Journal of Computational and Applied Mathematics 235 (2011) 2603–2614

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Controllability of impulsive neutral stochastic functional differential inclusions with infinite delay ☆

Yong Ren [a], Lanying Hu [a], R. Sakthivel [b,*]

[a] Department of Mathematics, Anhui Normal University, Wuhu, 241000, China

[b] Department of Mathematics, Sungkyunkwan University, Suwon 440-746, South Korea

ARTICLE INFO

Article history:
Received 8 May 2009
Received in revised form 7 October 2010

MSC:
34A37
60H20
93B05

Keywords:
Controllability
Multi-valued map
Impulsive neutral stochastic functional differential inclusion
Infinite delay
Mild solution

ABSTRACT

This paper deals with the controllability of a class of impulsive neutral stochastic functional differential inclusions with infinite delay in an abstract space. Sufficient conditions for the controllability are derived with the help of the fixed point theorem for discontinuous multi-valued operators due to Dhage. An example is provided to illustrate the obtained theory.

1. Introduction

In this paper, we consider the controllability of the following impulsive neutral stochastic functional differential inclusions with infinite delay

$$\begin{cases} \mathrm{d}\,[x(t) - g\,(t, x_t)] \in [Ax(t) + Bu(t)]\,\mathrm{d}t + F(t, x_t)\mathrm{d}w(t), & t \in J, t \neq t_k, k = 1, 2, \ldots, m, \\ \Delta x|_{t=t_k} \in I_k\left(x(t_k^-)\right), & k = 1, 2, \ldots, m, \\ x_0 = \varphi \in \mathcal{B}_h, & t \in J_0 := (-\infty, 0], \end{cases} \tag{1}$$

where $J := [0, b]$, A is the infinitesimal generator of an analytic semigroup of bounded linear operators $S(t), t \geq 0$ in the Hilbert space H. In the sequel, $\mathcal{P}(H)$ denotes the family of all nonempty subsets of H. Suppose that $g : J \times \mathcal{B}_h \to H$ is a continuous map, $F : J \times \mathcal{B}_h \to \mathcal{P}(L(K, H))$ is a bounded, closed, convex-valued multi-valued map and $I_k : H \to \mathcal{P}(H)$ $(k = 1, 2, \ldots, m)$ are multi-valued maps with closed graph. Moreover, the fixed times t_k satisfies $0 < t_1 < t_2 < \cdots < t_m < b$, $x(t_k^-)$ denotes the left limits of $x(t)$ at $t = t_k$. The control function $u(\cdot)$ takes its value in $L^2(J, U)$ of admissible control functions for a separable Hilbert space U, B is a bounded linear operator from U into H. The histories $x_t : \Omega \to \mathcal{B}_h, t \geq 0$,

☆ The first two authors are supported by the National Natural Science Foundation of China (Project. 10901003 and 11001029) and the Great Research Project of Natural Science Foundation of Anhui Provincial Universities (Project. KJ2010ZD02). The work of R. Sakthivel is supported by the Korean Research Foundation Grant funded by the Korean Government with grant number KRF 2010-0003495.

* Corresponding author.
E-mail addresses: brightry@hotmail.com, renyong@126.com (Y. Ren), lanyinghu@126.com (L. Hu), krsakthivel@yahoo.com (R. Sakthivel).

0377-0427/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2010.10.051

which are defined by setting $x_t = \{x(t+s), s \in (-\infty, 0]\}$, belong to the abstract phase space $\mathscr{B}_h$, which will be given in Section 2. The initial data $\varphi = \{\varphi(t) : -\infty < t \le 0\}$ is an $\mathcal{F}_0$-measurable, $\mathscr{B}_h$-valued stochastic process with finite second moment, and is independent of the Wiener process $\{w(t) : t \ge 0\}$ to be specified later.

In the past decades, the theory of impulsive differential equations or inclusions has become an active area of investigation due to their applications in fields such as mechanics, electrical engineering, medicine biology, ecology and so on. One can refer to [1,2] and the references therein. The development of the theory of functional differential equations or inclusions with infinite delay heavily depends on a choice of a phase space. In fact, various phase spaces have been considered and each different phase space requires a separate development of the theory [3]. The common space is the phase space $\mathscr{B}$ proposed by Hale and Kato in [4], which is widely applied in functional differential equations with infinite delay, one can refer to Hernández [5] and the references therein. However, the phase space $\mathscr{B}$ is not correct for the impulsive case. Generally, the theory of impulsive functional differential equations or inclusions is based on the phase space $\mathscr{B}_h$ defined later (see [1,6]).

In many cases, deterministic models often fluctuate due to noise, which is random or at least appears to be so. Therefore, we must move from deterministic problems to stochastic ones. Taking the disturbances into account, the theory of differential inclusions has been generalized to stochastic functional differential inclusions (see [7] and the references therein). The existence, uniqueness, stability, controllability and other quantitative and qualitative properties of solutions of stochastic evolution equations or inclusions have recently received a lot of attention (see [8–11] and the references therein).

As one of the fundamental concepts in mathematical control theory, controllability plays an important role both in deterministic and stochastic control theory. Roughly speaking, controllability generally means that it is possible to steer a dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. The controllability of nonlinear stochastic systems in infinite dimensional spaces has been extensively studied by several authors, see [12] and the references therein. Among them, Balachandran and Ntouyas [13] investigated the controllability for neutral functional differential inclusions with finite delay. Further, Liu [14] extended the result of [13] to the case of the infinite delay. Moreover, Balasubramaniam and Ntouyas [15] gave the controllability of a class of partial stochastic functional differential inclusions with infinite delay in the space $\mathscr{B}$ with the help of the Leray–Schauder nonlinear alternative. In addition, the model with multi-valued jump sizes may arise in a control problem where we want to control the jump sizes in order to achieve the given objectives.

To our best knowledge, there is no work reported on the controllability for the neutral stochastic functional differential inclusions with infinite delay and multi-valued jump operators, which is expressed in the form (1). To close the gap in this paper, we study this interesting problem. We derive the sufficient conditions for the controllability of the system (1) by means of the fixed point theorem for discontinuous multi-valued operators due to Dhage [16]. Especially, the known results that appear in [17] are generalized to the stochastic settings and the case of infinite delay. Based on the obtained result, we can establish the controllability of the following impulsive neutral stochastic partial differential inclusion

$$\begin{cases} \dfrac{\partial}{\partial t} v(t,x) \in \dfrac{\partial^2}{\partial x^2} v(t,x) + \dfrac{\partial}{\partial t} g(t, v(t-h,x)) + b(x)u(t) \\ \quad + [Q_1(t, v(t-h,x)), Q_2(t, v(t-h,x))]\mathrm{d}\beta(t), \quad 0 \le x \le \pi, t \in J, t \ne t_k, \\ v(t_k^+, x) - v(t_k^-, x) \in \left[-b_k|v(t_k^-,x)|, b_k|v(t_k^-,x)|\right], \quad t \in J, k = 1, 2, \ldots, m, \\ v(t,0) = v(t,\pi) = 0, \quad t \in J, \\ v(t,x) = \varphi(t,x), \quad -\infty < t \le 0, 0 \le x \le \pi, \end{cases} \tag{2}$$

where $J := [0, b]$, $b_k > 0$, $k = 1, 2, \ldots, m$, $v(t_k^+, x) = \lim_{(h,x)\to(0^+,x)} v(t_k + h, x)$, $v(t_k^-, x) = \lim_{(h,x)\to(0^-,x)} v(t_k + h, x)$, $Q_1, Q_2 : J \times \mathbf{R} \to \mathbf{R}$ are two given functions, and $\beta(t)$ is a one-dimensional standard Wiener process. We assume that for each $t \in J$, $Q_1(t, \cdot)$ is lower semi-continuous and for each $t \in J$, $Q_2(t, \cdot)$ is super semi-continuous.

The paper is organized as follows. In Section 2, we introduce some preliminaries. Section 3 is devoted to the main result. In Section 4, an example is given to illustrate the obtained result. In the last section, concluding remarks are given.

2. Preliminaries

Let $(K, \|\cdot\|_K)$ and $(H, \|\cdot\|_H)$ be two separable Hilbert spaces with inner product $\langle\cdot,\cdot\rangle_K$ and $\langle\cdot,\cdot\rangle_H$, respectively. In a case without confusion, we just use $\langle\cdot,\cdot\rangle$ for the inner product and $\|\cdot\|$ for the norm. Let $(\Omega, \mathcal{F}, P; \mathbb{F})$ $(\mathbb{F} = \{\mathcal{F}_t\}_{t\ge 0})$ be a complete filtered probability space satisfying that $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$. Suppose that $\{w(t) : t \ge 0\}$ is a cylindrical K-valued Wiener process with a finite trace nuclear covariance operator $Q \ge 0$. We also use the same notation $\|\cdot\|$ for the norm of $L(K, H)$, which denotes the space of all Q-Hilbert–Schmidt operators from K to H. For details of this paragraph, the reader may refer to [18] and the references therein.

Now, we present the abstract phase space $\mathscr{B}_h$. Assume that $h : (-\infty, 0] \to (0, \infty)$ is a continuous function with $l = \int_{-\infty}^0 h(t)\,\mathrm{d}t < \infty$. For each $a > 0$, define

$$\mathscr{B}_h = \Bigg\{\psi : (-\infty, 0] \to H : \left(E\|\psi(\theta)\|^2\right)^{1/2} \text{ is a bounded and measurable} \\ \text{function on } [-a, 0] \text{ and } \int_{-\infty}^0 h(s) \sup_{s\le\theta\le 0} \left(E\|\psi(\theta)\|^2\right)^{1/2} \mathrm{d}s < \infty\Bigg\}.$$

If $\mathcal{B}_h$ is endowed with the norm

$$\|\psi\|_{\mathcal{B}_h} = \int_{-\infty}^{0} h(s) \sup_{s\le\theta\le 0} \left(E\|\psi(\theta)\|^2\right)^{1/2} \mathrm{d}s, \quad \text{for all } \psi \in \mathcal{B}_h,$$

then $(\mathcal{B}_h, \|\cdot\|_{\mathcal{B}_h})$ is a Banach space [3].

Assume that $S(t)$ is a uniformly bounded and analytic semigroup with infinitesimal generator A such that $0 \in \rho(A)$. Then, it is possible to define the fractional power $(-A)^\alpha$, $0 < \alpha \le 1$ as a closed linear invertible operator with its domain $D((-A)^\alpha)$ being dense in H. We denote by H_α the Banach space $D((-A)^\alpha)$ endowed with the norm $\|x\|_\alpha = \|(-A)^\alpha x\|$, which is equivalent to the graph norm of $(-A)^\alpha$. In the sequel, we represent H_α the space $D((-A)^\alpha)$ with the norm $\|\cdot\|_\alpha$. Then, we have the following well-known properties that appear in [19].

Lemma 2.1. (i) *If $0 < \beta < \alpha \le 1$, then $H_\alpha \subset H_\beta$ and the embedding is compact whenever the resolvent operator of A is compact.*
(ii) *For each $0 < \alpha \le 1$, there exists a positive constant C_α such that*

$$\|(-A)^\alpha S(t)\| \le \frac{C_a}{t^\alpha}, \quad t > 0.$$

Let us introduce the following notations:

$$\mathcal{P}_{cl}(H) = \{y \in \mathcal{P}(H) : y \text{ is closed}\}, \qquad \mathcal{P}_{bd}(H) = \{y \in \mathcal{P}(H) : y \text{ is bounded}\},$$
$$\mathcal{P}_{cv}(H) = \{y \in \mathcal{P}(H) : y \text{ is convex}\}, \qquad \mathcal{P}_{cp}(H) = \{y \in \mathcal{P}(H) : y \text{ is compact}\}.$$

Consider $H_d : \mathcal{P}(H) \times \mathcal{P}(H) \to \mathbf{R} \cup \{\infty\}$ given by

$$H_d(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\right\},$$

where $d(A, b) = \inf_{a\in A} d(a, b)$, $d(a, B) = \inf_{b\in B} d(a, b)$. Then, $(\mathcal{P}_{bd,cl}(H), H_d)$ is a metric space and $(\mathcal{P}_{cl}(H), H_d)$ is a generalized metric space [20].

In what follows, we briefly introduce some facts on multi-valued analysis. For more details, one can see [21].

- A multi-valued map $\Gamma : H \to \mathcal{P}(H)$ is convex (closed) valued, if $\Gamma(x)$ is convex (closed) for all $x \in H$. Γ is bounded on bounded sets if $\Gamma(B) = \cup_{x\in B} \Gamma(x)$ is bounded in H, for any bounded set B of H, that is, $\sup_{x\in B} \sup\{\|y\| \in \Gamma(x)\} < \infty$.
- Γ is called upper semi-continuous (u.s.c., in short) on H, if for any $x \in H$, the set $\Gamma(x)$ is a non-empty, closed subset of H, and if for each open set B of H containing $\Gamma(x)$, there exists an open neighborhood N of x such that $\Gamma(N) \subseteq B$.
- Γ is said to be completely continuous if $\Gamma(B)$ is relatively compact, for every bounded subset $B \subseteq H$.
- If the multi-valued map Γ is completely continuous with nonempty compact values, then Γ is u.s.c. if and only if Γ has a closed graph, i.e., $x_n \to x$, $y_n \to y$, $y_n \in \Gamma(x_n)$ imply $y \in \Gamma(x)$.
- Γ has a fixed point if there is $x \in H$ such that $x \in \Gamma(x)$.
- A multi-valued map $\Gamma : J \to \mathcal{P}_{bd,cl,cv}$ is said to be measurable if for each $x \in H$, the mean-square distance between x and $\Gamma(t)$ is measurable.

The consideration of this paper is based on the following fixed point theorem due to Dhage [16].

Theorem 2.2. *Let H be a Hilbert space, $\Phi_1 : H \to \mathcal{P}_{cl,cv,bd}(H)$ and $\Phi_2 : H \to \mathcal{P}_{cp,cv}(H)$ be two multi-valued operators satisfying that*

(i) *Φ_1 is a contraction, and*
(ii) *Φ_2 is completely continuous.*

Then, either

(1) *the operator inclusion $x \in \Phi_1 x + \Phi_2 x$ has a solution, or*
(2) *the set $G = \{x \in H : x \in \lambda\Phi_1 x + \lambda\Phi_2 x\}$ is unbounded for $\lambda \in (0, 1)$.*

Definition 2.3. The multi-valued map $F : J \times \mathcal{B}_h \to \mathcal{P}(H)$ is said to be L^2-Carathéodory if

(i) $t \mapsto F(t, v)$ is measurable for each $v \in \mathcal{B}_h$;
(ii) $v \mapsto F(t, v)$ is u.s.c. for almost all $t \in J$;
(iii) for each $q > 0$, there exists $h_q \in L^1(J, \mathbf{R}_+)$ such that

$$\|F(t, v)\|^2 := \sup_{f\in F(t,v)} E\|f\|^2 \le h_q(t), \quad \text{for all } \|v\|^2_{\mathcal{B}_h} \le q \text{ and for a.e. } t \in J.$$

The following lemma that appears in [22] is crucial in the proof of our main result.

Lemma 2.4. *Let I be a compact interval and H be a Hilbert space. Let F be an L^2-Carathéodory multi-valued map with $N_{F,x} \neq \emptyset$ and let Γ be a linear continuous mapping from $L^2(I, H)$ to $C(I, H)$. Then, the operator*

$$\Gamma \circ N_F : C(I, H) \to \mathcal{P}_{cp,cv}(H), \qquad x \mapsto (\Gamma \circ N_F)(x) = \Gamma(N_{F,x}),$$

is a closed graph operator in $C(I, H) \times C(I, H)$, where $N_{F,x}$ is known as the selectors set from F and given by

$$f \in N_{F,x} = \left\{ f \in L^2\left(L(K, H)\right) : f(t) \in F(t, x_t) \text{ for a.e. } t \in J \right\}.$$

Now, we consider the space

$$\mathcal{B}_b = \Big\{ x : (-\infty, b] \to H, x_k \in C(J_k, H) \text{ and there exist } x(t_k^-) \text{ and } x(t_k^+) \\ \text{with } x(t_k) = x(t_k^-), x_0 = \varphi \in \mathcal{B}_h, k = 0, 1, 2, \ldots, m \Big\},$$

where x_k is the restriction of x to $J_k = (t_k, t_{k+1}]$, $k = 0, 1, 2, \ldots, m$. Set $\|\cdot\|_b$ a semi-norm in $\mathcal{B}_b$ defined by

$$\|x\|_b = \|x_0\|_{\mathcal{B}_h} + \sup_{0 \le s \le b} \left(E\|x(s)\|^2 \right)^{1/2}, \quad x \in \mathcal{B}_b.$$

Now, we give a useful lemma that appears in [23].

Lemma 2.5. *Assume that $x \in \mathcal{B}_b$, then for $t \in J$, $x_t \in \mathcal{B}_h$. Moreover,*

$$l\left(E\|x(t)\|^2\right)^{1/2} \le \|x_t\|_{\mathcal{B}_h} \le l \sup_{0 \le s \le t} \left(E\|x(s)\|^2\right)^{1/2} + \|x_0\|_{\mathcal{B}_h},$$

where $l = \int_{-\infty}^0 h(s)\,\mathrm{d}s < \infty$.

Lemma 2.6 ([5]). *Let $\nu(\cdot), \omega(\cdot) : [0, b] \to [0, \infty)$ be continuous functions. If $\omega(\cdot)$ is nondecreasing and there exist two constants $\theta > 0$ and $0 < \alpha < 1$ such that*

$$\nu(t) \le \omega(t) + \theta \int_0^t \frac{\nu(s)}{(t-s)^{1-\alpha}} \mathrm{d}s, \quad t \in J,$$

then

$$\nu(t) \le \mathrm{e}^{\theta^n (\Gamma(\alpha))^n t^{n\alpha} / \Gamma(n\alpha)} \sum_{j=1}^{n-1} \left(\frac{\theta b^\alpha}{\alpha} \right)^j \omega(t),$$

for every $t \in [0, b]$ and every $n \in \mathbf{N}$ such that $n\alpha > 1$ and $\Gamma(\cdot)$ is the Gamma function.

3. Main result

In the sequel, let $J_1 = (-\infty, b]$. Before stating and proving the main result, we present the definition of the mild solution to the system (1).

Definition 3.1. A stochastic process $x : J_1 \times \Omega \to H$ is called a mild solution of the system (1) if

- $x(t)$ is measurable and $\mathcal{F}_t$-adapted, for each $t \ge 0$;
- $x(t) \in H$ has càdlàg paths on $t \in [0, b]$ a.s., for every $0 \le s < t \le b$, the function $AS(t-s)g(s, x_s)$ is integrable and there exist $f \in N_{F,x}$ and $\mathcal{T}_k \in I_k\left(x(t_k^-)\right)$ $(k = 1, 2, \ldots, m)$ such that the following integral equation holds

$$x(t) = S(t)\left[\varphi(0) - g(0, \varphi)\right] + g(t, x_t) + \int_0^t AS(t-s)g(s, x_s)\,\mathrm{d}s \\ + \int_0^t S(t-s)(Bu)(s)\mathrm{d}s + \int_0^t S(t-s)f(s)\mathrm{d}w(s) + \sum_{0 < t_k < t} S(t - t_k)\mathcal{T}_k, \quad t \in J; \tag{3}$$

- $x_0(\cdot) = \varphi \in \mathcal{B}_h$ on J_0 satisfies $\|\phi\|_{\mathcal{B}_h} < \infty$.

Definition 3.2. The system (1) is said to be controllable on the interval J_1, if for every initial stochastic process $\varphi \in \mathcal{B}_h$ defined on J_0, there exists a stochastic control $u \in L^2(J, U)$, which is adapted to the filtration $\{\mathcal{F}_t\}_{t \ge 0}$ such that the mild solution $x(t)$ of the system (1) satisfies $x(b) = \varsigma$, where ς and b are preassigned the terminal state and time respectively.

In this paper, we will work under the following assumptions.

(H1) A is the infinitesimal generator of an analytic semigroup of bounded linear operators $S(t)$, $t \geq 0$ and there exists a constant M_1 such that

$$\|S(t)\|^2 \leq M_1, \quad \text{for all } t \geq 0.$$

(H2) The linear operator $W : L^2(J, U) \to L^2(\Omega; H)$, defined by

$$Wu = \int_0^b S(b-s)Bu(s)\mathrm{d}s$$

has an induced inverse W^{-1} which takes values in $L^2(J, U)/\ker \mathrm{W}$ (see [24]) and there exist two positive constants M_2 and M_3 such that

$$\|B\|^2 \leq M_2 \quad \text{and} \quad \|W^{-1}\|^2 \leq M_3.$$

(H3) $I_k : H \to \mathscr{P}_{c,cp}(H)$ $(k = 1, 2, \ldots, m)$ are multi-valued maps with closed graph and there exist constants $c_k > 0$, $k = 1, 2, \ldots, m$ such that for all $x, y \in H$

$$H_d\left(I_k(x), I_k(y)\right) \leq c_k\|x-y\|.$$

(H4) The function g is x_β-valued, $(-A)^\beta g : J \times \mathscr{B}_h \to H$ is completely continuous and such that the operator $g_1 : \mathscr{B}_h \to \mathscr{B}_h$ defined by $(g_1\varphi)(t) = g(t, \varphi)$ is compact and there exist constants M_g, β, θ_1 and θ_2 such that

$$E\|(-A)^\beta g(t, \varphi)\|^2 \leq \theta_1\|\varphi\|^2_{\mathscr{B}_h} + \theta_2, \quad t \in J, \varphi \in \mathscr{B}_h,$$
$$E\|g(t, \varphi_1) - g(t, \varphi_2)\|^2 \leq M_g\|\varphi_1 - \varphi_2\|^2_{\mathscr{B}_h} \quad t \in J, \varphi_1, \varphi_2 \in \mathscr{B}_h.$$

(H5) (i) $F : J \times \mathscr{B}_h \to \mathscr{P}(H)$ is an L^2-Carathédory function.

(ii) There exists a constant M_F such that

$$H_d(F(t, u_1), F(t, u_2)) \leq M_F\|u_1 - u_2\|, \quad t \in J, u_1, u_2 \in H.$$

(iii) There exists an integrable function $p : J \to [0, \infty)$ such that

$$E\|F(t, \varphi)\|^2 = \sup_{v \in F(t,\varphi)} \|v\|^2 \leq p(t)\Theta(\|v\|^2_{\mathscr{B}_h}), \quad t \in J, v \in \mathscr{B}_h,$$

where $\Theta : \mathbf{R}_+ \to (0, \infty)$ is a continuous nondecreasing function with

$$B_0K_3\int_0^b p(s)\mathrm{d}s < \int_{B_0K_3}^\infty \frac{\mathrm{d}s}{\Theta(s)},$$

where

$$B_0 = \mathrm{e}^{K_2^n(\Gamma(2\beta-1))^n b^{n(2\beta-1)}/\Gamma(n(2\beta-1))} \cdot \sum_{j=0}^{n-1}\left(\frac{K_3 b^{2\beta-1}}{2\beta-1}\right)^j, \tag{4}$$

$$N_0 = 2l^2\left\{49\|(-A)^{-\beta}\|^2\theta_1 + 49^2M_1M_2M_3b^2\left[\|(-A)^{-\beta}\|^2\theta_1M_1 + 2m^2M_1\sum_{k=1}^m c_k\right] + 98m^2M_1\sum_{k=1}^m c_k\right\}, \tag{5}$$

$$\begin{aligned} N_1 \;=\; & 2\|\varphi\|^2_{\mathscr{B}_h} + 2l^2\left\{\overline{M} + 49^2M_1M_2M_3b^2\left[bC^2_{1-\beta}\theta_1\int_0^b \frac{\mu(s)}{(b-s)^{2(1-\beta)}}\mathrm{d}s\right.\right. \\ & \left.\left. + M_1Tr(Q)\int_0^b p(s)\Theta(\mu(s))\,\mathrm{d}s\right]\right\}, \end{aligned} \tag{6}$$

$$N_2 = 98l^2bC^2_{1-\beta}\theta_1, \qquad N_3 = 98l^2M_1Tr(Q), \tag{7}$$

$$K_1 = N_1/(1-N_0), \qquad K_2 = N_2/(1-N_0), \qquad K_3 = N_3/(1-N_0), \tag{8}$$

$$\begin{aligned} \overline{M} \;=\; & 49M_1\|\varphi\|^2_{\mathscr{B}_h} + 49\|(-A)^{-\beta}\|^2M_1\left(\theta_1\|\varphi\|^2_{\mathscr{B}_h} + \theta_2\right) + 49\|(-A)^{-\beta}\|^2M_1\theta_2 + 49b\frac{C^2_{1-\beta}\theta_2b^{2\beta}}{2\beta-1} \\ & + 49^2M_1M_2M_3b^2\left\{\|\varsigma\|^2 + M_1\|\varphi\|^2_{\mathscr{B}_h}\|(-A)^{-\beta}\|^2 + M_1\left(\theta_1\|\varphi\|^2_{\mathscr{B}_h} + \theta_2\right) + \|(-A)^{-\beta}\|^2M_1\theta_2\right. \\ & \left. + \frac{C^2_{1-\beta}\theta_2b^{2\beta}}{2\beta-1} + 2m^2M_1\sum_{k=1}^m c_kE\|I_k(0)\|^2\right\} + 98m^2M_1\sum_{k=1}^m c_kE\|I_k(0)\|^2, \end{aligned} \tag{9}$$

and

$$C_0 = 32bM_1M_2M_3M_g + 32b^2M_1M_2M_3M_g\|(-A)^{-\beta}\|^2C_\beta\frac{b^{1-\beta}}{1-\beta} + 32bm^2M_1^2M_2M_3\sum_{k=1}^{m}c_k + 32b^2M_1^2M_2M_3M_F + 2m^2M_1\sum_{k=1}^{m}c_k. \tag{10}$$

The main result of the paper is the following theorem.

Theorem 3.3. *Assume that the assumptions* (H1)–(H5) *hold. If* $N_0 < 1$ *and* $C_0 < 1$, *then the system* (1) *is controllability on* J_1.

Proof. For an arbitrary process $x(\cdot)$, define the control process

$$u_x^b(t) = W^{-1}\left\{\varsigma - S(b)\left[\varphi(0) - g(0,\varphi)\right] - \int_0^b AS(b-s)g\left(s, x_s\right)\mathrm{d}s - g\left(b, x_b\right) - \int_0^b S(b-s)f(s)\mathrm{d}w(s) - \sum_{0<t_k<t} S(b-t_k)\mathcal{T}_k\right\}(t), \tag{11}$$

where $f \in N_{F,x}$ and $\mathcal{T}_k \in I_k\left(x(t_k^-)\right)$, $k = 1, 2, \ldots, m$.

Consider the multi-valued map $\Phi : \mathcal{B}_b \to \mathcal{P}(\mathcal{B}_b)$ defined by Φx the set of $\rho \in \mathcal{B}_b$ such that

$$\rho(t) = \begin{cases} \varphi(t), & t \in J_0 \\ S(t)\left[\varphi(0) - g(0,\varphi)\right] + g\left(t, x_t\right) - \int_0^t AS(t-s)g\left(s, x_s\right)\mathrm{d}s & \\ \quad + \int_0^t S(t-s)Bu_x^b(s)\mathrm{d}s + \int_0^t S(t-s)f(s)\mathrm{d}w(s) + \sum_{0<t_k<t} S(t-t_k)\mathcal{T}_k, & t \in J, \end{cases} \tag{12}$$

where $f \in N_{F,x}$ and $\mathcal{T}_k \in I_k\left(x(t_k^-)\right)$ $k = 1, 2, \ldots, m$. In what follows, we aim to show that the operator Φ has a fixed point, which is a solution of the system (1).

For $\varphi \in \mathcal{B}_h$, define

$$\widehat{\varphi}(t) = \begin{cases} \varphi(t), & t \in J_0 \\ S(t)\varphi(0), & t \in J, \end{cases} \tag{13}$$

then $\widehat{\varphi}(t) \in \mathcal{B}_b$. Set

$$x(t) = z(t) + \widehat{\varphi}(t), \quad -\infty < t \le b.$$

It is clear that x satisfies (3) if and only if z satisfies $z_0 = 0$ and

$$z(t) = -S(t)g(0,\varphi) + g\left(t, z_s + \widehat{\varphi}_s\right) + \int_0^t AS(t-s)g\left(s, z_s + \widehat{\varphi}_s\right)\mathrm{d}s + \int_0^t S(t-s)Bu_{z+\widehat{\varphi}}^b(s)\mathrm{d}s + \int_0^t S(t-s)f(s)\mathrm{d}w(s) + \sum_{0<t_k<t} S(t-t_k)\mathcal{T}_k, \quad t \in J, \tag{14}$$

where $u_{z+\widehat{\varphi}}$ is obtained from (11) by replacing x_t by $z_t + \widehat{\varphi}_t$.

Let $\mathcal{B}_b^0 = \{y \in \mathcal{B}_b : y_0 = 0 \in \mathcal{B}_h\}$. For each $y \in \mathcal{B}_b^0$, we have

$$\|y\|_b = \|y_0\|_{\mathcal{B}_h} + \sup_{0\le s\le b}\left(E\|y(s)\|^2\right)^{1/2} = \sup_{0\le s\le b}\left(E\|y(s)\|^2\right)^{1/2}.$$

Thus, $(\mathcal{B}_b^0, \|\cdot\|_b)$ is a Banach space. Set

$$\mathcal{B}_q = \{y \in \mathcal{B}_b^0 : \|y\|_b \le q\} \quad \text{for some } q \ge 0,$$

then $\mathcal{B}_q \subseteq \mathcal{B}_b^0$ is uniformly bounded. Moreover, for $z \in \mathcal{B}_q$, Lemma 2.5 shows that

$$\begin{aligned} \|z_t + \widehat{\varphi}_t\|_{\mathcal{B}_h}^2 &\le 2\left(\|z_t\|_{\mathcal{B}_h}^2 + \|\widehat{\varphi}_t\|_{\mathcal{B}_h}^2\right) \\ &\le 2l^2\sup_{0\le s\le t}E\|z(s)\|^2 + 2\|z_0\|_{\mathcal{B}_h}^2 + 2l^2\sup_{0\le s\le t}E\|\widehat{\varphi}(s)\|^2 + 2\|\widehat{\varphi}_0\|_{\mathcal{B}}^2 \\ &\le 2l^2q^2 + 2\|\varphi\|_{\mathcal{B}_h}^2 + 2l^2\sup_{0\le s\le t}\|S(s)\|^2E\|\varphi(0)\|^2 \\ &\le 2l^2\left(q^2 + M_1E\|\varphi(0)\|^2\right) + 2\|\varphi\|_{\mathcal{B}_h}^2 \\ &:= q'. \end{aligned} \tag{15}$$

Let the operator $\widehat{\Phi} : \mathcal{B}_b^0 \to \mathcal{P}(\mathcal{B}_b^0)$ defined by $\widehat{\Phi}z$ the set of $\hat{\rho} \in \mathcal{B}_b^0$ such that

$$\hat{\rho}(t) = \begin{cases} 0, \quad t \in J_0 \\ -S(t)g(0,\varphi) + g\left(t, z_t + \widehat{\varphi}_t\right) - \int_0^t AS(t-s)g\left(s, z_t + \widehat{\varphi}_t\right)\mathrm{d}s \\ \qquad + \int_0^t S(t-s)Bu^b_{z+\widehat{\varphi}}(s)\,\mathrm{d}s + \int_0^t S(t-s)f(s)\mathrm{d}w(s) + \sum_{0<t_k<t} S(t-t_k)\mathcal{T}_k, \quad t \in J. \end{cases} \tag{16}$$

Now, we consider the following multi-valued operators $\widehat{\Phi}_1$ and $\widehat{\Phi}_2$ defined by

$$\widehat{\Phi}_1 z(t) := \begin{cases} 0, \quad t \in J_0 \\ \int_0^t S(t-s)Bu^b_{z+\widehat{\varphi}}(s)\mathrm{d}s + \sum_{0<t_k<t} S(t-t_k)\mathcal{T}_k, \quad \mathcal{T}_k \in I_k\left(z(t_k^-) + \widehat{\varphi}(t_k^-)\right), t \in J \end{cases}$$

and

$$\widehat{\Phi}_2 z(t) := \begin{cases} 0, \quad t \in J_0 \\ -S(t)g(0,\varphi) + g\left(t, z_t + \widehat{\varphi}_t\right) - \int_0^t AS(t-s)g\left(s, z_t + \widehat{\varphi}_t\right)\mathrm{d}s + \int_0^t S(t-s)f(s)\mathrm{d}w(s), \quad t \in J. \end{cases}$$

It is clear that

$$\widehat{\Phi} = \widehat{\Phi}_1 + \widehat{\Phi}_2.$$

The problem of finding mild solutions of (1) is reduced to find the solutions of the operator inclusion $x \in \widehat{\Phi}_1(x) + \widehat{\Phi}_2(x)$. In what follows, we show that the operators $\widehat{\Phi}_1$ and $\widehat{\Phi}_2$ satisfy the conditions of Theorem 2.2.

Step 1. Φ_1 is a contraction.

Let $y_1, y_2 \in B_l$. By the assumptions, we have

$$\begin{aligned} EH_d^2\left(\widehat{\Phi}_1(y_1), \widehat{\Phi}_1(y_2)\right) \leq{}& 2EH_d^2\left(\sum_{0<t_k<t} S(t-t_k)I_k(y_1(t_k^-)), \sum_{0<t_k<t} S(t-t_k)I_k(y_2(t_k^-))\right) \\ &+ 2EH_d^2\Bigg(\int_0^t S(t-s)BW^{-1}\Bigg\{-g(b, y_{1,b}) - \int_0^b AS(b-s)g(s, y_{1,s})\mathrm{d}s \\ &- \int_0^b S(b-s)F(s, y_{1,s})\mathrm{d}W(s) - \sum_{0<t_k<s} S(b-t_k)I_k(y_1(t_k^-))\Bigg\}(s)\mathrm{d}s, \\ &\int_0^t S(t-s)BW^{-1}\Bigg\{-g(b, y_{2,b}) - \int_0^b AS(b-s)g(s, y_{2,s})\mathrm{d}s \\ &- \int_0^b S(b-s)F(s, y_{2,s})\mathrm{d}W(s) - \sum_{0<t_k<s} S(b-t_k)I_k(y_2(t_k^-))\Bigg\}(s)\mathrm{d}s\Bigg) \\ \leq{}& C_0 E\,||y_1 - y_2||^2, \end{aligned}$$

where C_0 is given in (10). Hence, $\widehat{\Phi}_1$ is a contraction.

Step 2. $\widehat{\Phi}_2$ has compact, convex values and it is completely continuous. This will be divided into the following claims.

Claim 1. *$\widehat{\Phi}_2 z$ is convex for each $z \in \mathcal{B}_b^0$.*

In fact, if $\hat{\rho}_1, \hat{\rho}_2 \in \widehat{\Phi}_2 z$, then, there exist $f_1, f_2 \in N_{F,z}$ such that

$$\begin{aligned} \hat{\rho}_i(t) ={}& -S(t)g(0,\varphi) + g\left(t, z_s + \widehat{\varphi}_s\right) + \int_0^t AS(t-s)g\left(s, z_s + \widehat{\varphi}_s\right)\,\mathrm{d}s \\ &+ \int_0^t S(t-s)f_i(s)\mathrm{d}w(s), \quad i = 1, 2, t \in J. \end{aligned} \tag{17}$$

Let $\lambda \in [0, 1]$. Since the operators B and W^{-1} are linear, we have

$$\begin{aligned} \left(\lambda\hat{\rho}_1(t) + (1-\lambda)\hat{\rho}_2(t)\right) ={}& -S(t)g(0,\varphi) + g\left(t, z_s + \widehat{\varphi}_s\right) + \int_0^t AS(t-s)g\left(s, z_s + \widehat{\varphi}_s\right)\mathrm{d}s \\ &+ \int_0^t S(t-s)\left[\lambda f_1(s) + (1-\lambda)f_2(s)\right]\mathrm{d}w(s). \end{aligned} \tag{18}$$

Since $N_{F,z}$ is convex (because F has convex values), we have $\lambda\hat{\rho}_1(t) + (1-\lambda)\hat{\rho}_2(t) \in \widehat{\Phi}_2$.

Claim 2. *$\widehat{\Phi}_2$ maps bonded sets into bounded sets in $\mathcal{B}_b^0$.*

Indeed, it is enough to show that there exists a positive constant Λ such that for each $\bar{\rho} \in \widehat{\Phi}_2 z, z \in \mathcal{B}_q = \{z \in \mathcal{B}_b^0 : \|z\|_b^0 \le q\}$, one has $\|\bar{\rho}\|_b^2 \le \Lambda$.

If $\hat{\rho} \in \widehat{\Phi}_2 z$, then there exists $f \in N_{F,z}$ such that, for each $t \in J$

$$\hat{\rho}(t) = -S(t)g(0,\varphi) + g\left(t, z_t + \widehat{\varphi}_t\right) + \int_0^t AS(t-s)g\left(s, z_s + \widehat{\varphi}_s\right) \mathrm{d}s + \int_0^t S(t-s)f(s)\mathrm{d}w(s). \tag{19}$$

Therefore, by the assumptions, for each $t \in J$, we have

$$\begin{aligned}
E\|\hat{\rho}(t)\|^2 \le\ & 16E\|S(t)g(0,\varphi)\|^2 + 16E\left\|g\left(t, z_t + \widehat{\varphi}_t\right)\right\|^2 + 16E\left\|\int_0^t AS(t-s)g\left(s, z_s + \widehat{\varphi}_s\right) \mathrm{d}s\right\|^2 \\
& + 16E\left\|\int_0^t S(t-s)f(s)\mathrm{d}w(s)\right\|^2 \\
\le\ & 16\|(-A)^{-\beta}\|^2 M_1\left(\theta_1\|\varphi\|_{\mathcal{B}_h}^2 + \theta_2\right) + 16\|(-A)^{-\beta}\|^2(\theta_1 q' + \theta_2) \\
& + 16bq'\int_0^b \frac{C_{1-\beta}^2\theta_1}{(t-s)^{2(t-s)}}\mathrm{d}s + \frac{16C_{1-\beta}^2\theta_2 b^{2\beta}}{2\beta-1} + 16M_1 Tr(Q)\|h_{q'}\|_{L^1} \\
:=\ & \Lambda.
\end{aligned}$$

Then, for each $\bar{\rho} \in \widehat{\Phi}_2 z$, we have $\|\bar{\rho}\|_b^2 \le \Lambda$.

Claim 3. *$\widehat{\Phi}_2$ maps bounded sets into equicontinuous sets of $\mathcal{B}_b^0$.*

Let $0 < \tau_1 < \tau_2 \le b$. Then, we have for each $z \in \mathcal{B}_q$ and $\bar{\rho} \in \widehat{\Phi}_2 z$, there exists $f \in N_{F,z}$ such that (19) holds. Therefore,

$$\begin{aligned}
E\|\hat{\rho}(\tau_2) - \hat{\rho}(\tau_1)\|^2 \le\ & 36\|(-A)^{-\beta}\|^2\left\|(S(\tau_1) - S(\tau_2))\left(\theta_1\|\varphi\|_{\mathcal{B}_h}^2 + \theta_2\right)\right\|^2 \\
& + 36\|(-A)^{\beta}\|^2\left\|(-A)^{-\beta}g\left(\tau_2, z_{\tau_2} + \widehat{\varphi}_{\tau_2}\right) - (-A)^{-\beta}g\left(\tau_1, z_{\tau_1} + \widehat{\varphi}_{\tau_1}\right)\right\|^2 \\
& + 36\|(-A)^{-\beta}\|^2 b\int_0^{\tau_2}\|S(\tau_1 - s) - AS(\tau_2 - s)\|^2\left(\theta_1 q' + \theta_2\right)\mathrm{d}s \\
& + 36(\tau_2 - \tau_1)\int_{\tau_1}^{\tau_2}\frac{C_{1-\beta}^2}{(\tau_1 - s)^{2(\beta-1)}}\left(\theta_1 q' + \theta_2\right)\mathrm{d}s \\
& + 36Tr(Q)\int_0^{\tau_2}\|S(\tau_1 - s) - S(\tau_2 - s)\|^2\|f(s)\|^2\mathrm{d}s \\
& + 36Tr(Q)E\int_{\tau_1}^{\tau_2}\|S(\tau_1 - s)\|^2\|f(s)\|^2\mathrm{d}s.
\end{aligned}$$

The right-hand side of the above inequality is independent of $z \in \mathcal{B}_q$ and tends to zero as $\tau_2 \to \tau_1$, the fact of g is completely continuous and the compactness of $S(t)$ for $t > 0$ imply the continuity in the uniform operator topology. Thus, the set $\{\widehat{\Phi}_2 z : z \in \mathcal{B}_q\}$ is equicontinuous.

Claim 4. *$\widehat{\Phi}_2$ is a compact multi-valued map.*

From the above claims, we see that family $\widehat{\Phi}_2\mathcal{B}_q$ is a uniformly bounded and equicontinuous collection. Therefore, it suffices to show by the Arzelá–Ascoli theorem that $\widehat{\Phi}_2$ maps $\mathcal{B}_q$ into a precompact set in $\mathcal{B}_q^0$. That is for each fixed $t \in J$, the set $V(t) = \{\widehat{\Phi}_2 z(t) : z \in \mathcal{B}_q\}$ is precompact in H.

Obviously, $V(0) = \{\hat{\Phi}(0)\}$. Let $t > 0$ be fixed and for $0 < \varepsilon < t$, define

$$\widehat{\Phi}_2^{\varepsilon} z(t) = -S(t)g(0,\varphi) + g(t-\varepsilon, z_{t-\varepsilon} + \widehat{\varphi}_{t-\varepsilon}) + \int_0^{t-\varepsilon} AS(t-s)g(s, z_s + \widehat{\varphi}_s)\mathrm{d}s + \int_0^{t-\varepsilon} S(t-s)f(s)\mathrm{d}w(s).$$

Since $S(t)$ is a compact operator, the set $V_\varepsilon(t) = \{\widehat{\Phi}_2^{\varepsilon} z(t) : z \in \mathcal{B}_q\}$ is precompact in z for each ε, $0 < \varepsilon < t$. Moreover,

$$\begin{aligned}
E\|\widehat{\Phi}_2 z(t) - \widehat{\Phi}_2^{\varepsilon} z(t)\|^2 \le\ & 16\|g(t, z_t + \widehat{\varphi}_t) - g(t-\varepsilon, z_{t-\varepsilon} + \widehat{\varphi}_{t-\varepsilon})\|^2 + 16\left\|\int_{t-\varepsilon}^t AS(t-s)g(s, z_s + \widehat{\varphi}_s)\mathrm{d}s\right\|^2 \\
& + 16M_1 Tr(Q)\int_{t-\varepsilon}^t h_{q'}(s)\mathrm{d}s.
\end{aligned}$$

Therefore,

$$E\|\widehat{\Phi}_2 z(t)-\widehat{\Phi}_2^{\varepsilon} z(t)\|^2 \to 0, \quad \text{as } \varepsilon \to 0^+,$$

and there are precompact sets arbitrary close to the set $\{\widehat{\Phi}_2 z(t) : z \in \mathcal{B}_q\}$. Hence, the Arzelá–Ascoli shows that $\widehat{\Phi}_2$ is a compact multi-valued map.

Claim 5. *$\widehat{\Phi}_2$ has a closed graph.*

Let $y^{(n)} \to y^*$, $\bar{\rho}_n \in \widehat{\Phi}_2 y^{(n)}$ and $\bar{\rho}_n \to \bar{\rho}_*$. We aim to show that $\bar{\rho}_* \in \widehat{\Phi}_2 y^*$. Indeed, $\bar{\rho}_n \in \widehat{\Phi}_2 y^{(n)}$ means that there exists $f_n \in N_{F,y^{(n)}}$ such that

$$\bar{\rho}_n(t) = -S(t)g(0,\varphi)+g(t,y_t^{(n)}+\widehat{\varphi}_t)+\int_0^t AS(t-s)g(s,y_s^{(n)}+\widehat{\varphi}_s)\mathrm{d}s+\int_0^t S(t-s)f_n(s)\mathrm{d}w(s), \quad t \in J.$$

We need to prove that there exists $f_* \in N_{F,y^*}$ such that

$$\bar{\rho}_*(t) = -S(t)g(0,\varphi)+g(t,y_t^*+\widehat{\varphi}_t)+\int_0^t AS(t-s)g(s,y_s^*+\widehat{\varphi}_s)\mathrm{d}s+\int_0^t S(t-s)f_*(s)\mathrm{d}w(s), \quad t \in J.$$

Since g is continuous, we get

$$\begin{aligned}&\Bigg\|\bar{\rho}_n(t)+S(t)g(0,\varphi)-g(t,y_t^{(n)}+\widehat{\varphi}_t)-\int_0^t AS(t-s)g(s,y_s^{(n)}+\widehat{\varphi}_s)\mathrm{d}s\\&\quad-\left(\bar{\rho}_*(t)+S(t)g(0,\varphi)-g(t,y_t^*+\widehat{\varphi}_t)-\int_0^t AS(t-s)g(s,y_s^*+\widehat{\varphi}_s)\mathrm{d}s\right)\Bigg\| \to 0, \quad \text{as } n \to \infty.\end{aligned}$$

Consider the linear continuous operator

$$\Gamma : L^2(J,H) \to C(J,H), \qquad f : \Gamma(f)(t) = \int_0^t S(t-s)f(s)\mathrm{d}w(s).$$

From Lemma 2.4, it follows that $\Gamma \circ N_F$ is a closed graph operator. Furthermore, we have

$$\bar{\rho}_n(t)+S(t)g(0,\varphi)-g(t,y_t^{(n)}+\widehat{\varphi}_t)-\int_0^t AS(t-s)g(s,y_s^{(n)}+\widehat{\varphi}_s)\mathrm{d}s \in \Gamma(N_{F,y^{(n)}}).$$

Since $y^{(n)} \to y^*$, it follows from Lemma 2.4 that

$$\bar{\rho}_*(t)+S(t)g(0,\varphi)-g(t,y_t^*+\widehat{\varphi}_t)-\int_0^t AS(t-s)g(s,y_s^*+\widehat{\varphi}_s)\mathrm{d}s = \int_0^t S(t-s)f^*(s)\mathrm{d}w(s),$$

for some $f^* \in N_{F,y^*}$.

Therefore, $\widehat{\Phi}_2$ is a completely continuous multi-valued map, u.s.c. with convex closed, compact values.

Step 3. A priori estimate.

Now it remains to show that the set

$$G = \{x \in H : x \in \lambda\Phi_1 x+\lambda\Phi_2 x \text{ for some } 0 < \lambda < 1\}$$

is bounded.

Let $x \in G$, then there exist $f \in N_{F,x}$ and $\mathcal{T}_k \in I_k(x(t_k^-))$ such that

$$\begin{aligned}x(t) &= \lambda S(t)\left[\varphi(0)-g(0,\varphi)\right]+\lambda g\left(t,x_t\right)+\lambda\int_0^t AS(t-s)g\left(s,x_s\right)\mathrm{d}s\\&\quad+\lambda\int_0^t S(t-s)(Bu)(s)\mathrm{d}s+\lambda\int_0^t S(t-s)f(s)\mathrm{d}w(s)+\lambda\sum_{0<t_k<t} S(t-t_k)\mathcal{T}_k, \quad t \in J\end{aligned}$$

for some $0 < \lambda < 1$. Then, by the assumptions, we have

$$\begin{aligned}E\|x(t)\|^2 &\le 49M_1\|\varphi\|_{\mathcal{B}_h}^2+49\|(-A)^{-\beta}\|^2M_1\left(\theta_1\|\varphi\|_{\mathcal{B}_h}^2+\theta_2\right)+49\|(-A)^{-\beta}\|^2M_1\left(\theta_1\|x_t\|_{\mathcal{B}_h}^2+\theta_2\right)\\&\quad+49b\left\{\int_0^t \frac{C_{1-\beta}^2\theta_1}{(t-s)^{2(1-\beta)}}\|x_s\|_{\mathcal{B}_h}^2\mathrm{d}s+\frac{C_{1-\beta}^2\theta_2 b^{2\beta}}{2\beta-1}\right\}+49^2M_1M_2M_3b^2\Bigg\{\|\varsigma\|^2+M_1\|\varphi\|_{\mathcal{B}_h}^2\|(-A)^{1-\beta}\|^2\\&\quad+M_1\left(\theta_1\|\varphi\|_{\mathcal{B}_h}^2+\theta_2\right)+\|(-A)^{-\beta}\|^2M_1\left(\theta_1\|x_b\|_{\mathcal{B}_h}^2+\theta_2\right)+b\int_0^b \frac{C_{1-\beta}^2\theta_1}{(b-s)^{2(1-\beta)}}\|x_s\|_{\mathcal{B}_h}^2\mathrm{d}s+\frac{C_{1-\beta}^2\theta_2 b^{2\beta}}{2\beta-1}\end{aligned}$$

$$
\begin{aligned}
&+ M_1 Tr(Q) \int_0^b p(s)\Theta(\|x_s\|^2_{\mathcal{B}_h})ds + 2m^2 M_1 \sum_{k=1}^m c_k E\|x(t_k^-)\|^2 \\
&+ 2m^2 M_1 \sum_{k=1}^m c_k E\|I_k(0)\|^2 \Bigg\} + 49 M_1 Tr(Q) \int_0^t p(s)\Theta(\|x_s\|^2_{\mathcal{B}_h})ds \\
&+ 98 m^2 M_1 \sum_{k=1}^m c_k E\|x(t_k^-)\|^2 + 98 m^2 M_1 \sum_{k=1}^m c_k E\|I_k(0)\|^2.
\end{aligned}
$$

Now, we consider the function μ defined by

$$\mu(t) = \sup_{0\le s\le t} E\|x(s)\|^2, \quad 0 \le t \le b.$$

From Lemma 2.4 and the above inequality, we have

$$E\|x(t)\|^2 \le 2l^2 \sup_{0\le s\le t} E\|x(s)\|^2 + 2\|\varphi\|^2_{\mathcal{B}_h}.$$

Therefore, we get

$$
\begin{aligned}
\mu(t) \le{}& 2\|\varphi\|^2_{\mathcal{B}_h} + 2l^2 \Bigg\{ \overline{M} + 49\|(-A)^{-\beta}\|^2\theta_1\mu(t) + 49bC^2_{1-\beta}\theta_1 \int_0^t \frac{\mu(s)}{(t-s)^{2(1-\beta)}}ds \\
&+ 49^2 M_1 M_2 M_3 b^2 \Bigg[\|(-A)^{-\beta}\|^2\theta_1\mu(t) M_1 + bC^2_{1-\beta}\theta_1 \int_0^b \frac{\mu(s)}{(b-s)^{2(1-\beta)}}ds \\
&+ M_1 Tr(Q) \int_0^b p(s)\Theta(\mu(s))ds + 2m^2 M_1 \sum_{k=1}^m c_k\mu(t) \Bigg] \\
&+ 49 M_1 Tr(Q) \int_0^t p(s)\Theta(\mu(s))ds + 98 m^2 M_1 \sum_{k=1}^m c_k\mu(t) \Bigg\},
\end{aligned}
$$

where $\overline{M}$ is given in (9). Thus, we obtain

$$\mu(t) \le K_1 + K_2 \int_0^t \frac{\mu(s)}{(t-s)^{2(1-\beta)}}ds + K_3 \int_0^t p(s)\Theta(\mu(s))ds,$$

where K_1, K_2 and K_3 are given in (8). By Lemma 2.6, we have

$$\mu(t) \le B_0 \left(K_1 + K_3 \int_0^t p(s)\Theta(\mu(s))ds \right),$$

where

$$B_0 = e^{K_2^n(\Gamma(2\beta-1))^n b^{n(2\beta-1)}/\Gamma(n(2\beta-1))} \cdot \sum_{j=0}^{n-1} \left(\frac{K_3 b^{2\beta-1}}{2\beta-1} \right)^j.$$

Let us take the right hand of the above inequality as $\nu(t)$. Then, $\nu(0) = B_0K_1$, $\mu(t) \le \nu(t)$, $0 \le t \le b$ and

$$\nu'(t) \le B_0 K_3 p(t)\Theta(\nu(t)).$$

This implies that

$$\int_{\nu(0)}^{\nu(t)} \frac{1}{\Theta(s)}ds \le B_0K_3 \int_0^b p(s)ds < \int_{B_0K_1}^{\infty} \frac{1}{\Theta(s)}ds.$$

This inequality shows that there is a constant K such that $\nu(t) \le K$, $t \in J$. So,

$$\|x_t\|^2_{\mathcal{B}_h} \le \mu(t) \le \nu(t), \quad t \in J,$$

where K depends only on b and on the functions $p(\cdot)$ and $\Theta(\cdot)$.

This indicates that the set G is bounded. As a consequence of Theorem 2.2, we deduce that $\widehat{\Phi}_1 + \widehat{\Phi}_2$ has a fixed point which is the mild solution of the system (1). Thus, the system (1) is controllable on J_1. □

4. An example

As an application, we consider the impulsive neutral stochastic partial differential inclusion of the following form

$$\begin{cases} \dfrac{\partial}{\partial t}v(t,x) \in \dfrac{\partial^2}{\partial x^2}v(t,x) + \dfrac{\partial}{\partial t}g(t, v(t-h,x)) + b(x)u(t) \\ \quad + [Q_1(t, v(t-h,x)), Q_2(t, v(t-h,x))]\mathrm{d}\beta(t), \quad 0 \le x \le \pi, t \in J, t \ne t_k, \\ v(t_k^+, x) - v(t_k^-, x) \in \left[-b_k|v(t_k^-, x)|, b_k|v(t_k^-, x)|\right], \quad t \in J, k = 1, 2, \ldots, m, \\ v(t,0) = v(t,\pi) = 0, \quad t \in J, \\ v(t,x) = \varphi(t,x), \quad -\infty < t \le 0, 0 \le x \le \pi, \end{cases} \tag{20}$$

where $J := [0, b]$, $b_k > 0$, $k = 1, 2, \ldots, m$, $v(t_k^+, x) = \lim_{(h,x)\to(0^+,x)} v(t_k + h, x)$, $v(t_k^-, x) = \lim_{(h,x)\to(0^-,x)} v(t_k + h, x)$, $Q_1, Q_2 : J \times \mathbf{R} \to \mathbf{R}$ are two given functions, and $\beta(t)$ is a one-dimensional standard Wiener process. We assume that for each $t \in J$, $Q_1(t, \cdot)$ is lower semi-continuous and for each $t \in J$, $Q_2(t, \cdot)$ is super semi-continuous.

Let $J_1 = (-\infty, b]$ and $H = L^2([0, \pi])$ with the norm $\|\cdot\|$. Define $A : H \to H$ by $Az = z''$ with domain

$$D(A) = \left\{z \in H, z, z' \text{ are absolutely continuous } z'' \in H, z(0) = z(\pi) = 0\right\}.$$

Then,

$$Az = \sum_{n=1}^{\infty} n^2(z, z_n), \quad z \in D(A),$$

where $z_n(s) = \sqrt{\frac{2}{\pi}}\sin(ns)$, $n = 1, 2, \ldots$ is the orthogonal set of eigenvectors in A. It is well known that A is the infinitesimal generator of an analytic semigroup $S(t)$, $t \ge 0$ in H given by

$$S(t)z = \sum_{n=1}^{\infty} \exp^{-n^2 t}(z, z_n)z_n, \quad z \in H.$$

Since the analytic semigroup $S(t)$ is compact, there exists a constant M_1 such that $\|S(t)\|^2 \le M_1$. In particular, $\|A^{-1/2}\|^2 = 1$.

Now, we give a special $\mathcal{B}_h$-space. Let $h(s) = \mathrm{e}^{2s}$, $s < 0$, then $l = \int_{-\infty}^0 h(s)\,\mathrm{d}s = \frac{1}{2}$ and let

$$\|\varphi\|_{\mathcal{B}_h} = \int_{-\infty}^0 h(s) \sup_{s\le\theta\le 0} \left(E\|\varphi(\theta)\|^2\right)^{1/2} \mathrm{d}s.$$

It follows from [3] that $(\mathcal{B}_h, \|\cdot\|_{\mathcal{B}_h})$ is a Banach space.

Let $B \in L(\mathbf{R}, H)$ be defined as

$$Bu(t) = b(x)u, \quad 0 \le x \le \pi, u \in \mathbf{R}, b(x) \in L^2([0, \pi]).$$

Moreover, the operator

$$Wu = \int_0^b \mathrm{e}^{-(t-s)}Bu(s)\mathrm{d}s,$$

is a bounded linear one. Let Ker $W = \{u \in L^2(J, U); Wu = 0\}$ be the null space of W, Then, the invertible operator W^{-1} is bounded and takes values in $L^2(J, U)/\ker \mathrm{W}$. For more details, one can see [15].

Hence, let

$$\varphi(\theta)x = \varphi(\theta, x), \quad (\theta, x) \in (-\infty, 0] \times [0, \pi], \quad v(t)(x) = v(t, x),$$
$$I_k(v(t_k^-))(x) = \left[-b_k|v(t_k^-, x)|, b_k|v(t_k^-, x)|\right], \quad x \in [0, \pi], k = 1, 2, \ldots, m,$$

and

$$F(t, \varphi)(x) = [Q_1(t, \varphi(\theta, x)), Q_2(t, \varphi(\theta, x))], \quad -\infty < \theta \le 0, x \in [0, \pi].$$

Then, (20) can be rewritten as the abstract form as the system (1). Moreover, we can define g, Q_1 and Q_2 as [15] to satisfy the assumptions stated in Theorem 3.3. We omit it here. Therefore, the system is controllable on J_1.

5. Conclusions

In this paper, we study the controllability of a class of neutral stochastic functional differential inclusions with infinite delay and multi-valued jump operators in Hilbert spaces. Sufficient conditions for the controllability are derived with the help of the fixed point theorem for discontinuous multi-valued operators due to Dhage. An example is provided to illustrate the feasibility of the obtained result.

Acknowledgement

The authors are deeply grateful to the anonymous referee for the careful reading of this paper and helpful comments, which have been very useful for improving the quality of this paper.

References

[1] Y.K. Chang, Controllability of impulsive functional differential systems with infinite delay in Banach spaces, Chaos Solitons Fractals 33 (2007) 1601–1609.
[2] Y.K. Chang, A. Anguraj, M.M. Mallika Arjunan, Existence results for impulsive neutral functional differential equations with infinite delay, Nonlinear Anal. Hybrid Syst. 2 (2008) 209–218.
[3] Y. Hino, S. Murakami, T. Naito, Functional-Differential Equations with Infinite Delay, in: Lecture Notes in Mathematics, vol. 1473, Springer-Verlag, Berlin, 1991.
[4] J. Hale, J. Kato, Phase spaces for retarded equations with infinite delay, Funkcial Ekvac. 21 (1978) 11–41.
[5] E. Hernández, Existence results for partial neutral functional integro–differential equations with unbounded delay, J. Math. Anal. Appl. 292 (2004) 194–210.
[6] B. Liu, Controllability of impulsive neutral functional differential inclusions with infinite delay, Nonlinear Anal. 60 (2005) 1533–1552.
[7] P. Balasubramaniam, D. Vinayagam, Existence of solutions of nonlinear neutral stochastic differential inclusions in a Hilbert space, Stochastic Anal. Appl. 23 (2005) 137–151.
[8] X. Mao, Stochastic Differential Equations and Applications, Horwood, Chichester, UK, 1997.
[9] Y. Ren, D.D. Sun, Second-order neutral impulsive stochastic evolution equations with delay, J. Math. Phys. 50 (2009) 102709.
[10] L. Hu, Y. Ren, Existence results for impulsive neutral stochastic functional integro-differential equations with infinite delays, Acta Appl. Math. 111 (2010) 303–317.
[11] R. Sakthivel, J. Luo, Asymptotic stability of impulsive stochastic partial differential equations with infinite delays, J. Math. Anal. Appl. 356 (2009) 1–6.
[12] R. Sakthivel, Approximate controllability of impulsive stochastic evolution equations, Funkcial Ekvac. 52 (2009) 381–393.
[13] M. Benchohra, S.K. Ntouyas, Controllability of neutral functional differential and integrodifferential inclusions in a Banach spaces with nonlocal conditions, Italian J. Pure Appl. Math. 14 (2003) 95–112.
[14] B. Liu, Controllability of neutral functional differential and integrodifferential inclusions with infinite delay, J. Optim. Theory Appl. 123 (2004) 573–593.
[15] P. Balasubramaniam, S.K. Ntouyas, Controllability for neutral stochastic functional differential inclusions with infinite delay in abstract space, J. Math. Anal. Appl. 324 (2006) 161–176.
[16] B.C. Dhage, Fixed-point theorems for discontinuous multi-valued operators on ordered spaces with applications, Comput. Math. Appl. 51 (2006) 589–604.
[17] N. Abada, M. Benchohra, H. Hammouche, Existence and controllability results for impulsive partial functional differential inclusions, Nonlinear Anal. 69 (2008) 2892–2909.
[18] G. Da Prato, J. Zabczyk, Stochastic Equations in Infinite Dimensions, Cambridge University Press, Cambridge, 1992.
[19] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, New York, 1983.
[20] M. Kisielewicz, Differential Inclusions and Optimal Control, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1991.
[21] K. Deimling, Multi-Valued Differential Equations, De Gruyter, Berlin, 1992.
[22] A. Lasota, Z. Opial, Application of the Kakutani-Ky-Fan theorem in the theory of ordinary differential equations or noncompact acyclic-valued map, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. 13 (1965) 781–786.
[23] Y. Li, B. Liu, Existence of solution of nonlinear neutral functional differential inclusions with infinite delay, Stochastic Anal. Appl. 25 (2007) 397–415.
[24] N. Carimichel, M.D. Quinn, Fixed Point Methods in Nonlinear Control, in: Lecture Notes in Control and Information Society, vol. 75, Springer, Berlin, 1984.