n+1})-j''\kappa_{n+1})$, and we know by Lemma 2.13 that $M_n\models$ "$\mathbb{P}^*_{n+1}$ is $<\aleph_{n+1}$-distributive and κ_n-Knaster."

We claim that $\mathbb{P}^*_{n+1}$ is still κ_n-Knaster in $M_n[g_{n+1}\times u_{n+1}\times g_{n+2}]$. We will use Lemma 2.6 to see this. $\aleph_{n+1}$ is still regular in $M_n[g_{n+1}\times u_{n+1}\times g_{n+2}]$, so to finish the proof of the claim suppose that $X\in M_n[g_{n+1}\times u_{n+1}\times g_{n+2}]$ is a set of ordinals with $|X|<\kappa_n$. Then $|X|\leqslant\aleph_{n+1}$ in $M_n[g_{n+1}\times u_{n+1}\times g_{n+2}]$, and so as we proved already $X\in M_n[g_{n+1}]$. Since g_{n+1} is generic over M_n for $\aleph_{n+2}$-c.c. (that is κ_n-c.c.) forcing and G_n is generic over M_{n-1} for κ_n-c.c. forcing, there is $Y\in M_{n-1}$ such that $|X|<\kappa_n$ in M_{n-1} and $X\subseteq Y$.

We also claim that $\mathbb{P}^*_{n+1}$ is still $<\aleph_{n+1}$-distributive in $M_n[g_{n+1}\times u_{n+1}\times g_{n+2}]$. To see this suppose that Y is an $\aleph_n$-sequence of ordinals in $M_n[h_{n+1}\times u_{n+1}\times g_{n+2}]$. By Easton's lemma $u_{n+1}\times g_{n+2}$ is generic for $<\aleph_{n+2}$-distributive forcing over $M_n[h_{n+1}]$, so $Y\in M_n[h_{n+1}]$. Since $j(\mathbb{P}_n)$ is $<\aleph_{n+1}$-distributive in M_n, $Y\in M_n$ and we are done.

It follows from these claims that κ_n is still a cardinal (in fact is still $\aleph_{n+2}$) in $M_n[h_{n+1}\times u_{n+1}\times g_{n+2}]$. Since $\mathbb{P}^*_{n+1}$ is κ_n-Knaster in $M_n[g_{n+1}\times u_{n+1}\times g_{n+2}]$, Lemma 2.3 implies that T is still a κ_n-Aronszajn tree in $M_n[h_{n+1}\times u_{n+1}\times g_{n+2}]$.

Finally we will force over $M_n[h_{n+1}\times u_{n+1}\times g_{n+2}]$ with $\mathbb{R}^*_n=j(\mathbb{P}_n)/G_n*(u_{n+1}\times g_{n+2})$, to get the model $M_{n-1}[H_n][h_{n+1}]$. We know from Lemmas 3.24 and 3.25 that $M_n[u_{n+1}\times g_{n+2}]\models$ "$\mathbb{R}^*_n$ is a projection of $\mathbb{P}^*_n\times\mathbb{U}^*_n$," where $\mathbb{P}^*_n=\text{Add}(\aleph_n, j(\kappa_n)-\kappa_n)_{M_{n-2}}$ and $M_n[u_{n+1}\times g_{n+2}]\models$ "$\mathbb{U}^*_n$ is $\aleph_{n+1}$-closed." We will finish the argument by showing that T can have no branch in any extension of $M_n[h_{n+1}\times u_{n+1}\times g_{n+2}]$ by $\mathbb{P}^*_n\times\mathbb{U}^*_n$.

We have already shown that every $\aleph_n$-sequence of ordinals in $M_n[h_{n+1}\times u_{n+1}\times g_{n+2}]$ is in M_n, so $\mathbb{U}^*_n$ is still $\aleph_{n+1}$-closed in

$M_n[h_{n+1} \times u_{n+1} \times g_{n+2}]$. Since $2^{\aleph_n} = \aleph_{n+2} = \kappa_n$ in $M_n[h_{n+1} \times u_{n+1} \times g_{n+2}]$, another application of Lemma 2.4 shows that T has no cofinal branch in $M_n[h_{n+1} \times u_{n+1} \times g_{n+2}][\mathbb{U}_n^*]$. Forcing with $\mathbb{U}_n^*$ preserves cardinals up to $\aleph_{n+1}$ and collapses $\aleph_{n+2}$ (that is κ_n) to be an ordinal of cofinality $\aleph_{n+1}$, so that in the extension by $\mathbb{U}_n^*$ we may find a tree T_0 of height $\aleph_{n+1}$ and a cofinal map from T_0 to T.

We now claim that $\mathbb{P}_n^*$ is $\aleph_{n+1}$-Knaster in the model $M_n[h_{n+1} \times u_{n+1} \times g_{n+2}][\mathbb{U}_n^*]$. To see this we apply Lemma 2.6. $\aleph_n$ is still regular in $M_n[h_{n+1} \times u_{n+1} \times g_{n+2}][\mathbb{U}_n^*]$, so let $X \in M_n[h_{n+1} \times u_{n+1} \times g_{n+2}][\mathbb{U}_n^*]$ be a set of ordinals of size less than $\aleph_{n+1}$ (that is, κ_{n-1}). Then $|X| \leqslant \aleph_n$ in $M_n[h_{n+1} \times u_{n+1} \times g_{n+2}][\mathbb{U}_n^*]$, and so by the closure of $\mathbb{U}_n^*$ we have $X \in M_n[h_{n+1} \times u_{n+1} \times g_{n+2}]$. We saw already that this implies $X \in M_n$, and actually we can go further and see that $X \in M_{n-1}[g_n]$; this is a κ_{n-1}-c.c. extension of M_{n-2}, so that X is covered by a set in M_{n-2} of size less than κ_{n_1}.

By Lemma 2.3, forcing with $\mathbb{P}_n^*$ over $M_n[h_{n+1} \times u_{n+1} \times g_{n+2}][\mathbb{U}_n^*]$ adds no cofinal branch to T_0, and therefore adds no cofinal branch to T. This concludes the proof of Theorem 1.

PART 2. THE TREE PROPERTY AT THE DOUBLE SUCCESSOR OF A SINGULAR

In this part of the paper we will give Foreman's proof that it is consistent to have a double successor of a singular cardinal with the tree property. It is possible to modify the construction along the lines of [3] or [4] to get a model in which $\aleph_{\omega+2}$ has the tree property.

5. PRELIMINARIES

Starting with $\kappa < \lambda$, where κ is supercompact and λ is weakly compact, we will build a generic extension in which

1. κ is a singular cardinal of cofinality ω.
2. κ^+ is preserved.
3. $2^\kappa = \kappa^{++} = \lambda$.
4. λ has the tree property.

We start by doing Laver's construction from [10] to make κ indestructibly supercompact under any κ-directed-closed forcing. Laver's poset has cardinality κ, so it will preserve the weak compactness of λ. We will denote by V the resulting model in which κ is indestructibly supercompact and λ is weakly compact.

Now let $\mathbb{P} =_{\text{def}} \text{Add}(\kappa, \lambda)$, so that by construction κ is supercompact in $V[\mathbb{P}]$. Let $\dot{U}$ be a $\mathbb{P}$-name for a normal ultrafilter on κ in the model $V[\mathbb{P}]$.

LEMMA 5.1. *There is $A \subseteq \lambda$ a set of Mahlo cardinals such that*

1. *If $\alpha \in A$ and G is* $\text{Add}(\kappa, \lambda)$*-generic over V then $\dot{U}^G \cap V[G \restriction \alpha] \in V[G \restriction \alpha]$.*
2. *A is in the weakly compact filter on λ.*

Proof. Given $\beta < \lambda$, we can use the inaccessibility of λ and the κ^+-c.c. for $\mathbb{P}$ to find $F(\beta) < \lambda$ such that for every canonical $\mathbb{P} \restriction \beta$-name $\dot{A}$ for a subset of κ we have $[\dot{A} \in \dot{U}]_{RO(\mathbb{P})} \in RO(\mathbb{P} \restriction F(\beta))$. A standard argument shows that the set of Mahlo limit points of F is in the weakly compact filter, so we define A to be this set. ∎

For each $\alpha \in A$, let $U_\alpha =_{\text{def}} \dot{U}^G \cap V[G \restriction \alpha]$. Clearly, U_α is a normal measure on κ in the model $V[G \restriction \alpha]$.

Working in $V[\mathbb{P}]$ we define $\mathbb{Q}$ to be the Prikry forcing constructed from the normal measure U. We also define $\mathbb{Q}_\alpha$ to be the Prikry forcing constructed from the measure U_α in $V[\mathbb{P} \restriction \alpha]$.

LEMMA 5.2. *Let x be a cofinal ω-sequence in κ, which is $\mathbb{Q}$-generic over $V[G]$. Then for each $\alpha \in A$, x is $\mathbb{Q}_\alpha$-generic over $V[G_\alpha]$.*

Proof. This is immediate from Mathias' theorem [13] that an ω-sequence is Prikry generic for a measure U if and only if every set of U-measure one contains a final segment of the ω-sequence. ∎

Let π_α denote the natural projection map from $P * \dot{\mathbb{Q}}$ to $RO(\mathbb{P} \restriction \alpha * \mathbb{Q}_\alpha)$.

LEMMA 5.3. *$\mathbb{P} * \mathbb{Q}$ has the κ^+-Knaster property.*

Proof. Let $\langle (p_\alpha, q_\alpha) : \alpha < \kappa^+ \rangle$ be a κ^+-sequence of conditions. Refining if necessary, we may as well assume that p_α decides the lower part of q_α for each α. Since there are only κ possible lower parts, we may find a subsequence of length κ^+ on which the lower part of q_α is constant, say with value s.

Now $\text{Add}(\kappa, \lambda)$ has the κ^+-Knaster property, so we may thin out even further to find A unbounded in κ^+ such that if $\alpha, \beta \in A$ then p_α is compatible with p_β and the lower parts of q_α and q_β are equal. This implies that (p_α, q_α) is compatible with (p_β, q_β), so we are done. ∎

Notice that the same proof applies to $\mathbb{P} \restriction \alpha * \mathbb{Q}_\alpha$.

6. THE MAIN FORCING

Let G be $\mathbb{P}$-generic over V and let x be $\mathbb{Q}$-generic over $V[G]$. We have seen that for each $\alpha \in A$ we may view the submodel $V[G \upharpoonright \alpha][x]$ of $V[G][x]$ as a generic extension of V by $\mathbb{P} \upharpoonright \alpha * \mathbb{Q}_\alpha$. For each α, we define $\mathbb{R}_\alpha = \mathrm{Add}(\kappa^+, 1)_{V[\mathbb{P} \upharpoonright \alpha * \mathbb{Q}_\alpha]}$.

Now we are ready to define the main forcing $\mathbb{R}$.

DEFINITION 6.1. Conditions in $\mathbb{R}$ are triples of the form (p, q, r), where

1. $(p, q) \in \mathbb{P} * \dot{\mathbb{Q}}$.
2. r is a partial function with $\mathrm{dom}(r) \subseteq A$, $|\mathrm{dom}(r)| \leqslant \kappa$.
3. For each $\alpha \in \mathrm{dom}(r)$, $r(\alpha)$ is a $\mathbb{P} \upharpoonright \alpha * \mathbb{Q}_\alpha$-name for a condition in $\mathbb{R}_\alpha$.

Given conditions (p_0, q_0, r_0) and (p_1, q_1, r_1), $(p_1, q_1, r_1) \leqslant (p_0, q_0, r_0)$ iff

1. $(p_1, q_1) \leqslant (p_0, q_0)$ in $\mathbb{P} * \mathbb{Q}$.
2. $\mathrm{dom}(r_0) \subseteq \mathrm{dom}(r_1)$, and $\pi_\alpha((p_1, q_1)) \models r_1(\alpha) \leqslant r_0(\alpha)$ for every $\alpha \in \mathrm{dom}(r_0)$.

We observe that the definition is very similar to that of the main forcing from Mitchell's paper [14]. The only difference is that $\mathbb{P} \upharpoonright \alpha$ is replaced by the more complex forcing $\mathbb{P} \upharpoonright \alpha * \mathbb{Q}_\alpha$. Lemmas 6.2, 6.3, and 6.4 are proved as for the main forcing from [14] (see [1] or the first part of this paper).

LEMMA 6.2. *Let $\mathbb{U}$ be the partial ordering consisting of elements of $\mathbb{R}$ of the form $(0, 0, r)$ with the induced partial ordering. Let $\rho: (\mathbb{P} * \mathbb{Q}) \times \mathbb{U} \to \mathbb{R}$ be given by $\rho: ((p, q), (0, 0, r)) \mapsto (p, q, r)$. Then*

1. *$\mathbb{U}$ is κ^+-closed.*
2. *ρ is a projection map which commutes with the natural projections from $\mathbb{R}$ and $\mathbb{P} * \mathbb{Q} \times \mathbb{U}$ to $\mathbb{P} * \mathbb{Q}$ (so that in a natural way $V[\mathbb{P} * \mathbb{Q}] \subseteq V[\mathbb{R}] \subseteq V[\mathbb{P} * \mathbb{Q} \times \mathbb{U}]$).*
3. *$V[\mathbb{R}]$ and $V[\mathbb{P} * \mathbb{Q}]$ have the same κ-sequences.*

LEMMA 6.3. *$\mathbb{R}$ is λ-Knaster. It preserves all cardinals except for those in the interval (κ^+, λ), which it collapses to κ^+. In the model $V[\mathbb{R}]$, $2^\kappa = \lambda = \kappa^{++}$.*

Given $\beta \in A$, we will define $\mathbb{R} \upharpoonright \beta$ in the following way: conditions are triples of the form (p, q, r), where

1. $(p, q) \in \mathbb{P} \restriction \beta * \dot{\mathbb{Q}}_\beta$.
2. r is a partial function with $\mathrm{dom}(r) \subseteq A \cap \beta$, $|\mathrm{dom}(r)| \leqslant \kappa$.
3. For each $\alpha \in \mathrm{dom}(r)$, $r(\alpha)$ is a $\mathbb{P} \restriction \alpha * \mathbb{Q}_\alpha$-name for a condition in $\mathbb{R}_\alpha$.

The ordering is defined just as for $\mathbb{R}$. It is easy to see that if β is in A then $\mathbb{R} \restriction \beta$ preserves all cardinals except those in (κ^+, β) and forces $2^\kappa = \kappa^{++} = \beta$. Moreover, $V[\mathbb{R} \restriction \beta]$ is a submodel of $V[\mathbb{R}]$, and if $X \in V[\mathbb{R}]$ is a bounded subset of λ then $X \in V[\mathbb{R} \restriction \beta]$ for all sufficiently large β.

LEMMA 6.4. *Let β be in A. Then $\mathbb{R}/\mathbb{R} \restriction \beta$ is (in $V[\mathbb{R} \restriction \beta]$) a projection of $(\mathbb{P} * \mathbb{Q}/\mathbb{P} \restriction \beta * \mathbb{Q}_\beta) \times \mathbb{U}^*$ for some κ^+-closed forcing $\mathbb{U}^*$.*

To finish the proof it will suffice to prove $(\mathbb{P} * \mathbb{Q}/\mathbb{P} \restriction \beta * \mathbb{Q}_\beta)$ is κ^+-Knaster in $V[\mathbb{R} \restriction \beta * \mathbb{U}^*]$.

LEMMA 6.5. *If $(\mathbb{P} * \mathbb{Q}/\mathbb{P} \restriction \beta * \mathbb{Q}_\beta)$ is κ^+-Knaster in $V[\mathbb{R} \restriction \beta * \mathbb{U}^*]$ for every $\beta \in A$, then there is no κ^{++}-Aronszajn tree in $V[\mathbb{R}]$.*

Proof. The proof follows exactly the same lines as the proof of Theorem 1. If T is a κ^{++}-Aronszajn tree in $V[\mathbb{R}]$ then, by the reflection properties of λ, there is $\beta \in A$ such that $T \restriction \beta$ is an Aronszajn tree in $V[\mathbb{R} \restriction \beta]$.

Clearly $T \restriction \beta$ has a branch in $V[\mathbb{R}]$ (fix a point on level β of T and look at the points below). Since $\mathbb{R}/\mathbb{R} \restriction \beta$ is a projection of $(\mathbb{P} * \mathbb{Q}/\mathbb{P} \restriction \beta * \mathbb{Q}_\beta) \times \mathbb{U}^*$, $T \restriction \beta$ will have a branch in $V[\mathbb{R} \restriction \beta][\mathbb{P} * \mathbb{Q}/\mathbb{P} \restriction \beta * \mathbb{Q}_\beta \times \mathbb{U}^*]$.

Since $\mathbb{U}^*$ is κ^+-closed, T will have no branch in $V[\mathbb{R} \restriction \beta][\mathbb{U}^*]$. $\mathbb{U}^*$ collapses β to be some ordinal of cardinality and cofinality κ^+, so that there is a tree T_0 of height κ^+-which embeds cofinally into $T \restriction \beta$. Now supposedly forcing with $\mathbb{P} * \mathbb{Q}/\mathbb{P} \restriction \beta * \mathbb{Q}_\beta$ will add a branch to T and (therefore a branch to T_0), but this is impossible because κ^+-Knaster forcing cannot add a branch to a branchless tree of height κ^+. ∎

7. ANALYSING $\mathbb{P} * \mathbb{Q}/\mathbb{P} \restriction \beta * \mathbb{Q}_\beta$

We saw in the last section that to finish the proof of Theorem 2 we just need to prove the following result.

LEMMA 7.1. *$(\mathbb{P} * \mathbb{Q}/\mathbb{P} \restriction \beta * \mathbb{Q}_\beta)$ is κ^+-Knaster in $V[\mathbb{R} \restriction \beta * \mathbb{U}^*]$ for every $\beta \in A$.*

Proof. We start with a claim in the same spirit as Lemma 2.6.

Claim. *Let $V \subseteq W$ be two inner models of set theory. Suppose that*

1. *κ is a limit cardinal in W.*
2. *$\kappa^+_V = \kappa^+_W$.*
3. *If $X \in W$ is a set of ordinals, $W \models |X| \leqslant \kappa$, then $\exists Y \in V$ such that $X \subseteq Y$ and $V \models |Y| \leqslant \kappa$.*
4. *$V \models \kappa^{<\kappa} = \kappa$.*

Let $\langle x_\alpha : \alpha < \kappa^+ \rangle \in K$ be a κ^+-sequence of sets of ordinals such that $x_\alpha \in V$, $|x_\alpha| < \kappa$. Then there exists $X \subseteq \kappa^+$ unbounded such that $\langle x_\alpha : \alpha \in X \rangle$ forms a Δ-system.

Proof. We work in W. Let $A = \bigcup_{\alpha < \kappa^+} x_\alpha$, and let $F \in W$ be a bijection $F: A \cong \kappa^+$. Let $y_\alpha = F``x_\alpha$. Thinning out we may assume that $|y_\alpha|_W = \mu$ for some fixed $\mu < \kappa$.

Let T_0 be the stationary set $\{\alpha < \kappa^+ : \mathrm{cf}(\alpha) = \mu^+\}$. For each $\alpha \in T_0$ we have $\sup(y_\alpha \cap \alpha) < \alpha$, so by Fodor's lemma there is $T_1 \subseteq T_0$ stationary and a fixed $\beta < \kappa^+$ such that $\forall \alpha \in T_1 \sup(y_\alpha \cap \alpha) = \beta$.

Now $F^{-1}``\beta$ is a set of size $\kappa$ lying in $W$, so by our assumptions there is $Z \supseteq F^{-1}``\beta$ such that $|Z| = \kappa$ and $Z \in V$. Now since $V \models \kappa^{<\kappa} = \kappa$ there are only κ possibilities for $x_\alpha \cap Z$, so we may find $T_2 \subseteq T_1$ stationary and a fixed v such that $\forall \alpha \in T_2\ x_\alpha \cap Z = v$. Then $\forall \alpha \in T_2\ y_\alpha \wedge \alpha = y_\alpha \cap \beta = F``x_\alpha \cap Z = F``v$.

It is now easy to finish, building inductively an unbounded set $T_3 \subseteq T_2$ such that $\langle y_\alpha : \alpha \in T_3 \rangle$ (and hence $\langle x_\alpha : \alpha \in T_3 \rangle$) is a Δ-system. This finishes the proof of the claim. ∎

We observe that the assumptions of the claim hold if we set $W = V[\mathbb{R} \restriction \beta][\mathbb{U}^*]$.

Our next task is to analyze the forcing $\mathbb{P} * \mathbb{Q}/\mathbb{P} \restriction \beta * \mathbb{Q}_\beta$. It is important to notice that this forcing is not just $\mathrm{Add}(\kappa, \lambda - \beta)_V$, although it does add a generic object for that forcing. The point is that given $g * x$ which is $\mathbb{P} \restriction \beta * \mathbb{Q}_\beta$-generic, the forcing $\mathbb{P} * \mathbb{P}/\mathbb{P} \restriction \beta * \mathbb{Q}_\beta$ will add an $\mathrm{Add}(\kappa, \lambda - \beta)$-generic object H with the property that x is actually $\mathbb{Q}$-generic over $V[g \times H]$.

Claim. *Let $p^* = (p, (x, \dot{X})) \in \mathbb{P} \restriction \beta * \mathbb{Q}_\beta$, and let $q^* = (q, (s, \dot{A})) \in \mathbb{P} * \mathbb{Q}$. Then p^* forces that q^* is not a condition in $\mathbb{P} * \mathbb{Q}/\mathbb{P} \restriction \beta * \mathbb{Q}_\beta$ if and only if one of the following conditions holds:*

1. *$q \restriction \beta$ is incompatible with p.*
2. *$q \restriction \beta$ is compatible with p, $s \not\subseteq x$, and $x \not\subseteq s$.*

3. *$q \restriction \beta$ is compatible with p, x extends s and $q \cup p$ forces that $x - s$ is not a subset of $\dot{A}$.*

4. *$q \restriction \beta$ is compatible with p, s extends x and $(q \restriction \beta) \cup p$ forces that $s - x$ is not a subset of $\dot{X}$.*

Proof. To say that $p^* \Vdash q^* \notin \mathbb{P} * \mathbb{Q}/\mathbb{P} \restriction \beta * \mathbb{Q}_\beta$ is to say that there is no generic object $G * x \ni q^*$ such that $G \restriction \beta * x \ni p^*$. It is easy to check that each of the conditions above rules out the existence of such a G. On the other hand, if they all fail then we can force below an appropriate condition r to manufacture an appropriate G; r is $(p \cup q, (t, \dot{A} \cap \dot{X}))$, where t is the longer of x and s. ∎

To finish the proof of Lemma 7.1, suppose for a contradiction that we have a condition $((p, (x, \dot{X}), f), u) \in \mathbb{R} \restriction \beta * \mathbb{U}^*$ which forces that $\langle \dot{r}_\alpha : \alpha < \lambda^+ \rangle$ is a counterexample to the κ^+-Knaster property for $\mathbb{P} * \mathbb{P}/\mathbb{P} \restriction \beta * \mathbb{Q}_\beta$.

Working in $V[\mathbb{R} \restriction \beta * \mathbb{U}^*]$ for the moment, suppose that r_α is a name for a condition $(q_\alpha, (s_\alpha, \dot{A}_\alpha))$. We may refine and thin out to assume that there is a fixed s such that $q_\alpha \Vdash s_\alpha = s$ for all α. Refining $((p, (x, \dot{X}), f), u)$ if necessary we may assume that it decides the value of s and, what is more, that x extends s and $p \cup q_\alpha \Vdash x - s \subseteq \dot{X}$ for every α.

So now $((p, (x, \dot{X}), f), u)$ forces $\langle r^*_\alpha : \alpha < \kappa^+ \rangle$ is a counterexample to the Knaster property, where r^*_α names a condition of the form $(q_\alpha, (s, \dot{A}_\alpha))$ and $s \subseteq x$. Applying the first claim in $V[\mathbb{R} \restriction \beta * \mathbb{U}^*]$ we may assume that $((p, (x, \dot{X}), f), u)$ forces the sequence $\langle q_\alpha : \alpha < \kappa^+ \rangle$ to consist of mutually compatible conditions in $\mathbb{P}$.

We claim that $((p, (x, \dot{X}), f), u)$ forces $\langle r^*_\alpha : \alpha < \kappa^+ \rangle$ to be mutually compatible. Otherwise we can find α and β and a refinement $((q, (y, \dot{Y}), g), v)$ of $(p, (x, \dot{X}), f), u)$ which decides r^*_α and r^*_β and forces them incompatible. Let us say that the condition $((q, (y, \dot{Y}), g), v)$ forces that $r^*_\alpha = (q_\alpha, (s, \dot{A}_\alpha))$ and $r^*_\beta = (q_\beta, (s, \dot{A}_\beta))$.

It follows that $(q, (y, \dot{Y}))$ must force that $(q_\alpha, (s, \dot{A}_\alpha))$ and $(q_\beta, (s, \dot{A}_\beta))$ are conditions in $\mathbb{P} * \mathbb{Q}/\mathbb{P} \restriction \beta * \mathbb{Q}_\beta$, and that $(q_\alpha \cup q_\beta, (s, \dot{A}_\alpha \cap A_\beta))$ is not a condition. But now $q \cup q_\alpha \cup q_\beta$ is a condition in $\mathbb{P}$ because q, q_α, and q_β are pairwise compatible, so that it must be the case that $q \cup q_\alpha \cup q_\beta \Vdash y - s \nsubseteq \dot{Y}$. This is absurd because $(q \cup q_\alpha \cup q_\beta, (y, \dot{Y}))$ refines $(q, (y, \dot{Y}))$ so it should force that $(q_\alpha, s, \dot{A}_\alpha)) \in \mathbb{P} * \mathbb{Q}/\mathbb{P} \restriction \beta * \mathbb{Q}_\beta$. This concludes the proof of Lemma 7.1 and, with it, the proof of Theorem 2. ∎

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