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Refinement type equations and Grincevičjus series ☆

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ABSTRACT

We consider L^1-solutions of the following refinement type equations

$$f(x)=\sum_{n\in\mathbb{Z}}c_{n,1}f(kx-n)+\sum_{n\in\mathbb{Z}}c_{n,-1}f(-kx-n),$$

where $k\geqslant 2$ is an integer and for all $n\in\mathbb{Z}$ reals $c_{n,1}$, $c_{n,-1}$ are non-negative with $\sum_{n\in\mathbb{Z}}(c_{n,1}+c_{n,-1})=k$ and $\sum_{n\in\mathbb{Z}}\log|n|(c_{n,1}+c_{n,-1})<+\infty$. Necessary and sufficient conditions for the existence of non-trivial L^1-solutions in several special cases are given.

1. Introduction

In this paper we are interested in L^1-solutions $f:\mathbb{R}\to\mathbb{R}$ of the following refinement type equation

$$f(x)=\sum_{n\in\mathbb{Z}}c_{n,1}f(kx-n)+\sum_{n\in\mathbb{Z}}c_{n,-1}f(-kx-n) \tag{1.1}$$

assuming that $k>1$ is a fixed integer and the following hypothesis concerning the given reals $c_{n,\varepsilon}$'s.

(H) For all $n\in\mathbb{Z}$ the non-negative reals $c_{n,1}$, $c_{n,-1}$ satisfy

$$\sum_{n\in\mathbb{Z}}(c_{n,1}+c_{n,-1})=k, \tag{1.2}$$

$$\sum_{n\in\mathbb{Z}}\log|n|(c_{n,1}+c_{n,-1})<+\infty, \tag{1.3}$$

$$c_{\alpha(k-1),1}+c_{-\alpha(k+1),-1}<k \quad \text{for all } \alpha\in\mathbb{R} \text{ such that } \alpha(k+1)\in\mathbb{Z} \text{ and } \alpha(k-1)\in\mathbb{Z}. \tag{1.4}$$

Note that if $\alpha(k+1)\in\mathbb{Z}$ and $\alpha(k-1)\in\mathbb{Z}$, then either $\alpha\in\mathbb{Z}$ (if k is even) or $2\alpha\in\mathbb{Z}$ (if k is odd).

Since we are interested in the existence of non-trivial L^1-solutions of (1.1), there is no loss of generality in assuming that (1.4) holds. In fact, if (1.4) does not hold, then from [3] (cf. also [25]) we can deduce that Eq. (1.1) has no non-trivial L^1-solutions. Moreover, condition (1.4) simplify formulations of our results.

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One of the most interesting and intensively studied case of (1.1) is the refinement equation

$$f(x)=\sum_{n\in\mathbb{Z}} c_n f(kx-n) \tag{1.5}$$

with $\sum_{n\in\mathbb{Z}} c_n = k$. It plays an important role in the wavelets theory (see [2,5–8]). Non-trivial L^1-solutions of (1.5) have been also used in such fields as splines (see [13,19,24]), subdivision schemes in approximation theory and curve design (see [4,14,20]), probability theory (see [11,12]), combinatorial number theory (see [23]).

There are known necessary and sufficient conditions for Eq. (1.5) with finitely many non-zero terms to have non-trivial L^1-solutions (see [16,18]). Unfortunately, in general, these conditions are very difficult to apply. In the case of non-negative c_n's a nice criterion can be found in [22] (see also [10]).

The aim of this paper is to give necessary and sufficient conditions for the existence of non-trivial L^1-solutions for several special cases of (1.1) (for instance, when $c_{n,1}=0$ for all $n\in\mathbb{Z}$, or if $c_{n,1}=c_{n,-1}$ for all $n\in\mathbb{Z}$); in the general case the problem remains open. Our main idea follows closely the idea from [22], but we work under weaker assumptions. On the one hand, we consider equations of a more general type. On the other hand, we trace the connection between the solutions and the probability distribution functions of random Grincevičjus series and by this observation the decreasing condition on the coefficients $c_{n,\varepsilon}$'s is slightly relaxed from the linear decay ($\sum_{n\in\mathbb{Z}}|n|c_{n,\varepsilon}<+\infty$) in the work [22] to the logarithmic one (see (1.3)). The main approach to the problem of solvability of (1.5) was elaborated in [22] and in [10]. We borrow from these papers the idea of use of trees and blocking sets, extending that technique to more general equations.

This paper is organized as follows. In Section 2 we show a connection between Grincevičjus series and Eq. (1.1). In Section 3 we introduce the notion of blocking sets. Section 4 contains key lemmas to formulate main results included in Section 5. Finally we formulate conditions for several special cases of (1.1) to have non-trivial L^1-solutions.

2. From Grincevičjus series to Eq. (1.1)

First of all we would like to show how L^1-solutions of (1.1) arise from the Grincevičjus series. Let $(\Omega,\mathcal{A},P)$ be a probability space and let $L,M:\Omega\to\mathbb{R}$ be random variables such that

$$L\neq 0 \quad \text{almost everywhere}, \qquad 0<\int_\Omega \log\bigl|L(\omega)\bigr|\,dP(\omega)<+\infty, \tag{2.1}$$

$$\int_\Omega \log\max\left\{\frac{|M(\omega)|}{|L(\omega)|},1\right\}dP(\omega)<+\infty, \tag{2.2}$$

$$P(Lc-M=c)<1 \quad \text{for all } c\in\mathbb{R}. \tag{2.3}$$

Fix a sequence $((\xi_n,\eta_n)\colon n\in\mathbb{N})$ of independent identically distributed vectors of random variables distributed as $(\frac{1}{L},\frac{M}{L})$ and consider the Grincevičjus series

$$\sum_{n=1}^{\infty}\eta_n\prod_{m=1}^{n-1}\xi_m. \tag{2.4}$$

It is known (see [15]) that under the above assumptions series (2.4) converges almost surely and its probability distribution function F is either absolutely continuous or purely singular, and satisfies the functional-integral equation

$$F(x)=\int_{L>0}F\bigl(L(\omega)x-M(\omega)\bigr)\,dP(\omega)+\int_{L<0}\bigl[1-F\bigl(L(\omega)x-M(\omega)\bigr)\bigr]\,dP(\omega). \tag{2.5}$$

Assume $M(\Omega)\subseteq\mathbb{Z}$ and $L(\Omega)\subseteq\{-k,k\}$. Put $p_{n,\varepsilon}=P(L=\varepsilon k, M=n)$ and $c_{n,\varepsilon}=kp_{n,\varepsilon}$ for all $n\in\mathbb{Z}$ and $\varepsilon\in\{-1,1\}$. Since P is a probability measure, it follows that (1.2) is satisfied. Clearly, (2.1) also holds. Conditions (2.2) and (2.3) can be written as (1.3) and (1.4), respectively. Eq. (2.5) takes now the form

$$F(x)=\sum_{n\in\mathbb{Z}}p_{n,1}F(kx-n)+\sum_{n\in\mathbb{Z}}p_{n,-1}\bigl[1-F(-kx-n)\bigr]. \tag{2.6}$$

Let (H_1) denote (H) with $c_{n,\varepsilon}=kp_{n,\varepsilon}$ for all $n\in\mathbb{Z}$ and $\varepsilon\in\{-1,1\}$.

We begin with a result on Eq. (2.6) which can be found in [17].

Theorem 2.1. *Assume* (H_1). *Then Eq.* (2.6) *has exactly one solution in the class of all continuous probability distribution functions. Moreover, this solution is either absolutely continuous or purely singular.*

From now on we will assume (H_1) and we will denote by F_* the unique probability distribution solution of (2.6).

Taking the Fourier–Stieltjes transform of F_* we get

$$\widehat{F_*}(t) = \sum_{\varepsilon=\pm 1} m_\varepsilon\left(\frac{\varepsilon t}{k}\right)\widehat{F_*}\left(\frac{\varepsilon t}{k}\right), \tag{2.7}$$

where $m_{-1}, m_1 : \mathbb{R} \to \mathbb{C}$ are masks defined by

$$m_\varepsilon(t) = \sum_{n\in\mathbb{Z}} p_{n,\varepsilon} e^{-int}.$$

We conclude this section with a corollary which extends a result on L^1-solutions of (1.5) from [9] (and, consequently, also from [1]).

Corollary 2.2. *The vector space of all L^1-solutions of* (1.1) *is at most one-dimensional and it is one-dimensional if and only if F_* is absolutely continuous. Moreover, any L^1-solution of* (1.1) *is of constant sign.*

Proof. It is easy to see that if F_* is absolutely continuous, then its density is a non-trivial L^1-solution of (1.1).

Fix a non-trivial L^1-solution f of (1.1) and put $a = \int_{\mathbb{R}} f(t)\,dt$.

We first show that $a \neq 0$. On the contrary, suppose that $a = 0$. Define $F : \mathbb{R} \to \mathbb{R}$ by $F(x) = \int_{-\infty}^{x} f(t)\,dt$ and observe that

$$F(x) = \sum_{n\in\mathbb{Z}} p_{n,1} F(kx-n) - \sum_{n\in\mathbb{Z}} p_{n,-1} F(-kx-n)$$

for $x \in \mathbb{R}$. Let $M = \sup\{|F(x)|\colon\ x \in \mathbb{R}\}$. Clearly, $M > 0$. Put $\mathcal{S} = \{x \in \mathbb{R}\colon\ |F(x)| = M\}$, $x_0 = \inf\mathcal{S}$, $y_0 = \sup\mathcal{S}$. Since $\lim_{x\to\pm\infty} F(x) = 0$, it follows that both x_0 and y_0 are finite. If $x \in \mathcal{S}$, then

$$M = \left|F(x)\right| \leqslant \sum_{n\in\mathbb{Z}} p_{n,1}\left|F(kx-n)\right| + \sum_{n\in\mathbb{Z}} p_{n,-1}\left|F(-kx-n)\right| \leqslant M,$$

whence $kx - n \in \mathcal{S}$ for all $n \in \mathbb{Z}$ such that $p_{n,1} > 0$ and $-kx - n \in \mathcal{S}$ for all $n \in \mathbb{Z}$ such that $p_{n,-1} > 0$. In particular, for all $n \in \mathbb{Z}$ we have

$$\begin{aligned} &\text{if } p_{n,1} > 0, \quad \text{then } x_0 \leqslant kx_0 - n \leqslant ky_0 - n \leqslant y_0, \\ &\text{if } p_{n,-1} > 0, \quad \text{then } x_0 \leqslant k^2x_0 + kn - n \leqslant k^2y_0 + kn - n \leqslant y_0. \end{aligned}$$

Hence $y_0 \leqslant x_0$ and we conclude that $\mathcal{S} = \{x_0\}$. Consequently, $p_{x_0(k-1),1} + p_{-x_0(k+1),-1} = 1$. This contradicts (1.4) and so $a \neq 0$.

Now simple calculations show that $F_*(x) = \frac{1}{a}\int_{-\infty}^{x} f(t)\,dt$. Hence F_* is absolutely continuous. Moreover, by Theorem 2.1, f has a constant sign and it is the unique L^1-solution of (1.1) up to a multiplicative constant. □

3. Blocking sets

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Put

$$\begin{aligned} V_0 &= \{2\pi\}, \\ V_N &= \left\{2\pi \sum_{n=1}^{N} \frac{d_n}{k^n}\colon\ d_1, \ldots, d_{N-1} \in \{0, \ldots, k-1\},\ d_N \in \{1, \ldots, k-1\}\right\} \quad \text{for } N \in \mathbb{N}, \\ V &= \bigcup_{N\in\mathbb{N}_0} V_N, \\ E &= \left\{(v, w)\colon \bigvee_{N\in\mathbb{N}_0} \bigvee_{j\in\{0,\ldots,k-1\}} v \in V_N,\ w = \frac{v+j}{k}\right\}. \end{aligned}$$

The pair $T = (V, E)$ is a tree of order k with the root 2π. A path from a vertex v_0 to a vertex v_N of T is a sequence $(v_0, v_1, \ldots, v_N)$ such that $(v_n, v_{n+1}) \in E$ for $n \in \{0, \ldots, N-1\}$. An infinite path from a vertex v_0 of T is a sequence $(v_n\colon\ n \in \mathbb{N}_0)$ such that $(v_n, v_{n+1}) \in E$ for $n \in \mathbb{N}_0$. All paths are without backtracking.

For a given integer $M \geqslant 2$ put

$$\mathcal{B}_M = \bigcup_{N\in\mathbb{N}} \left\{2\pi \sum_{n=1}^{M+N} \frac{d_n}{k^n} \in V\colon\ d_1 = \cdots = d_M \in \{0, k-1\}\right\}.$$

Definition 3.1. A subset $\mathcal{V}$ of vertices of the tree T is said to be weakly blocking if the following three conditions are satisfied:

(i) $2\pi \notin \mathcal{V}$;
(ii) $v \in \mathcal{V}$ if and only if $2\pi - v \in \mathcal{V}$;
(iii) There exists an integer $M \geqslant 2$ such that for any $v \in \mathcal{B}_M$ the path from the root of T to the vertex v contains exactly one element of $\mathcal{V}$.

Definition 3.2. A subset $\mathcal{V}$ of vertices of the tree T is said to be strongly blocking if the following three conditions are satisfied:

(i) $2\pi \notin \mathcal{V}$;
(ii) $v \in \mathcal{V}$ if and only if $2\pi - v \in \mathcal{V}$;
(iii) Any infinite path from the root of T contains exactly one element of $\mathcal{V}$.

It is easy to see that every strongly blocking set is weakly blocking, and if a weakly blocking set is finite, then it is strongly blocking. Moreover, there are weakly blocking sets which are not strongly blocking.

4. Key lemmas

To formulate our criterion for the absolute continuity of F_* we need some lemmas. The first two of them can be derived from [22, Lemmas 3 and 2].

Lemma 4.1. *Let $F:\mathbb{R} \to \mathbb{C}$ be continuous at the origin with $F(0) \neq 0$, let $M:\mathbb{R} \to \mathbb{C}$ be 2π-periodic with $M(-t) = \overline{M(t)}$ for $t \in \mathbb{R}$, and let $\psi_1, \psi_2 : \mathbb{R} \to \mathbb{R}$ satisfy*

$$\left|\psi_1(t)\right| = \left|\psi_2(t)\right| = |t| \quad \textit{for } t \in \mathbb{R}.$$

If

$$F(t) = (M \circ \psi_1)\left(\frac{t}{k}\right)(F \circ \psi_2)\left(\frac{t}{k}\right) \quad \textit{for } t \in \mathbb{R},$$

then

$$F(2n\pi) = 0 \quad \textit{for } n \in \mathbb{Z} \setminus \{0\}$$

if and only if there exists a weakly blocking set $\mathcal{V}$ of the tree T of order k such that

$$\mathcal{V} \subset \left\{t \in (0, 2\pi):\ M(t) = 0\right\}.$$

Lemma 4.2. *Let F be a continuous probability distribution function. If $\widehat{F}(2n\pi) = 0$ for $n \in \mathbb{Z} \setminus \{0\}$, then F is a contraction; in particular, F is absolutely continuous.*

Lemma 4.3. *If F_* is absolutely continuous, then*

$$\operatorname{Re} \widehat{F_*}(2n\pi) = 0 \quad \textit{for } n \in \mathbb{Z} \setminus \{0\}. \tag{4.1}$$

Proof. Fix $l \in \mathbb{N}$ and $n \in \mathbb{Z}$. By (2.7) we get

$$\begin{aligned}
\operatorname{Re} \widehat{F_*}\left(2nk^l\pi\right) &= \frac{1}{2}\left[\widehat{F_*}\left(2nk^l\pi\right) + \widehat{F_*}\left(-2nk^l\pi\right)\right] \\
&= \frac{1}{2}\left[m_1(0)\widehat{F_*}\left(2nk^{l-1}\pi\right) + m_{-1}(0)\widehat{F_*}\left(-2nk^{l-1}\pi\right) + m_1(0)\widehat{F_*}\left(-2nk^{l-1}\pi\right) + m_{-1}(0)\widehat{F_*}\left(2nk^{l-1}\pi\right)\right] \\
&= \operatorname{Re} \widehat{F_*}\left(2nk^{l-1}\pi\right) = \cdots = \operatorname{Re} \widehat{F_*}(2n\pi).
\end{aligned}$$

Since F_* is absolutely continuous, we see that $\widehat{F_*}(2nk^l\pi)$ tends to zero as l tends to $+\infty$. Hence (4.1) holds. □

The proof of the next lemma is similar to the above one, so we omit it.

Lemma 4.4. *For $n \in \mathbb{Z}$ and $l \in \mathbb{N}$ we have*

$$\operatorname{Im} \widehat{F_*}\left(2nk^l\pi\right) = \left[m_1(0) - m_{-1}(0)\right]^l \operatorname{Im} \widehat{F_*}(2n\pi).$$

Moreover, if F_* is absolutely continuous and if either $m_1(0) = 0$ or $m_{-1}(0) = 0$, then

$$\operatorname{Im} \widehat{F_*}(2n\pi) = 0 \quad \textit{for } n \in \mathbb{Z}. \tag{4.2}$$

Since in general we cannot deduce that absolute continuity of F_* implies (4.2), we are looking for conditions under which (4.2) is satisfied.

Lemma 4.5. *Assume*

$$m_1\left(\frac{2n\pi}{k}\right) = m_{-1}\left(\frac{2n\pi}{k}\right) \quad \text{for } n \in \{0, \ldots, k-1\}. \tag{4.3}$$

Then (4.2) *holds.*

Proof. Fix $n \in \mathbb{Z}$. By (2.7) and (4.3) we have

$$\begin{aligned}\widehat{F_*}(2n\pi) &= m_1\left(\frac{2n\pi}{k}\right)\widehat{F_*}\left(\frac{2n\pi}{k}\right) + m_1\left(-\frac{2n\pi}{k}\right)\widehat{F_*}\left(-\frac{2n\pi}{k}\right)\\ &= 2\operatorname{Re}\left[m_1\left(\frac{2n\pi}{k}\right)\widehat{F_*}\left(\frac{2n\pi}{k}\right)\right] \in \mathbb{R}.\end{aligned}$$

Hence (4.2) holds. $\square$

To see that condition (4.3) is close to (4.2) observe that iterating (2.7) we get

$$\begin{aligned}\operatorname{Im}\widehat{F_*}(t) &= -\lim_{l\to\infty}\sum_{\varepsilon_1,\ldots,\varepsilon_l=\pm 1}\ \sum_{n_1,\ldots,n_l\in\mathbb{Z}} p_{n_1,\varepsilon_1}\cdots p_{n_l,\varepsilon_l}\sin t\left(n_1\frac{\varepsilon_1}{k} + n_2\frac{\varepsilon_2\varepsilon_1}{k^2} + \cdots + n_l\frac{\varepsilon_l\cdots\varepsilon_1}{k^l}\right)\\ &= \left[\sum_{n_1\in\mathbb{Z}} p_{n_1,-1}\sin n_1\frac{t}{k} - \sum_{n_1\in\mathbb{Z}} p_{n_1,1}\sin n_1\frac{t}{k}\right]\\ &\quad\times\lim_{l\to\infty}\sum_{\varepsilon_2,\ldots,\varepsilon_l=\pm 1}\ \sum_{n_2,\ldots,n_l\in\mathbb{Z}} p_{n_2,\varepsilon_2}\cdots p_{n_l,\varepsilon_l}\cos t\left(n_2\frac{\varepsilon_2}{k^2} + \cdots + n_l\frac{\varepsilon_l\cdots\varepsilon_2}{k^l}\right)\\ &\quad+\left[\sum_{n_1\in\mathbb{Z}} p_{n_1,-1}\cos n_1\frac{t}{k} - \sum_{n_1\in\mathbb{Z}} p_{n_1,1}\cos n_1\frac{t}{k}\right]\\ &\quad\times\lim_{l\to\infty}\sum_{\varepsilon_2,\ldots,\varepsilon_l=\pm 1}\ \sum_{n_2,\ldots,n_l\in\mathbb{Z}} p_{n_2,\varepsilon_2}\cdots p_{n_l,\varepsilon_l}\sin t\left(n_2\frac{\varepsilon_2}{k^2} + \cdots + n_l\frac{\varepsilon_l\cdots\varepsilon_2}{k^l}\right)\end{aligned}$$

for all $t \in \mathbb{R}$. It is clear that if we want to have (4.2) we should assume that

$$\sum_{n\in\mathbb{Z}} p_{n,-1}\sin n\frac{2l\pi}{k} = \sum_{n\in\mathbb{Z}} p_{n,1}\sin n\frac{2l\pi}{k} \tag{4.4}$$

and

$$\sum_{n\in\mathbb{Z}} p_{n,-1}\cos n\frac{2l\pi}{k} = \sum_{n\in\mathbb{Z}} p_{n,1}\cos n\frac{2l\pi}{k} \tag{4.5}$$

for all $l \in \mathbb{Z}$. Obviously, conditions (4.4) and (4.5) are equivalent to (4.3). Let us also observe that if $k = 2$, then (4.4) holds true.

5. Main results

We can now formulate necessary and sufficient conditions for absolute continuity of F_*.

Theorem 5.1. *Assume $\varepsilon \in \{-1, 1\}$ and let $m_\varepsilon(0) = 0$. Then F_* is absolutely continuous if and only if there exists a weakly blocking set of the tree T of order k that consists of roots of the equation $m_{-\varepsilon}(t) = 0$.*

Proof. From Lemmas 4.2–4.4 it follows that F_* is absolutely continuous if and only if $\widehat{F_*}(2n\pi) = 0$ for every $n \in \mathbb{Z} \setminus \{0\}$. Since $m_\varepsilon(0) = 0$, (2.7) gives

$$\widehat{F_*}(t) = m_{-\varepsilon}\left(-\frac{\varepsilon t}{k}\right)\widehat{F_*}\left(-\frac{\varepsilon t}{k}\right)$$

for $t \in \mathbb{R}$. Lemma 4.1 completes the proof. $\square$

Theorem 5.1 with $\varepsilon = -1$ generalizes Theorem 1 from [22].

Theorem 5.2. *Assume $p_{n,1} = p_{n,-1}$ for $n \in \mathbb{Z}$. Then F_* is absolutely continuous if and only if there exists a weakly blocking set of the tree T of order k that consists of roots of the equation $\operatorname{Re} m_1(t) = 0$.*

Proof. We have now $\widehat{F_*}(t) = \widehat{F_*}(-t)$ which jointly with (2.7) gives

$$\widehat{F_*}(t) = 2 \operatorname{Re} m_1\left(\frac{t}{k}\right) \cdot \widehat{F_*}\left(\frac{t}{k}\right)$$

for $t \in \mathbb{R}$. Lemmas 4.1–4.3 complete the proof. $\square$

Theorem 5.3. *Assume $p_{n,1} = p_{-n,-1}$ for $n \in \mathbb{Z}$ and let*

$$\sum_{n\in\mathbb{Z}} p_{kn+j,1} = \sum_{n\in\mathbb{Z}} p_{kn-j,1} \quad \text{for } j \in \{1, \dots, k-1\}. \tag{5.1}$$

Then F_* is absolutely continuous if and only if there exists a weakly blocking set of the tree T of order k that consists of roots of the equation $\operatorname{Re} m_1(t) = 0$.

Proof. It is evident that $m_1(0) = m_{-1}(0)$. Fix $l \in \{1, \dots, k-1\}$. Then

$$\begin{aligned} m_1\left(\frac{2l\pi}{k}\right) - m_{-1}\left(\frac{2l\pi}{k}\right) &= \sum_{n\in\mathbb{Z}} (p_{n,1} - p_{n,-1}) e^{-i\frac{2ln\pi}{k}} \\ &= \sum_{j=0}^{k-1} \sum_{n\in\mathbb{Z}} (p_{kn+j,1} - p_{-kn-j,1}) e^{-i\frac{2l(kn+j)\pi}{k}} \\ &= \sum_{j=0}^{k-1} e^{-i\frac{2lj\pi}{k}} \left(\sum_{n\in\mathbb{Z}} p_{kn+j,1} - \sum_{n\in\mathbb{Z}} p_{kn-j,1} \right) = 0. \end{aligned}$$

Lemma 4.5 now implies (4.2). Since

$$m_1(t) = \sum_{n\in\mathbb{Z}} p_{n,1} e^{-int} = \sum_{n\in\mathbb{Z}} p_{-n,-1} e^{-int} = m_{-1}(-t),$$

we see that

$$\widehat{F_*}(t) = m_1\left(\frac{t}{k}\right) \widehat{F_*}\left(\frac{t}{k}\right) + m_{-1}\left(-\frac{t}{k}\right) \widehat{F_*}\left(-\frac{t}{k}\right) = 2 m_1\left(\frac{t}{k}\right) \operatorname{Re} \widehat{F_*}\left(\frac{t}{k}\right),$$

and, consequently,

$$\operatorname{Re} \widehat{F_*}(t) = 2 \operatorname{Re} m_1\left(\frac{t}{k}\right) \operatorname{Re} \widehat{F_*}\left(\frac{t}{k}\right)$$

for $t \in \mathbb{R}$. Lemmas 4.1–4.3 complete the proof. $\square$

It is worth pointing out that (5.1) is satisfied in the case $k = 2$.

The next result gives only sufficient conditions for absolute continuity of F_* but concerns the general case of (2.6).

Theorem 5.4. *Assume*

$$\sum_{n\in\mathbb{Z}} p_{kn+j,\varepsilon} = \sum_{n\in\mathbb{Z}} p_{kn,\varepsilon} \quad \text{for } j \in \{1, \dots, k-1\} \text{ and } \varepsilon \in \{-1, 1\}.$$

Then F_* is absolutely continuous.

Proof. According to Lemma 4.2 it is enough to prove that $\widehat{F_*}(2l\pi) = 0$ for every $l \in \mathbb{Z} \setminus \{0\}$.

Fix $l \in \mathbb{Z}$ and $q \in \{0, \dots, k-1\}$. Using (2.7) we obtain

$$\begin{aligned} \widehat{F_*}\big(2(kl+q)\pi\big) &= \sum_{\varepsilon=\pm1} \sum_{n\in\mathbb{Z}} p_{n,\varepsilon} e^{-i\varepsilon \frac{2(kl+q)\pi}{k} n} \widehat{F_*}\left(\varepsilon \frac{2(kl+q)\pi}{k}\right) \\ &= \sum_{\varepsilon=\pm1} \widehat{F_*}\left(\varepsilon \frac{2(kl+q)\pi}{k}\right) \sum_{j=0}^{k-1} e^{-i\varepsilon \frac{2q\pi j}{k}} \sum_{n\in\mathbb{Z}} p_{kn+j,\varepsilon} \\ &= \sum_{\varepsilon=\pm1} \widehat{F_*}\left(\varepsilon \frac{2(kl+q)\pi}{k}\right) \sum_{n\in\mathbb{Z}} p_{kn,\varepsilon} \sum_{j=0}^{k-1} e^{-i\varepsilon \frac{2qj\pi}{k}}. \end{aligned}$$

Since

$$\sum_{j=0}^{k-1} e^{-i\frac{2qj\pi}{k}} = 0 \quad \text{for } q \in \{1, \dots, k-1\},$$

it follows that

$$\widehat{F_*}\big(2(kl+q)\pi\big) = 0 \quad \text{for } l \in \mathbb{Z} \text{ and } q \in \{0, \dots, k-1\}. \tag{5.2}$$

Putting $(l, q) = (0, 1)$ and $(l, q) = (1, k-1)$ in (5.2) we see that $\widehat{F_*}(2\pi) = 0$ and $\widehat{F_*}(-2\pi) = 0$, and hence that

$$\widehat{F_*}(2k\pi) = \widehat{F_*}(-2k\pi) = 0.$$

Fix now $l \in \mathbb{Z} \setminus \{0\}$ and suppose that $\widehat{F_*}(2kj\pi) = 0$ for any $j \in \mathbb{Z}$ such that $0 < |j| < |l|$. Then

$$\widehat{F_*}(2kl\pi) = \sum_{\varepsilon=\pm 1} \sum_{n\in\mathbb{Z}} p_{n,\varepsilon} \widehat{F_*}(\varepsilon 2l\pi) = 0.$$

The proof is completed. □

6. Consequences of the main results

Using Corollary 2.2 we reformulate here all the results concerning absolute continuity of F_* on results concerning the existence of non-trivial L^1-solutions of (1.1). Theorem 5.1 reformulates as follows.

Theorem 6.1. *Assume $\varepsilon \in \{-1, 1\}$ and let $m_\varepsilon(0) = 0$. Then Eq.* (1.1) *has a non-trivial L^1-solution if and only if there exists a weakly blocking set of the tree T of order k that consists of roots of the equation $m_{-\varepsilon}(t) = 0$.*

As a consequence of Theorem 6.1 and the fact that every finite weakly blocking set is strongly blocking we get the following corollary.

Corollary 6.2. *Assume $\varepsilon \in \{-1, 1\}$ and let $c_0, \dots, c_N$ be non-negative reals such that $c_0 c_N \neq 0$ and $\sum_{n=0}^{N} c_n = k$. Then the equation*

$$f(x) = \sum_{n=0}^{N} c_n f(\varepsilon kx - n)$$

has a non-trivial L^1-solution if and only if there exists a strongly blocking set of the tree T of order k of cardinality less or equal to N that consists of roots of the equation

$$\sum_{n=0}^{N} c_n e^{-int} = 0.$$

Theorem 5.2 can be stated in the following form.

Theorem 6.3. *Assume $c_{n,1} = c_{n,-1}$ for $n \in \mathbb{Z}$. Then Eq.* (1.1) *has a non-trivial L^1-solution if and only if there exists a weakly blocking set of the tree T of order k that consists of roots of the equation* $\operatorname{Re} m_1(t) = 0$.

An obvious consequence of Theorem 6.3 is the following corollary.

Corollary 6.4. *Assume $c_0, \dots, c_N$ are non-negative reals such that $c_0 c_N \neq 0$ and $\sum_{n=0}^{N} c_n = \frac{k}{2}$. Then the equation*

$$f(x) = \sum_{n=0}^{N} c_n \big[f(kx - n) + f(-kx - n) \big]$$

has a non-trivial L^1-solution if and only if there exists a strongly blocking set of the tree T of order k of cardinality less or equal to N that consists of roots of the equation

$$\sum_{n=0}^{N} c_n \cos(nt) = 0. \tag{6.1}$$

Theorem 5.3 we reformulate as follows.

Theorem 6.5. *Assume $c_{n,1} = c_{-n,-1}$ for $n \in \mathbb{Z}$ and let*

$$\sum_{n\in\mathbb{Z}} c_{kn+j,1} = \sum_{n\in\mathbb{Z}} c_{kn-j,1} \quad \text{for } j \in \{1, \dots, k-1\}.$$

Then Eq. (1.1) *has a non-trivial L^1-solution if and only if there exists a weakly blocking set of the tree T of order k that consists of roots of the equation* $\operatorname{Re} m_1(t) = 0$.

From Theorem 6.5 we get the following corollary.

Corollary 6.6. *Assume $c_0, \dots, c_N$ are non-negative reals such that $c_0 c_N \neq 0$, $\sum_{n=0}^{N} c_n = \frac{k}{2}$ and*

$$\sum_{n=0}^{[\frac{N-j}{k}]} c_{kn+j} = \sum_{n=1}^{[\frac{N+j}{k}]} c_{kn-j} \quad \text{for } j \in \{1, \dots, k-1\}.$$

Then the equation

$$f(x) = \sum_{n=0}^{N} c_n \big[f(kx-n) + f(-kx+n) \big]$$

has a non-trivial L^1-solution if and only if there exists a strongly blocking set of the tree T of order k of cardinality less or equal to N that consists of roots of Eq. (6.1)*.*

Theorem 5.4 allows us to generalize the well-known result on L^1-solutions of (1.5) (see [10,21,26,27]) to the case of Eq. (1.1).

Theorem 6.7. *Assume*

$$\sum_{n\in\mathbb{Z}} c_{kn+j,\varepsilon} = \sum_{n\in\mathbb{Z}} c_{kn,\varepsilon} \quad \text{for } j \in \{1, \dots, k-1\} \text{ and } \varepsilon \in \{-1, 1\}.$$

Then Eq. (1.1) *has a non-trivial L^1-solution.*

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