Asymptotic behavior for differences of eigenvalues of two Sturm–Liouville problems with smooth potentials

Guangsheng Wei*,1, Yan Wang

College of Mathematics and Information Science, Shaanxi Normal University, Xi’an 710062, PR China

Article info

Article history:
Received 27 July 2010
Available online 18 November 2010
Submitted by Goong Chen

Keywords:
Sturm–Liouville problem
Eigenvalue
Boundary condition

1. Introduction

In this paper, we consider the Sturm–Liouville (SL) problem consisting of the differential equation

$$Ly := -y'' + q(x)y = \lambda y,$$  \hspace{1cm} (1.1)

defined on $[0, \pi]$, associated with the following boundary conditions

$$y'(0) = 0, \quad y'(\pi) + hy(\pi) = 0.$$  \hspace{1cm} (1.2)

Here the potential $q \in W^{m-1}_2[0, \pi]$ with $m \in \mathbb{N}$ is a real-valued function and $h \in \mathbb{R} \cup \{\infty\}$, where, in obvious, notation, $h = \infty$ singles out the Dirichlet boundary condition $y(\pi) = 0$. As is well known, the operator $L$ corresponding to the SL problem is self-adjoint and bounded below in Hilbert space $L^2[0, \pi]$, and has discrete spectrum consisting of simple real eigenvalues $\{\lambda_n(h)\} =: \sigma(L)$ for $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Consider a second Sturm–Liouville problem:

$$\tilde{L}y := -y'' + \tilde{q}(x)y = \lambda y,$$  \hspace{1cm} (1.3)

with the same boundary conditions (1.2), where $\tilde{q}$ is also a real-valued function belonging to $W^{m-1}_2[0, \pi]$ with $m \in \mathbb{N}$. We denote by $\{\tilde{\lambda}_n(h)\}_{n=0}^{\infty} := \sigma(\tilde{L})$ the set of the eigenvalues of the SL problem.

It is well known [see, for example, [3,8,9,11,12]] that asymptotic formulas for the eigenvalues of the SL problems have been known for a long time. To be precise, if $q$ is a $W^{m-1}_2$ function with $m \in \mathbb{N}$, then the asymptotic behavior of $\lambda_n$ always takes the form (see [11] for a rigorous analysis):

$$\sqrt{\lambda_n} = n + \frac{a_0}{n} + \frac{a_1}{n^2} + \cdots + \frac{a_{(m-1)/2}}{n^{(m-1)/2}} + \frac{\gamma_n}{n^{(m-1)/2}+1},$$  \hspace{1cm} (1.4)

* Corresponding author.
E-mail addresses: weimath@vip.sina.com (G. Wei), wysnu@126.com (Y. Wang).
1 This research is supported by the NNSF of PR China (No. 10771165).

© 2010 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2010.11.033
where all the coefficients \(a_k\) for \(k = 0, 1, \ldots, 2(m - 1)/2\) are constants independent of \(n\) and \([\alpha]\) denotes the integral part of \(\alpha\). Further, when \(m\) is even, then \(\gamma_n = \alpha(1/n)\), and when \(m\) is odd, then \(\gamma_n = o(1)\). It should be noted that the coefficients \(a_k\) are dependent on \(q^{(i)}\) and, in general, they cannot be expressed in an analytical form except for the first two terms.

The aim of the present paper is to provide the asymptotic expansions for the differences of the eigenvalues \(\lambda_n(h)\) and \(\tilde{\lambda}_n(h)\) of two SL problems with different smooth potentials. This is motivated by the inverse eigenvalues problem of SL problems, for which the smoothness of potentials can instead of some eigenvalues determine uniquely the potential functions and boundary conditions (see [1,5,6]). Whereas the proof of the result in [5] relies on the high-energy asymptotics of the Weyl–Titchmarsh m-function \(m(\lambda, \pi)\) in any sector \(\varepsilon < \text{Arg}(\lambda) < \pi - \varepsilon\) for \(\varepsilon > 0\). This high-energy asymptotics has been considered by a number of authors, see, for example, [2,7] and references therein. It should be noted that the coefficients of the asymptotic expansion of \(m(\lambda, \pi)\) are determined completely by \(q(\pi)\) and its derivatives \(q^{(i)}(\pi)\) for \(i = 1, \ldots, m - 2\). In contrast to this fact, it is a natural question to ask if one can determine completely the coefficients \(a_k\) of \(\sqrt{\lambda_n}\) using only the \(q^{(i)}(\pi)\). Indeed, it is easy to check that \(a_0 = 2/\pi (2h + \int_0^\pi q(t) dt)\) (see [4, p. 5]). This shows that all the \(q^{(i)}(\pi)\) are not sufficient to determine all \(a_k\) and, therefore, another condition needs to be added. In this paper, we attempt to treat this problem.

Here we give the main result of this paper, its proof should be given subsequently.

**Theorem 1.1.** Assume \(h_1, h \in \mathbb{R} \cup \{\infty\}\) with \(h_1 \neq h\), and \(q, \tilde{q} \in W^{m-1}_2[0, \pi]\) with \(m \in \mathbb{N}\) and \(q^{(k)}(\pi) = \tilde{q}^{(k)}(\pi)\) for \(k = 0, 1, \ldots, m - 2\). If the eigenvalues \(\lambda_j(h) = \tilde{\lambda}_j(h_1)\) with some \(h_1 \in \mathbb{R}\) and for all \(j \in \mathbb{N}_0\), then the following formulas for the differences of \(\sqrt{\lambda_n(h)}\) and \(\sqrt{\tilde{\lambda}_n(h)}\) hold for all \(h \in \mathbb{R} \cup \{\infty\}\):

(1) for odd \(m = 2s + 1\),

\[
\sqrt{\lambda_n(h)} - \sqrt{\tilde{\lambda}_n(h)} = (h - h_1) \left(-1\right)^{s+1} \frac{a_{2n} + 2\Delta G_{2s+2}(\pi)}{\pi (2n)^{2s+2}} + \frac{\gamma_{2n}}{n^{2s+3}},
\]

(1.5)

(2) for even \(m = 2s\),

\[
\sqrt{\lambda_n(h)} - \sqrt{\tilde{\lambda}_n(h)} = (h - h_1) \frac{-1}{2(2n)^{2s+1}} + \frac{\gamma_{2n}}{n^{2s+2}},
\]

(1.7)

\[
\sqrt{\lambda_n(\infty)} - \sqrt{\tilde{\lambda}_n(\infty)} = \frac{\gamma_{2n}}{2(2n)^{2s+1}} + \frac{\Delta G_{2s+1}(\pi)}{\pi (2n)^{2s+1}} + \frac{\gamma_{2n}}{2(2n+1)^{2s+1}}.
\]

(1.8)

If the eigenvalues \(\lambda_j(\infty) = \tilde{\lambda}_j(\infty)\) for all \(j \in \mathbb{N}_0\), then

(1) for odd \(m = 2s + 1\),

\[
\sqrt{\lambda_n(h)} - \sqrt{\tilde{\lambda}_n(h)} = (h - h_1) \frac{-1}{2(2n)^{2s+1}} + \frac{\gamma_{2n}}{n^{2s+2}},
\]

(1.9)

(2) for even \(m = 2s\),

\[
\sqrt{\lambda_n(h)} - \sqrt{\tilde{\lambda}_n(h)} = (h - h_1) \left(-1\right)^{s} \frac{a_{2n}}{2(2n)^{2s}} + \frac{2\Delta F_{2s}(\pi)}{\pi (2n)^{2s}} + \frac{\gamma_{2n}}{n^{2s+1}}.
\]

(1.10)

In these formulas \((\gamma_{2n})_1^{\infty}\) is a sequence of \(l^2\) and the numbers \(a_i := a_i((q - \tilde{q})^{(m-1)})\) and \(b_i := b_i(((q - \tilde{q})^{(m-1)})\) are defined by

\[
a_i = 2 \frac{\pi}{\cos(l t) dt}, \quad b_i = 2 \frac{\pi}{\sin(l t) dt}.
\]

(1.11)

While \(\Delta F_{2s}(\pi)\) and \(\Delta G_{2s}(\pi)\) should be defined as in (2.39) and (2.41) below.

**Remark.** As a typical example, if \(q, \tilde{q} \in L^2[0, \pi]\) and \(h_1, h \in \mathbb{R}\) with \(h_1 \neq h\) and \(\lambda_j(h_1) = \tilde{\lambda}_j(h_1)\) for all \(j \in \mathbb{N}_0\), then

\[
\sqrt{\lambda_n(h)} - \sqrt{\tilde{\lambda}_n(h)} = (h_1 - h) \frac{b_{2n}}{4n^2} + \frac{\gamma_{2n}}{n^3},
\]

(1.12)

where \((\gamma_{2n})_1^{\infty}\). To the best of our knowledge, the asymptotic expansion for the differences of the eigenvalues of two SL problems seems to be new.
Similar results may be obtained also for Dirichlet boundary condition at the endpoint \( x = 0 \). Furthermore, this approach may be used also to deal with the asymptotics of the differences of norming constants \( \alpha_n \) of two SL problems, where
\[
\alpha_n = \int_0^\pi \varphi_n^2(t) \, dt \quad \text{and} \quad \varphi_n = \varphi(\cdot, \lambda_n) \quad \text{are the eigenfunctions corresponding to the eigenvalues } \lambda_n, \text{ normalized by } \varphi(0, \lambda_n) = 1.
\]

This article is organized as follows. In Section 2 we recall the solutions of Eq. (1.1) in the form of a power series and present the asymptotic expressions of characteristic functions associated with the SL problems. The proof of Theorem 1.1 is given in Section 3.

2. Asymptotic expressions of characteristic functions

Throughout this paper, \( N \) denotes the set of positive integral and \( W^m_2[0, \pi] \) the Sobolev space of function \( f \) defined on \([0, \pi]\) such that \( f^{(k)} \) for \( k = 0, 1, \ldots, m - 1 \) are absolutely continuous on \([0, \pi]\) and \( f^{(m)} \in L^2[0, \pi] \). We always express \( \lambda = \rho^2 \) and \( \sqrt{X} \) is the square root branch with \( \text{Im}(\sqrt{X}) \geq 0 \). Unless explicitly stated otherwise, we always assume that all following estimates hold in the strip \( |\text{Im}(\rho)| < C \) and \( \text{Re}(\rho) > 0 \), where \( C \) is a positive constant.

In this section, we shall provide the asymptotic expressions of the differences of two characteristic functions of the operators \( L \) and \( \tilde{L} \) associated with different potentials \( q \) and \( \tilde{q} \) under suitable circumstances.

Let \( \varphi(x, \lambda) \) be the solution of the equation
\[
- y'' + q(x)y = \lambda y
\]
with initial conditions \( \varphi(0, \lambda) = 1 \) and \( \varphi'(0, \lambda) = 0 \). Then by the method of successive approximations (see [10, p. 5]), we express the solution \( \varphi(x, \lambda) \) and its derivative \( \varphi'(x, \lambda) \) with respect to the variable \( x \) in the forms of the following power series:
\[
\varphi(x, \lambda) = \sum_{k=0}^\infty \varphi_k(x, \rho), \quad \varphi'(x, \lambda) = \sum_{k=0}^\infty \varphi'_k(x, \rho),
\]
where
\[
\varphi_0(x, \rho) = \cos \rho x, \quad \varphi_k(x, \rho) = \frac{1}{\rho} \int_0^x \sin \rho(x-t)q(t)\varphi_{k-1}(t, \rho) \, dt,
\]
\[
\varphi'_0(x, \rho) = - \rho \sin \rho x, \quad \varphi'_k(x, \rho) = \int_0^x \cos \rho(x-t)q(t)\varphi_{k-1}(t, \rho) \, dt.
\]
In view of the estimates (see, for example, [10, p. 14])
\[
|\varphi_k(x, \rho)| \leq \frac{\|q\| k^{1/2} e^{k|x|/|\rho|}}{\sqrt{k!}|\rho|^{k-1}}, \quad \left| \varphi'_k(x, \rho) \right| \leq \frac{\|q\| k^{1/2} e^{k|x|/|\rho|}}{\sqrt{k!}|\rho|^{k-1}},
\]

it is easily verified by induction that the series in (2.2) are uniformly convergent on \([0, \pi]\).

For our purpose, we need to have exact information on the summands \( \varphi_k \) for \( k \leq m + 1 \) and \( \varphi'_k \) for \( k \leq m \). Just as above, the notation \( O(\rho^{-k}) \) or \( o(n^{-k}) \) is used for expressions whose absolute value admits estimates \( \leq C|\rho|^{-k} \) or \( \leq C n^{-k}, n \geq 1 \), in which the constant \( C \) only depends on \( R = \|q\| \). For the sum of the other summands, from (2.4) we obtain the estimates
\[
\left| \sum_{k=m+2}^\infty \varphi_k(x, \rho) \right| = O(\rho^{-m-2}), \quad \left| \sum_{k=m+2}^\infty \varphi'_k(x, \rho) \right| = O(\rho^{-m-1}),
\]
for \( |\rho| \) sufficiently large and \( |\text{Im}(\rho)| < C \), where \( C \) is a positive constant.

We adopt the following notations
\[
\sigma(x) = \int_0^x q(t) \, dt, \quad j(\pm) = \begin{cases} -1 & \text{if } j = 4s, 4s + 1, \\ 1 & \text{if } j = 4s + 2, 4s + 3, \end{cases}
\]
\[
u_{2s}(x, \rho) = (2\rho)^{-2s} \cos \rho x, \quad \nu_{2s+1}(x, \rho) = (2\rho)^{-2s-1} \sin \rho x.
\]
In should be noted that \( j(\pm) = (-1)^{(j+1)(j+2)}/2 \). In terms of the above notations, letting \( q(x) \in W^{m-1}_2[0, \pi] \) with \( m \geq 1 \), according to [11, Lemma 4.2], for similar computations, we have the following representations for \( \varphi_k \) with \( k = 1, 2, \ldots, m \).
where the functions \( g_{1j}(x) \) for \( j = 1, 2, \ldots, m \) are defined by

\[
g_{11}(x) = \sigma(x) - \sigma(0),
\]

\[
g_{1j}(x) = (j + 1)^{(\pm)}(\sigma^{(j-1)}(x) - (-1)^j\sigma^{(j-1)}(0)),
\]

and, for \( k \geq 2 \), the functions \( g_{kj}(x) \) with \( j = k, k+1, \ldots, m+1 \) are determined by the recurrence relations

\[
g_{kj}(x) = (-1)^{j-1}\int_0^x q(t)g_{k-1,j-1}(t)\,dt - \sum_{s=1}^{j-2}(s+1)^{(\pm)}(j+1)^{(\pm)}[qg_{k-1,s}](j-2-s)(x)
\]

\[
\quad - (-1)^j(qg_{k-1,s})(j-2-s)(0)\right].
\]

From the above recurrence relations, it is easily seen that \( g_{kj}(x) \equiv 0 \) for all \( k > j \) and \( g_{kj}(x) \) belong to \( W_{2}^{m+k-j-1}[0, \pi] \) for \( k \leq j \). Furthermore, we have for \( \phi_j'(x) \) with \( j = 1, 2, \ldots, m \)

\[
\phi_j'(x) = \sum_{j=0}^{m-1} u_j(x, \rho)f_{1j}(x) + \frac{(m+1)^{(\pm)}}{2}\int_0^x u_{m-1}(x-2t, \rho)q^{(m-1)}(t)\,dt,
\]

\[
\phi_j'(x) = \sum_{j=0}^{m} u_j(x, \rho)f_{2j}(x) - \frac{(m+2)^{(\pm)}}{2}\int_0^x u_{m}(x-2t, \rho)(q\sigma)^{(m-1)}(t)\,dt + O(\rho^{-m-1}),
\]

\[
\phi_k'(x, \rho) = \sum_{j=0}^{m} u_j(x, \rho)f_{kj}(x) + O(\rho^{-m-1}), \quad k = 3, 4, \ldots, m+1,
\]

where the functions \( f_{1j}(x) \) with \( j = 0, 1, \ldots, m-1 \) are defined by

\[
f_{10}(x) = \frac{\sigma(x) - \sigma(0)}{2},
\]

\[
f_{1j}(x) = \frac{(j + 1)^{(\pm)}(\sigma^{(j)}(x) - (-1)^j\sigma^{(j)}(0))}{2},
\]

and, for \( k \geq 2 \), the functions \( f_{kj}(x) \) with \( j = k-1, \ldots, m \) are determined by

\[
f_{kj}(x) = \frac{1}{2}\int_0^x q(t)g_{k-1,j}(t)\,dt - \frac{1}{2}\sum_{s=1}^{j-2}(s+1)^{(\pm)}(j+1)^{(\pm)}[qg_{k-1,s}](j-2-s)(x)
\]

\[
\quad - (-1)^j(qg_{k-1,s})(j-2-s)(0)\right].
\]

It should be also noted that \( f_{kj}(x) \equiv 0 \) for all \( k > j \) and \( f_{kj}(x) \) belong to \( W_{2}^{m+k-j-1}[0, \pi] \) for \( k < j \).

Summing up formulas (2.9) and (2.12), respectively, we have from (2.11) and (2.14) that

\[
\phi(x, \rho) = \sum_{j=0}^{m+1} G_j(x)u_j(x, \rho) + R_1(x, \rho, q) + O(\rho^{-m-2}),
\]

\[
\phi'(x, \rho) = -\rho \sin \rho x + \sum_{j=0}^{m} F_j(x)u_j(x, \rho) + R_2(x, \rho, q) + O(\rho^{-m-1}).
\]

where
\[ G_0(x) = 1, \quad G_j(x) = \sum_{i=1}^{j} g_{ij}(x), \quad G_{m+1}(x) = \sum_{i=2}^{m+1} g_{i,m+1}(x). \]  
(2.17)

\[ F_j(x) = \sum_{i=1}^{j+1} f_{ij}(x), \quad F_m(x) = \sum_{i=2}^{m} f_{im}(x), \]  
(2.18)

and

\[
R_1(x, \rho, q) = (m + 2)^{[\pm]} \int_0^x u_m(x - 2t, \rho) q^{(m-1)}(t) \, dt - (m + 3)^{[\pm]} \int_0^x u_{m+1}(x - 2t, \rho) (q\sigma)^{(m-1)}(t) \, dt,  
(2.19)
\]

\[
R_2(x, \rho, q) = \frac{(m + 1)^{[\pm]}}{2} \int_0^x u_{m-1}(x - 2t, \rho) q^{(m-1)}(t) \, dt - \frac{(m + 2)^{[\pm]}}{2} \int_0^x u_m(x - 2t, \rho) (q\sigma)^{(m-1)}(t) \, dt.  
(2.20)
\]

We will provide some relations of the coefficients \( G_j(x) \) and \( F_j(x) \) in formulas (2.15) and (2.16), which should be used subsequential. It is easy to check that \( G_j \in W_2^{m-j+1}[0, \pi] \) and \( F_j \in W_2^{m-1}[0, \pi] \).

**Lemma 2.1.** Suppose that \( q \in W_2^{m-1}[0, \pi] \) with \( m \geq 1 \). Then the functions \( G_j(x) \) for \( j = 1, 2, \ldots, m \) given in (2.17) satisfy

\[ G_j'(x) = -(j + 1)^{[\pm]} \left( \sum_{s=1}^{j-1} (s + 1)^{[\pm]} (qG_s)(j-1-s)(x) + \sigma^{(j)}(x) \right). \]  
(2.21)

**Proof.** By differentiating (2.17) with respect to the variable \( x \), we have

\[ G_j'(x) = \sum_{i=1}^{j} g'_{ij}(x) \]  
(2.22)

and, by (2.10) and (2.11),

\[ g'_{ij}(x) = (j + 1)^{[\pm]} \sigma^{(j)}(x), \]

\[ g'_{ij}(x) = \sum_{s=1}^{j-1} (s + 1)^{[\pm]} (qg_{i-1,s})(j-1-s)(x) \]

\[ = - \sum_{s=1}^{j-1} (s + 1)^{[\pm]} (qg_{i-1,s})(j-1-s)(x). \]  
(2.23)

Substituting (2.23) into (2.22) we obtain

\[ G_j'(x) = - \sum_{i=1}^{j} \left( \sum_{s=1}^{j-1} (s + 1)^{[\pm]} (qg_{i-1,s})(j-1-s)(x) \right) + (j + 1)^{[\pm]} \sigma^{(j)}(x) \]

\[ = - \sum_{s=1}^{j-1} (s + 1)^{[\pm]} (qg_{s})(j-1-s)(x) + (j + 1)^{[\pm]} \sigma^{(j)}(x). \]

This completes the proof of Lemma 2.1. \( \square \)

**Lemma 2.2.** We have

\[ G_j(x) = (-1)^{j-1} \int_0^x q(t) G_{j-1}(t) \, dt + (-1)^j G_{j-1}'(x) + G_j^0(0), \]  
(2.24)

where

\[ G_j^0(0) = \sum_{s=1}^{j-2} (s + 1)^{[\pm]} j^{[\pm]} (qG_s)(j-2-s)(0) - j^{[\pm]} \sigma^{(j-1)}(0). \]  
(2.25)
Proof. Substituting $g_{ij}(x)$ given in (2.11) into (2.17), we obtain the following formula

$$G_j(x) = (-1)^{j-1} \sum_{i=2}^{j} \int_{0}^{x} q(t)g_{i-1,j-1}(t) \, dt$$

$$+ (-1)^{j+1} \sum_{i=2}^{j} \left( \sum_{s=1}^{j-2} (s + 1)^{(\pm)} j^{(\pm)} (qg_{i-1,s})^{(j-2-s)}(x) \right)$$

$$+ \sum_{i=2}^{j} \left( \sum_{s=1}^{j-2} (s + 1)^{(\pm)} j^{(\pm)} (qg_{i-1,s})^{(j-2-s)}(0) \right)$$

$$+ (-1)^{j} j^{(\pm)} \sigma^{(j-1)}(x) - j^{(\pm)} \sigma^{(j-1)}(0)$$

$$= (-1)^{j-1} \int_{0}^{x} q(t)G_{j-1}(t) \, dt$$

$$+ (-1)^{j} \left( - \sum_{s=1}^{j-2} (s + 1)^{(\pm)} j^{(\pm)} (qG_{s})^{(j-2-s)}(x) + j^{(\pm)} \sigma^{(j-1)}(x) \right)$$

$$+ \sum_{s=1}^{j-2} (s + 1)^{(\pm)} j^{(\pm)} (qG_{s})^{(j-2-s)}(0) - j^{(\pm)} \sigma^{(j-1)}(0).$$

(2.26)

By (2.21) we complete the proof. $\square$

**Lemma 2.3.** Let $F_j(x)$ be given as in (2.18). Then we have

$$F_j(x) = \frac{(-1)^{j}}{2} G_{j+1}(x) + G'_j(x), \quad j = 0, 1, \ldots, m,$$

where $j = m$, the function $G'_m(x) = \sum_{i=2}^{m} g'_{i,m}(x)$.

**Proof.** By (2.14) we rewrite these as

$$f_{ij}(x) = \frac{1}{2} \int_{0}^{x} q(t)g_{i-1,j}(t) \, dt + (-1)^{j} \frac{1}{2} \sum_{s=1}^{j-1} (s + 1)^{(\pm)} (j + 1)^{(\pm)} (qg_{i-1,s})^{(j-1-s)}(0)$$

$$- \frac{1}{2} \sum_{s=1}^{j-1} (s + 1)^{(\pm)} (j + 1)^{(\pm)} (qg_{i-1,s})^{(j-1-s)}(x),$$

(2.28)

where $i = 2, 3, \ldots, m + 1$. Note that $(j + 2)^{(\pm)} = -(j + 1)^{(\pm)}(1)^j$ by (2.7). Thus, by (2.11) we conclude

$$(-1)^{j} \int_{0}^{x} q(t)g_{i-1,j}(t) \, dt + \sum_{s=1}^{j-1} (s + 1)^{(\pm)} (j + 1)^{(\pm)} (qg_{i-1,s})^{(j-1-s)}(0)$$

$$= g_{i,j+1}(x) + (-1)^{j+1} \sum_{s=1}^{j-1} (s + 1)^{(\pm)} (j + 1)^{(\pm)} (qg_{i-1,s})^{(j-1-s)}(x).$$

(2.29)

Multiplying (2.29) by $(-1)^{j}/2$ and substituting it into (2.28), we have

$$f_{ij}(x) = \frac{(-1)^{j}}{2} g_{i,j+1}(x) - \sum_{s=1}^{j-1} (s + 1)^{(\pm)} (j + 1)^{(\pm)} (qg_{i-1,s})^{(j-1-s)}(x).$$

(2.30)

Note that when $i = 1$, the functions $f_{ij}(x)$ and $g_{ij}(x)$ have the following relations

$$f_{1j}(x) = \frac{(-1)^{j}}{2} g_{1,j+1}(x) + (j + 1)^{(\pm)} \sigma^{(j)}(x).$$

(2.31)
Substituting (2.30) and (2.31) into (2.28) and applying (2.14), we have

\[ F_j(x) = \frac{(-1)^j}{2} \sum_{i=1}^{j+1} G_{i+j+1}(x) - \sum_{i=2}^{j+1} (s + 1)^{(\pm)} (j + 1)^{(\pm)} (q g_{i-1})^{(j-1-s)}(x) + (j + 1)^{(\pm)} \sigma^{(j)}(x) \]

\[ = \frac{(-1)^j}{2} G_{j+1}(x) - \sum_{s=1}^{j-1} (s + 1)^{(\pm)} (j + 1)^{(\pm)} (q G_s)^{(j-1-s)}(x) + (j + 1)^{(\pm)} \sigma^{(j)}(x) \]

\[ = \frac{(-1)^j}{2} G_{j+1}(x) + G'_j(x), \] (2.32)

where \( j = m \), the function \( G_m(x) = \sum_{i=2}^{m} G_{im}(x) \). This completes the proof. □

Let us define the characteristic function corresponding the operator \( L(q, h) \) by

\[ \omega(\rho, h) = \varphi'(\tau, \lambda) + h \varphi(\tau, \lambda), \] (2.33)

where \( \rho = \sqrt{h} \). Then all the zeros of \( \omega(\rho, h) \) are the eigenvalues \( \{\lambda_n\}_{n=0}^{\infty} := \sigma(L) \). It is well known [4] that \( \omega(\rho, h) \) is an entire function of \( \rho \) for each \( h \in \mathbb{R} \cup \{\infty\} \). In this case, by the expansions (2.15) and (2.16) of \( \varphi(x, \lambda) \) and \( \varphi'(x, \lambda) \) we infer that if \( h = \infty \), then

\[ \omega(\rho, \infty) = \sum_{j=0}^{m+1} G_j(\tau) u_j(\tau, \rho) + R_1(\tau, \rho, q) + O\left(\rho^{-m-2}\right), \] (2.34)

where \( G_j(\tau) \) and \( R_1(\tau, \rho, q) \) are defined by (2.17) and (2.19) with \( x = \tau \); and if \( h \in \mathbb{R} \) then

\[ \omega(\rho, h) = -\rho \sin(\pi \rho) + \sum_{j=0}^{m} H_j(\tau) u_j(\tau, \rho) + H_0(\tau, \rho, q) + O\left(\rho^{-m-1}\right), \] (2.35)

where \( H_0 = f_{10}(\tau) + h \) and \( H_j(\tau) = F_j(\tau) + h G_j(\tau) \) for \( j \geq 1 \) and

\[ H_0(\tau, \rho, q) = \frac{(m + 1)^{(\pm)}}{2} \int_0^{\pi} u_{m-1}(\tau - 2t, \rho) q^{(m-1)}(t) dt \]

\[ + \frac{(m + 2)^{(\pm)}}{2} \int_0^{\pi} u_m(\tau - 2t, \rho) (2h q - q \sigma)^{(m)}(t) dt. \] (2.36)

Let us consider the second operator \( \tilde{L} := \tilde{L}(\tilde{q}, h) \) associated with the potential \( \tilde{q} \in W_{2}^{m-1}[0, \pi] \) with \( m \geq 1 \) and the same boundary conditions (1.2). Then its eigenvalues \( \{\tilde{\lambda}_n\}_{n=0}^{\infty} := \sigma(\tilde{L}) \) are precisely the zeros of the characteristic function

\[ \tilde{\omega}(\rho, \infty) = \tilde{\varphi}'(\tau, \lambda) + h \tilde{\varphi}(\tau, \lambda), \] (2.37)

where \( \tilde{\varphi}(x, \rho) \) is the solution of Eq. (2.1) replaced \( q \) by \( \tilde{q} \) with initial conditions \( \tilde{\varphi}(0, \lambda) = 1 \) and \( \tilde{\varphi}'(0, \lambda) = 0 \). Similar argument also enables us to obtain the expansions of \( \tilde{\varphi}'(\tau, \lambda) \) and \( \tilde{\varphi}(\tau, \lambda) \) and therefore that of \( \tilde{\omega}(\rho, h) \), see (2.34) and (2.35), in which \( G_j(\tau) \) and \( F_j(\tau) \) are replaced by \( G_j(\tau) \) and \( F_j(\tau) \).

Here is our main result of this section.

**Theorem 2.4.** Let \( h_1, h \in \mathbb{R} \cup \{\infty\} \) with \( h_1 \neq h \). Assume that \( q, \tilde{q} \in W_{2}^{m-1}[0, \pi] \) with \( m \geq 1 \) and \( q^{(k)}(\tau) = \tilde{q}^{(k)}(\tau) \) (\( k = 0, 1, \ldots, m - 2 \)). If \( \omega(\rho, \infty) = \tilde{\omega}(\rho, \infty) \) then

\[ \omega(\rho, h) - \tilde{\omega}(\rho, h) = \Delta F_m(\tau) u_m(\tau, \rho) + \Delta H_0(\tau, \rho) + O\left(\frac{1}{\rho^{m+1}}\right), \] (2.38)

as \( \rho \to \infty \), where \( \Delta H_0(\tau, \rho) = H_0(\tau, \rho, q - \tilde{q}) \) (see (2.36)) and

\[ \Delta F_m(\tau) = F_m(\tau) - \tilde{F}_m(\tau). \] (2.39)

If \( \omega(\rho, h_1) = \tilde{\omega}(\rho, h_1) \) with \( h_1 \in \mathbb{R} \), then

\[ \omega(\rho, h) - \tilde{\omega}(\rho, h) = A(h) \left( \Delta G_{m+1}(\tau) u_{m+1}(\tau, \rho) + \Delta R_1(\tau, \rho) \right) + O\left(\frac{1}{\rho^{m+2}}\right), \] (2.40)
as $\rho \to \infty$, where $A(h) = h - h_1$ as $h \in \mathbb{R}$ and $A(h) = 1$ as $h = \infty$, and $\Delta R_1(\pi, \rho) = R_1(\pi, \rho, q - \tilde{q})$ (see (2.19)) and

$$\Delta G_{m+1}(\pi) = G_{m+1}(\pi) - \tilde{G}_{m+1}(\pi).$$  \hspace{1cm} (2.41)

**Proof.** We first consider the case of $h_1 = \infty$ and $h \in \mathbb{R}$. In this case, letting $\rho_{1,n} = 2n$ and $\rho_{2,n} = 2n + 1/2$ in (2.34), it is easy to see from (2.8) that $u_{2s}(\pi, \rho_{1,n}) = 1$ and $u_{2s-1}(\pi, \rho_{2,n}) = 1$. This shows

$$G_j(\pi) = \tilde{G}_j(\pi)$$  \hspace{1cm} (2.42)

for $j = 0, 1, \ldots, m$, since $o(\rho, \infty) = \tilde{o}(\rho, \infty)$. Further, we differentiate two sides of Eq. (2.24) $k$ times with respect to the variable $x$, and obtain the following formulas

$$G_j^{(k)}(\pi) = (-1)^{j-1}(qG_{j-1})(\pi) + (-1)^j G_j^{(k+1)}(\pi),$$  \hspace{1cm} (2.43)

where $k = 1, 2, \ldots, m - j$. Results analogous to (2.43), we have for $\tilde{G}_j$ that

$$\tilde{G}_j^{(k)}(\pi) = (-1)^{j-1}(\tilde{q}\tilde{G}_{j-1})(\pi) + (-1)^j \tilde{G}_j^{(k+1)}(\pi).$$

We proceed by induction to show $G_j^{(k)}(\pi) = \tilde{G}_j^{(k)}(\pi)$ for $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, m - j$. It is noted that for $j = 1$ we easily ensure

$$G_1^{(k)}(\pi) = \tilde{G}_1^{(k)}(\pi) \quad \text{for } k = 1, \ldots, m - 1$$  \hspace{1cm} (2.44)

since $q^{(k)}(\pi) = \tilde{q}^{(k)}(\pi)$ ($k = 0, 1, \ldots, m - 2$) and $G_1(\pi) = \sigma(\pi) - \sigma(0)$. If $j = 2$ then by (2.24) and (2.43) we have

$$G_2^{(k)}(\pi) = -(qG_1)(\pi) + G_1^{(k+1)}(\pi)$$  \hspace{1cm} (2.45)

for $k = 1, 2, \ldots, m - 2$. This combined with (2.42) shows that $G_2^{(k)}(\pi) = \tilde{G}_2^{(k)}(\pi)$. Suppose that when $j = 2, 3, \ldots, m - 2$, the formulas $G_j^{(k)}(\pi) = \tilde{G}_j^{(k)}(\pi)$ hold for $k = 1, 2, \ldots, m - j$. By $G_{m-2}(\pi) = \tilde{G}_{m-2}(\pi)$ and (2.43) we infer

$$G_{m-1}(\pi) = (-1)^{j-1}(qG_{m-2})(\pi) + (-1)^j G_{m-2}'(\pi) = \tilde{G}_{m-1}'(\pi).$$  \hspace{1cm} (2.46)

Based on the above results, by Lemma 2.3 we have

$$F_j(\pi) = \frac{(-1)^{j-1}}{2} G_{j+1}(\pi) + G_j'(\pi) = \tilde{F}_j(\pi)$$

for $j = 0, 1, \ldots, m - 1$. This together with (2.35) and (2.42) shows that (2.38) holds.

Secondly, we consider the case of $h_1 \in \mathbb{R}$ and $h \in \mathbb{R} \cup \{\infty\}$. It should be noted that if $h \in \mathbb{R}$ then

$$o(\rho, h) - \tilde{o}(\rho, h) = (h - h_1)(o(\rho, \infty) - \tilde{o}(\rho, \infty)).$$  \hspace{1cm} (2.47)

Thus we only need to treat the case of $h = \infty$. In this case, by (2.35) we easily conclude that $H_j(\pi) = \tilde{H}_j(\pi)$ for $j = 0, 1, \ldots, m - 1$ and therefore $G_1(\pi) = \tilde{G}_1(\pi)$. From (2.43) and the conditions of $q^{(k)}(\pi) = \tilde{q}^{(k)}(\pi)$ ($k = 0, 1, \ldots, m - 2$) we have

$$G_j^{(k)}(\pi) = \tilde{G}_j^{(k)}(\pi) \quad \text{for } j = 1, \ldots, m - 1 \text{ and } k = 0, \ldots, m - j.$$  \hspace{1cm} (2.48)

Furthermore, by Lemma 2.3, we have

$$\frac{(-1)^{j-1}}{2} G_j(\pi) = F_{j-1}(\pi) + G_{j-1}'(\pi) = H_{j-1}(\pi) - h_1 G_{j-1}(\pi) + G_{j-1}'(\pi).$$  \hspace{1cm} (2.49)

Proceeding by induction, we conclude $G_j(\pi) = \tilde{G}_j(\pi)$ for $j = 0, 1, \ldots, m$ by (2.48) and (2.49). The proof is completed. \qed

3. **Proof of Theorem 1.1**

In this section, we shall prove Theorem 1.1. The proof is based on the following lemma and Theorem 2.4.
Lemma 3.1. Assume \( q \in L^2[0, \pi] \). If \( h = \infty \) and \( \zeta_n = n + 1/2 + O(1/n) \) for sufficiently larger \( n \), then

\[
\frac{\partial \omega}{\partial \rho}(\zeta_n, \infty) = (-1)^{n+1} \pi + O\left(\frac{1}{n}\right).
\]  

(3.1)

If \( h \in \mathbb{R} \) and \( \zeta_n = n + O(1/n) \) for sufficiently larger \( n \), then

\[
\frac{\partial \omega}{\partial \rho}(\zeta_n, h) = (-1)^{n+1} n \pi + O\left(\frac{1}{n}\right).
\]

(3.2)

Proof. Let us only consider the case of \( h = \infty \). The same way can treat the case of \( h \in \mathbb{R} \). Let \( \psi := \varphi(x, \rho^2) \) and \( \psi := \psi(x, \rho^2) \) be the solutions of Eq. (2.1) satisfying the following initial conditions

\[
\varphi(0, \rho^2) = 1, \quad \varphi'(0, \rho^2) = 0; \quad \psi(\pi, \rho^2) = 0, \quad \psi'(\pi, \rho^2) = 1.
\]

Then

\[
(\rho^2 - \zeta_n^2) \int_0^\pi \varphi(x, \rho^2) \psi(x, \zeta_n^2) \, dx = \left[ \frac{\psi(x, \zeta_n^2)}{\varphi(x, \rho^2)} \right]_0^\pi - \omega(\rho^2, \infty) + \omega(\zeta_n^2, \infty).
\]

Letting \( \rho \to \zeta_n \), this deduces that

\[
\frac{\partial \omega}{\partial \rho^2}(\zeta_n, \infty) = -\int_0^\pi \varphi(x, \zeta_n^2) \psi(x, \zeta_n^2) \, dx.
\]

(3.3)

Note that the following asymptotic formulas hold (see [4, pp. 4–9]):

\[
\varphi(x, \rho^2) = \cos(\rho x) + O\left(\frac{1}{\rho}\right), \quad \psi(x, \rho^2) = \frac{\sin(\rho(\pi - x))}{\rho} + O\left(\frac{1}{\rho}\right)
\]

uniformly with respect to \( x \in [0, \pi] \), as \( \rho \to +\infty \). This yields

\[
\int_0^\pi \varphi(x, \zeta_n^2) \psi(x, \zeta_n^2) \, dx = \frac{1}{\zeta_n} \int_0^\pi \cos(\zeta_n x) \sin((\pi - \zeta_n) x) \, dx + O\left(\frac{1}{\zeta_n}\right)
\]

\[
= \frac{1}{2\zeta_n} \int_0^\pi [\sin(\zeta_n \pi) - \sin((\pi - \zeta_n) x)] \, dx + O\left(\frac{1}{\zeta_n}\right)
\]

\[
= \frac{\pi - \sin(\zeta_n \pi) + O\left(\frac{1}{\zeta_n}\right)}{2\zeta_n}.
\]

Since \( \zeta_n = n + 1/2 + O(1/n) \), it follows from (3.3) that

\[
\frac{\partial \omega}{\partial \rho^2}(\zeta_n, \infty) = (-1)^{n+1} \frac{\pi}{2n+1} + O\left(\frac{1}{n}\right).
\]

Note that \( \partial \omega/\partial \rho = 2\rho \partial \omega/\partial \rho^2 \). This yields (3.1) and the proof is completed. \( \square \)

We now prove Theorem 1.1.

Proof of Theorem 1.1. Let us treat the case of \( h_1 \in \mathbb{R} \). In this case, let \( \rho_n = \sqrt{\lambda_n(h)} \) and \( \tilde{\rho}_n = \sqrt{\lambda_n(h)} \) and, without loss of generality, we always assume that \( \rho_0 \neq \tilde{\rho}_n \). Since \( \lambda_n(h_1) = \lambda_n(h_1) \) for all \( n \in \mathbb{N}_0 \), it follows from [4, p. 12] that

\[
\omega(\rho, h_1) = \pi \left( \lambda_0(h_1) - \rho^2 \right) \prod_{n=1}^\infty \left( \frac{\lambda_n(h_1) - \rho^2}{n^2} \right) = \tilde{\omega}(\rho, h_1).
\]

(3.4)
Thus, by Lagrange Theorem, we have for all $n \in \mathbb{N}_0$ that
\[
0 = \omega(\rho_n, h) - \dot{\omega}(\tilde{\rho}_n, h) \\
= (\omega(\rho_n, h) - \omega(\tilde{\rho}_n, h)) + (\omega(\tilde{\rho}_n, h) - \dot{\omega}(\tilde{\rho}_n, h)) \\
= (\rho_n - \tilde{\rho}_n) \frac{\partial \omega}{\partial \rho}(\zeta_n, h) + A(h)(\omega(\tilde{\rho}_n, \infty) - \dot{\omega}(\tilde{\rho}_n, \infty)),
\]
(3.5)
where $\zeta_n \in (\rho_n, \tilde{\rho}_n)$ exist and $A(h)$ is defined as in Theorem 2.4 (see (2.40)).

Let $m = 2s + 1$ and $h \in \mathbb{R}$. Then
\[
\tilde{\rho}_n = n + O \left( \frac{1}{n} \right)
\]
and therefore $\cos(\tilde{\rho}_n \pi) = (-1)^n + 0(1/n)$ and $\sin(\tilde{\rho}_n \pi) = O(1/n)$. This shows
\[
\begin{align*}
0 = \int_0^\pi u_m(\pi - 2t, \tilde{\rho}_n) (\Delta q)^{(m-1)}(t) \, dt &= \int_0^\pi \cos \pi \tilde{\rho}_n \sin 2t \tilde{\rho}_n (\Delta q)^{(m-1)}(t) \, dt + \gamma_{2n} \left( \frac{1}{2n} \right)^{2s+2} \sin 2t \tilde{\rho}_n \, dt \\
&= \frac{\pi (-1)^n}{2} b_{2n} + \gamma_{2n} \left( \frac{1}{2n} \right)^{2s+2}, \\
&= \frac{\pi (-1)^n}{2} b_{2n} + \gamma_{2n} \left( \frac{1}{2n} \right)^{2s+2}.
\end{align*}
\]
(3.6)
(3.7)
\[
\int_0^\pi u_{m+1}(\pi - 2t, \tilde{\rho}_n) (\Delta q)(q\sigma)^{(m-1)}(t) \, dt = \frac{\gamma_{2n}}{2} \left( \frac{1}{2n} \right)^{2s+2}.
\]
(3.8)
where $\Delta q = q - \bar{q}$ and $\Delta(q\sigma) = \sigma q - \bar{\sigma} \bar{q}$. Note that $(m + 2)^{(d)} = (-1)^d$. By Theorem 2.4 and (3.6)-(3.8) and (2.19) we have
\[
\omega(\tilde{\rho}_n, \infty) - \dot{\omega}(\tilde{\rho}_n, \infty) = \Delta G_{2s+2}(\pi) u_{m+1}(\pi, \tilde{\rho}_n) + \Delta R_1(\pi, \tilde{\rho}_n) + O \left( \frac{1}{\rho_n^{m+2}} \right) \\
= (-1)^n \Delta G_{2s+2}(\pi) + \pi(-1)^d (-1)^{n+1} b_{2n} + \gamma_{2n} \left( \frac{1}{2n} \right)^{2s+2}.
\]
and, therefore, by Lemma 3.1 and (3.5),
\[
\begin{align*}
\rho_n - \tilde{\rho}_n &= - \frac{h - h_1}{\frac{\partial \omega}{\partial \rho}(\zeta_n, h)} (\omega(\tilde{\rho}_n, \infty) - \dot{\omega}(\tilde{\rho}_n, \infty)) \\
&= (h - h_1) \left( \frac{(-1)^{d+1} b_{2n}}{(2n)^{2s+2}} + \frac{2 \Delta G_{2s+2}(\pi)}{(2n)^{2s+3}} \right) + \gamma_{2n} \left( \frac{1}{n^{2s+3}} \right).
\end{align*}
\]
(3.9)
This yields that (1.5) holds.

Let $m = 2s + 1$ and $h = \infty$. Then
\[
\tilde{\rho}_n = n + \frac{1}{2} + O \left( \frac{1}{n} \right)
\]
(3.10)
and therefore $\cos(\tilde{\rho}_n \pi) = O(1/n)$ and $\sin(\tilde{\rho}_n \pi) = (-1)^n + O(1/n)$. This shows
\[
u_{m+1}(\pi, \tilde{\rho}_n) = \frac{1}{(2\tilde{\rho}_n)^{2s+2}} \cos(\tilde{\rho}_n \pi) = O \left( \frac{1}{(2n + 1)^{2s+3}} \right)
\]
(3.11)
and
\[
\int_0^\pi u_m(\pi - 2t, \tilde{\rho}_n) (\Delta q)^{(m-1)}(t) \, dt = \frac{1}{(2\tilde{\rho}_n)^{2s+1}} \int_0^\pi \sin((\pi - 2t) \tilde{\rho}_n) (\Delta q)^{(m-1)}(t) \, dt \\
= \frac{\pi (-1)^n a_{2n+1}}{2} + \frac{\gamma_{2n}}{2} \left( \frac{1}{(2n + 1)^{2s+2}} \right).
\]
(3.12)
By Theorem 2.4 and Lemma 3.1 we have from (2.40), (3.11)–(3.12) that
\[\rho_n - \tilde{\rho}_n = \frac{1}{\partial \omega / \partial \rho (\zeta_n, h)} \left( \Delta G_{m+1}(\pi) u_{m+1}(\pi, \rho) + \Delta R_1(\pi, \rho) + O\left(\frac{1}{\rho^{m+2}}\right) \right)\]
\[= \frac{(-1)^{s+1} a_{2n+1}}{(2n + 1)2^{s+1}} + \frac{\gamma_{2n}}{(2n + 1)2^{s+2}}.\]

This yields that (1.6) holds. Formally, we must take other steps: letting \(m = 2s\) and \(h \in \mathbb{R}\) and \(h = \infty\). This can be done in the same way.

For the case of \(h_1 = \infty\) and \(h \in \mathbb{R}\), from the fact of \(\omega(\rho, \infty) = \tilde{\omega}(\rho, \infty)\) and (3.5) we only need to emphasize that
\[\rho_n - \tilde{\rho}_n = -\frac{1}{\partial \omega / \partial \rho (\zeta_n, h)} \left( \omega(\tilde{\rho}_n, 0) - \tilde{\omega}(\tilde{\rho}_n, 0) \right).\]

where
\[
\omega(\tilde{\rho}_n, 0) - \tilde{\omega}(\tilde{\rho}_n, 0) = \frac{(-1)^n \Delta H_{2s}(\pi)}{\pi (2n)^{2s}} + \frac{\pi (-1)^{s+b_{2n}}}{2(2n)^{2s-1}} + \frac{\gamma_{2n}}{n^{2s}} \quad \text{if} \quad m = 2s,
\]
\[
\omega(\tilde{\rho}_n, 0) - \tilde{\omega}(\tilde{\rho}_n, 0) = \frac{\pi (-1)^{s+b_{2n+1}}}{2(2n)^{2s+1}} + \frac{\gamma_{2n}}{(2n)^{2s+1}} \quad \text{if} \quad m = 2s + 1.
\]

By Lemma 3.1, the formulas (1.9) and (1.10) hold. The proof is completed. □

References