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A Conjecture of Regev about the Capelli Polynomial

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Let $X_1,..., X_{n^2}$, $Y_1,..., Y_{n^2}$ be generic $n \times n$ matrices over a field k of characteristic zero. If $f(X_1,..., X_n)$ is a multilinear invariant of $X_1,..., X_n$, then

$$\sum_{\pi \in S_{n^2}} (\operatorname{sign} \pi) f(X_1 Y_{\pi(1)}, X_2 Y_{\pi(2)} Y_{\pi(3)} Y_{\pi(4)}, X_3 Y_{\pi(5)} \cdots Y_{\pi(9)},..., X_n Y_{\pi(n^2-2n+2)} \cdots Y_{\pi(n^2)}) = \hat{f}(X_1,..., X_n)\, \Delta(Y),$$

where S_{n^2} is the symmetric group of degree n^2, $\Delta(Y)$ is the discriminant of $Y_1,..., Y_{n^2}$, and $\hat{f}(X_1,..., X_n)$ is a uniquely defined multilinear invariant of $X_1,..., X_n$. Thus $f \to \hat{f}$ defines a function from the vector space of multilinear invariants of $X_1,..., X_n$ to itself. An analysis of this function is used to prove Regev's conjecture that

$$\sum_{\pi,\rho \in S_{n^2}} (\operatorname{sign} \pi\rho) X_{\pi(1)} Y_{\rho(1)} X_{\pi(2)} X_{\pi(3)} X_{\pi(4)} Y_{\rho(2)} Y_{\rho(3)} Y_{\rho(4)} X_{\pi(5)} \cdots X_{\pi(n^2-2n+2)} \cdots X_{\pi(n^2)} Y_{\rho(n^2-2n+2)} \cdots Y_{\rho(n^2)}$$

is nonzero. In addition, a variant of the above function is used to evaluate the Capelli polynomial. © 1987 Academic Press, Inc.

1. INTRODUCTION

Let K be a field of characteristic zero, and let $X_1,..., X_{n^2}$, $Y_1,..., Y_{n^2}$ be generic $n \times n$ matrices over K. Regev [7, p. 1429] conjectured that the polynomial

$$F(X, Y) = \sum_{\pi,\rho S_{n^2}} (\operatorname{sign} \pi\rho) X_{\pi(1)} Y_{\rho(1)} X_{\pi(2)} X_{\pi(3)} X_{\pi(4)} Y_{\rho(2)} Y_{\rho(3)} Y_{\rho(4)} X_{\pi(5)} \cdots X_{\pi(n^2-2n+2)} \cdots X_{\pi(n^2)} Y_{\rho(n^2-2n+2)} \cdots Y_{\rho(n^2)}$$

is nonzero. Since $F(X, Y)$ is alternating and multilinear as a function of

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$X_1,\ldots,X_{n^2}$ and likewise as a function of $Y_1,\ldots, Y_{n^2}$, it is a scalar multiple of $\Delta(X)\,\Delta(Y)$, where $\Delta(X)$ is the discriminant of $X_1,\ldots, X_{n^2}$ and $\Delta(Y)$ is the discriminant of $Y_1,\ldots, Y_{n^2}$. Thus, proving the conjecture amounts to showing that the scalar is nonzero. Some consequences of the conjecture for the quantitative study of $n\times n$ matrices can be found in [2, pp. 212–214].

The main result of this article is Theorem 16, a proof of Regev's conjecture. The proof is based on the Procesi–Razmyslov theory of trace identities, which identifies $C(m)$, the space of multilinear invariants of $X_1,\ldots, X_m$, with a certain homomorphic image, $\overline{KS_m}$, of the group algebra of the symmetric group S_m. Via this identification $C(m)$ becomes an S_m-bimodule. This identification, although not formalized, was first used by Kostant [4] to give a proof of the Amitsur–Levitzki Theorem. Many of the arguments of this paper are related to those of Kostant.

The most important feature of the proof, which I believe has independent interest, is the construction of an isomorphism of $C(n)$ with itself as follows. Suppose that $f(X_1,\ldots, X_n)\in C(n)$. Then

$$\sum_{\pi\in S_{n^2}} (\operatorname{sign}\pi)\, f(X_1 Y_{\pi(1)}, X_2 Y_{\pi(2)} Y_{\pi(3)} Y_{\pi(4)}, X_3 Y_{\pi(5)}\cdots Y_{\pi(9)},\ldots,$$
$$X_n Y_{\pi(n^2-2n+2)}\cdots Y_{\pi(n^2)}) = \hat{f}(X_1,\ldots, X_n)\,\Delta(Y),$$

where $\Delta(Y)$ is the discriminant of $Y_1,\ldots, Y_{n^2}$ and $\hat{f}(X_1,\ldots, X_n)$ is a uniquely defined element of $C(n)$ (Lemma 6). This gives rise to a map $\Phi\colon C(n)\to C(n)$ defined by $\Phi(f)=\hat{f}$. A quite simple formal argument shows that Φ is a left S_n-module homomorphism (Theorem 8). Then an explicit formula for Φ is found which implies that Φ is an S_n-bimodule isomorphism (Theorem 11). Many variants of the map Φ can be defined; they are left S_n-module homomorphisms, but in general they are neither isomorphisms nor right S_n-module homomorphisms.

To prove Regev's conjecture that $F(X, Y)\neq 0$, let $g(X_1,\ldots, X_n)= T(X_1\cdots X_n)$, where T denotes trace. Then

$$T(F(X, Y)) = \sum_{\pi\in S_{n^2}} (\operatorname{sign}\pi)\, \Phi(g)(X_{\pi(1)}, X_{\pi(2)}X_{\pi(3)}X_{\pi(4)}, X_{\pi(5)}\cdots X_{\pi(9)},\ldots,$$
$$X_{\pi(n^2-2n+2)}\cdots X_{\pi(n^2)})\,\Delta(Y).$$

The proof is completed by showing that the right-hand side of the above equation is not zero. This requires the explicit formula for Φ as well as combinatorial results about $n\times n$ matrices and the group ring of S_n.

Regev's polynomial is a linear combination of specializations of the Capelli polynomial

$$C_{n^2}(X, Y) = \sum_{\pi\in S_{n^2}} (\operatorname{sign}\pi)\, X_1 Y_{\pi(1)} X_2 Y_{\pi(2)} X_3 Y_{\pi(3)}\cdots X_{n^2} Y_{\pi(n^2)}.$$

The argument of Lemma 6 shows that

$$T(C_{n^2}(X, Y)) = \hat{C}_{n^2}(X_1,..., X_{n^2})\, \Delta(Y),$$

where $\hat{C}_{n^2}(X_1,..., X_{n^2})$ is a multilinear invariant of $X_1,..., X_{n^2}$. Section 8 employs some of the above ideas to find an explicit formula for $\hat{C}_{n^2}(X_1,..., X_{n^2})$.

The final section of the paper determines a constant which occurs throughout the paper, beginning with Theorem 4(2), which says that

$$\sum_{\pi \in S_{n^2}} (\text{sign } \pi)\, T(X_{\pi(1)})\, T(X_{\pi(2)} X_{\pi(3)} X_{\pi(4)}) \cdots T(X_{\pi(n^2-2n+2)} \cdots X_{\pi(n^2)})$$
$$= C_n\, \Delta(X_1,..., X_n),$$

where C_n is a nonzero constant. In order to establish Regev's conjecture, it is enough to know that C_n is nonzero. Its exact value (up to a sign) is given by Theorem 24,

$$\pm C_n = \frac{1!3!5! \cdots (2n-1)!}{1!2! \cdots (n-1)!}$$

2. Fundamentals

Let K be a field of characteristic zero, and let $X_1, X_2,...,$ be generic $n \times n$ matrices over K. That is, $X_r = (x_{ij}(r))$ $(1 \leqslant i,\ j \leqslant n,\ r = 1, 2,...)$ are $n \times n$ matrices whose entries are independent commuting indeterminates over K. We assume throughout that the size, $n \times n$, of the matrices is fixed.

There is a homogeneous action of $GL(n, K)$ on $K[x_{ij}(r)]$ induced by $x_{ij}(r) \to \bar{x}_{ij}(r)$, where $P \in GL(n, K)$ and $PX_rP^{-1} = (\bar{x}_{ij}(r))$. The *ring of invariants* (or *ring of simultaneous polynomial invariants*), denoted C, is $K[x_{ij}(r)]^{GL(n,K)}$, the fixed ring of this action. For more information about C, see [1, pp. 67–71] or [2, pp. 195–199].

The trace of an $n \times n$ matrix U is denoted $T(U)$. Since $T(PUP^{-1}) = T(U)$ if $P \in GL(n, K)$, $T(X_{i_1} \cdots X_{i_k})$ lies in C whenever $X_{i_1} \cdots X_{i_k}$ is a monomial in the generic matrices $X_1, X_2,...$. Conversely

THEOREM 1 (First fundamental theorem of matrix invariants [5, Theorem 1.3]). *C is generated as a K-algebra by the traces of monomials $X_{i_1} \cdots X_{i_k}$ in the generic matrices $X_1, X_2,...$.* ∎

The second fundamental theorem describes all multilinear relations among the traces. In order to state it we need to introduce various objects associated with the group algebra of the symmetric group. Some knowledge of the representation theory of S_r will be assumed, above all the correspondence between simple factors of KS_r and partitions of r. We use [3] as reference.

Notation

S_r = symmetric group of permutations of $\{1,..., r\}$,

KS_r = group algebra of S_r over K,

$\lambda = (\lambda_1,..., \lambda_k)$ denotes a partition of r of length k, where $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k > 0$, $\lambda_1 + \cdots + \lambda_k = r$,

$I(\lambda)$ = minimal two-sided ideal of KS_r corresponding to λ,

$J(n, r) = \sum \{I(\lambda) \mid \lambda \text{ is a partition of } r \text{ of length} \geqslant (n+1)\}$,

$\overline{KS_r} = KS_r/J(n, r)$.

If $\pi \in S_r$, write π as a product of disjoint cycles

$$\pi = (a_1 \cdots a_{k_1})(b_1 \cdots b_{k_2})(c_1 \cdots c_{k_3})...,$$

where cycles of length one are included, so that each of the digits $1,..., r$ occurs exactly once. The *trace monomial* $T_\pi \in C$ associated with π is defined by

$$T_\pi(X_1,..., X_r) = T(X_{a_1} \cdots X_{a_{k_1}})\, T(X_{b_1} \cdots X_{b_{k_2}})\, T(X_{c_1} \cdots X_{c_{k_3}}) \cdots.$$

There is a K-vector space homomorphism from KS_r to $\bar{C}$ defined by $\sum a_\pi \pi \to \sum a_\pi T_\pi(X_1,..., X_r)$. The Procesi–Razmyslov Theorem describes its kernel.

THEOREM 2 (Second fundamental theorem of matrix invariants (Procesi [5, Theorem 4.3]; Razmyslov [6, p. 755 of English translation]). *$\sum a_\pi T_\pi(X_1,..., X_r) = 0$ in C if and only if $\sum a_\pi \pi \in J(n, r)$.* ∎

3. Multilinear Elements of C and the Discriminant

Let $C(r)$ denote the elements of C which are multilinear in $X_1,..., X_r$. In other words, $C(r)$ is the image of the map $KS_r \to C$ defined by $\sum a_\pi \pi \to \sum a_\pi T_\pi(X_1,..., X_r)$. By Theorem 2, this map induces a K-vector space isomorphism

$$\overline{KS_r} \xrightarrow{\theta_r} C(r),$$

where $\overline{KS_r} = KS_r/J(n, r)$. *We will write elements of* $\overline{KS_r}$ *as linear combinations* $\sum a_\pi \pi$ $(a_\pi \in K,\ \pi \in S_r)$, *even though such a representation is not unique when* $J(n, r) \neq 0$.

We make $C(r)$ into an S_r-bimodule via θ_r. Note that neither the left nor the right action is the usual permuation action of S_r on $C(r)$. For example, if $r = 3$ and we regard permutations as functions acting on the left (so that $[(12)(3)][(1)(23)] = (123)$), then

$$\begin{aligned}(1)(23) &\xrightarrow{\theta} T(X_1)\, T(X_2X_3),\\ (12)(3)[T(X_1)\, T(X_2X_3)] &= T(X_1X_2X_3),\\ [T(X_1)\, T(X_2X_3)](12)(3) &= T(X_1X_3X_2),\\ (12)(3)[T(X_1)\, T(X_2X_3)](12)(3) &= T(X_2)\, T(X_1X_3).\end{aligned}$$

The usual permutation action of S_r on $C(r)$ is induced by the action of S_r on $\overline{KS_r}$ by conjugation. This implies that θ_r carries the center of the ring $\overline{KS_r}$ to elements of $C(r)$ which are symmetric functions of $X_1, \ldots, X_r$.

The *discriminant* of $n \times n$ matrices $U_1, \ldots, U_{n^2}$ is the determinant of the $n^2 \times n^2$ matrix whose i-th row is $(u_i(1, 1), \ldots, u_i(1, n), u_i(2, 1), \ldots, u_i(n, n))$. It is denoted $\Delta(U_1, \ldots, U_{n^2})$ or $\Delta(U)$. The following properties of the discriminant are well known.

LEMMA 3. (1) *As a function* Δ: $M_n(K)^{n^2} \to K$, *the discriminant is characterized by the following two properties*:

(a) Δ *is an alternating multilinear function.*

(b) $\Delta(e_{11}, \ldots, e_{1n},\ e_{21}, \ldots, e_{nn}) = 1$, *where the* e_{ij} *are the standard matrix units.*

(2) *If* $P \in GL(n, K)$, $\Delta(PU_1P^{-1}, \ldots, PU_{n^2}P^{-1}) = \Delta(U_1, \ldots, U_{n^2})$. *Hence* $\Delta(X_1, \ldots, X_{n^2})$ *lies in* C, *by Theorem* 1. ∎

The rest of this section is an expanded reprise of [2, p. 211]. Consider the isomorphism θ_{n^2}: $\overline{KS_{n^2}} \to C(n^2)$. For an element u of $C(n^2)$ to be alternating as a function of $X_1, \ldots, X_{n^2}$ means precisely that

$$\pi u \pi^{-1} = (\operatorname{sign} \pi) u \qquad \text{for all} \quad \pi \in S_{n^2}. \tag{$*$}$$

Let us also call an element of $\overline{KS_{n^2}}$ *alternating* if it satisfies $(*)$. It is clear that every alternating element of $\overline{KS_{n^2}}$ is a K-linear combination of the elements

$$\sum_{\pi \in S_{n^2}} (\operatorname{sign} \pi)\, \pi \sigma \pi^{-1},$$

as σ varies over S_{n^2}. However, we do not need all $\sigma \in S_{n^2}$, just a representative from each conjugacy class.

The conjugacy classes of S_{n^2} are in one-to-one correspondence with the partitions of n^2. For each partition $\lambda = (\lambda_1, ..., \lambda_k)$ of n^2, set

$$\sigma(\lambda) = (1 \cdots \lambda_k)(\lambda_k + 1 \cdots \lambda_k + \lambda_{k-1}) \cdots (\lambda_k + \lambda_{k-1} + \cdots + \lambda_2 + 1 \cdots n^2),$$

$$a_\lambda = \sum_{\pi \in S_{n^2}} (\operatorname{sign} \pi)\, \pi\sigma(\lambda)\pi^{-1}.$$

Since the set of all such $\sigma(\lambda)$ is a full set of representatives for the conjugacy classes of S_{n^2}, every alternating element in $\overline{KS_{n^2}}$ is a K-linear combination of the a_λ.

Most a_λ are zero (even regarded as elements of KS_{n^2}), for if $\lambda = (\lambda_1, ..., \lambda_k)$ has either an even part or two equal parts, then the S_{n^2}-conjugacy class of $\sigma(\lambda)$ equals its A_{n^2}-conjugacy class [3, 1.2.10, p. 12], and the summation defining a_λ collapses. Such a_λ are already zero in KS_{n^2} and in fact the set

$$\{a_\lambda \mid \lambda \text{ is a partition of } n^2 \text{ into distinct odd parts}\}$$

is a K-basis for the alternating elements of KS_{n^2}.

In $\overline{KS_{n^2}}$, however, all the a_λ are zero except one. Define $A_\lambda = A_\lambda(X_1, ..., X_{n^2}) \in C(n^2)$ by

$$A_\lambda = \theta_{n^2}(a_\lambda) = \sum_{\pi \in S_{n^2}} (\operatorname{sign} \pi)\, T(X_1 \cdots X_{\lambda_k})\, T(X_{\lambda_k+1} \cdots X_{\lambda_k+\lambda_{k-1}}) \cdots T(X_{\lambda_k+\lambda_{k-1}+\cdots+\lambda_2+1} \cdots X_{n^2}).$$

The Amitsur–Levitzki Theorem [1, p. 45] asserts that if $t \geqslant 2n$, then the *standard polynomial*

$$S_t(U_1, ..., U_t) = \sum_{\pi \in S_t} (\operatorname{sign} \pi)\, U_{\pi(1)} \cdots U_{\pi(t)}$$

vanishes on $n \times n$ matrices. The above formula for A_λ can be rearranged so that A_λ is expressed as a linear combination of products, each containing the trace of a standard polynomial of degree λ_1 as the final factor. Hence $A_\lambda = 0$ if λ_1, the largest part of λ, satisfies $\lambda_1 \geqslant 2n$.

In conjunction with the preceding analysis of a_λ, this implies that $a_\lambda \in \overline{KS_{n^2}}$ and $A_\lambda \in C(n^2)$ are zero unless the parts of λ are odd, distinct, and $< 2n$. There is only one such partition, $\lambda_0 = (2n-1, 2n-3, ..., 5, 3, 1)$.

Since the discriminant $\Delta(X_1, ..., X_{n^2})$ is alternating, it is a K-linear combination of the A_λ and thus a scalar multiple of A_{λ_0}. Furthermore, A_{λ_0} is not zero since $\Delta(X_1, ..., X_{n^2})$ is not zero. The following theorem summarizes our discussion.

THEOREM 4 ([2, p. 211]). *For each partition* λ *of* n^2, *let* $a_\lambda \in \overline{KS_{n^2}}$ *and* $A_\lambda = \theta_{n^2}(a_\lambda) \in C(n^2)$ *be defined as above, and let* $\lambda_0 = (2n-1, 2n-3, \ldots, 5, 3, 1)$. *Then*

(1) *If* $\lambda \neq \lambda_0$, $a_\lambda = 0$ *and* $A_\lambda = 0$.

(2) $A_{\lambda_0}(X_1, \ldots, X_{n^2}) = C_n \Delta(X_1, \ldots, X_{n^2})$, *where* C_n *is a nonzero constant depending only on n.* ∎

We will show later (Theorem 24) that

$$\pm C_n = \frac{1!3!5! \cdots (2n-1)!}{1!2! \cdots (n-1)!}.$$

The main results of the paper only require knowing that C_n is not zero.

4. THE FUNCTION Φ: $C(n) \to C(n)$

The object of this and the next two sections is to define and analyze a function Φ: $C(n) \to C(n)$, which is the main tool in proving Regev's conjecture. The definition involves an auxiliary map Γ: $C(n) \to C(n+n^2)$, and it is convenient to rename the first $n+n^2$ generic matrices $X_1, \ldots, X_n$, $Y_1, \ldots, Y_{n^2}$, and to set

$$S_n = \text{permutations of } \{1, \ldots, n\},$$
$$S_{n^2} = \text{permutations of } \{1^*, \ldots, (n^2)^*\},$$
$$S_{n+n^2} = \text{permutations of } \{1, \ldots, n, 1^*, \ldots, (n^2)^*\},$$
$$C(n) = \text{multilinear invariants of } X_1, \ldots, X_n,$$
$$C(n^2) = \text{multilinear invariants of } Y_1, \ldots, Y_{n^2},$$
$$C(n+n^2) = \text{multilinear invariants of } X_1, \ldots, X_n, Y_1, \ldots, Y_{n^2}.$$

Then $S_n \times S_{n^2} \subseteq S_{n+n^2}$, $KS_n \overline{KS_{n^2}} \subseteq \overline{KS_{n+n^2}}$, $C(n)\, C(n^2) \subseteq C(n+n^2)$, and there are isomorphisms θ_n: $KS_n \to C(n)$, θ_{n^2}: $\overline{KS_{n^2}} \to C(n^2)$, θ_{n+n^2}: $\overline{KS_{n+n^2}} \to C(n+n^2)$. Note that we have written KS_n instead of $\overline{KS_n}$ because $KS_n = \overline{KS_n}$ (i.e., $J(n, n) = 0$). The θ's are obviously not ring homomorphisms individually, but they do have the following multiplicative property.

LEMMA 5. *Let* $f(X_1, \ldots, X_n) \in C(n)$ *and* $g(Y_1, \ldots, Y_{n^2}) \in C(n^2)$. *Then* $(\theta_{n+n^2})^{-1}(fg) = (\theta_n)^{-1}(f) \cdot (\theta_{n^2})^{-1}(g)$. ∎

Define a function Γ: $C(n) \to C(n+n^2)$ by

$$\Gamma(f) = \sum_{\pi \in S_{n^2}} (\operatorname{sign} \pi) f(X_1 Y_{\pi(1^*)}, X_2 Y_{\pi(2^*)} Y_{\pi(3^*)} Y_{\pi(4^*)}, \ldots, X_n Y_{\pi((n^2-2n+2)^*)} \cdots Y_{\pi((n^2)^*)}).$$

LEMMA 6. *Let* $f = f(X_1, \ldots, X_n) \in C(n)$. *Then there is a unique* $\hat{f} = \hat{f}(X_1, \ldots, X_n) \in C(n)$ *such that* $\Gamma(f) = \hat{f}\Delta(Y)$, *where* $\Delta(Y)$ *is the discriminant of* $Y_1, \ldots, Y_{n^2}$.

Proof. Regard the entries of $X_1, \ldots, X_n$ as constants. Since $\Gamma(f)$ is an alternating multilinear function of $Y_1, \ldots, Y_{n^2}$, aplying Lemma 3(1) with field of constants $L = K(x_{ij}(r))$ shows that $\Gamma(f) = \hat{f}\Delta(Y)$, where $\hat{f} \in L$. Routine arguments show that in fact $\hat{f} \in C(n)$. Finally, $\hat{f}$ is unique because C is a domain. ∎

Define Φ: $C(n) \to C(n)$ by $\Phi(f) = \hat{f}$. It is clear that Φ is a K-linear map. In fact it is a left S_n-module homomorphism. To see this, we translate Lemma 6 into a statement about KS_n via the θ isomorphisms.

Let $\Gamma_0 = (\theta_{n+n^2})^{-1} \Gamma \theta_n$.

$$\begin{array}{ccc} KS_n & \xrightarrow{\Gamma_0} & \overline{KS_{n+n^2}} \\ {\scriptstyle \theta_n} \downarrow & & \downarrow {\scriptstyle \theta_{n+n^2}} \\ C(n) & \xrightarrow{\Gamma} & C(n+n^2) \end{array}$$

To define Γ_0 directly in terms of KS_n, let

$$\alpha = (11^*)(22^*3^*4^*) \cdots (n(n^2-2n+2)^* \cdots (n^2)^*),$$

$$\beta = \sum_{\pi \in S_{n^2}} (\operatorname{sign} \pi)\, \pi\alpha\pi^{-1}.$$

If $\sigma \in S_n$, then the disjoint cycle decomposition of $\sigma\alpha \in S_{n+n^2}$ is obtained from that of σ by replacing each $i \in \{1, \ldots, n\}$ by the "string" $[i(i^2-2i+2)^* \cdots (i^2)^*]$, and for any $\pi \in S_{n^2}$ the disjoint cycle decomposition of $\sigma\pi\alpha\pi^{-1}$ is obtained from that of σ by replacing each $i \in \{1, \ldots, n\}$ by the string $[i, \pi((i^2-2i+2)^*) \cdots \pi((i^2)^*)]$. Having made this observation, it is not difficult to see that

(1) For any $f_0 \in KS_n$, $\Gamma_0(f_0) = f_0\beta$.

Let $\Delta_0 = (\theta_{n^2})^{-1}(\Delta(Y)) \in \overline{KS_{n^2}}$. (Theorem 4 gives an explicit formula for Δ_0, but we do not need it.) By Lemma 5,

(2) For any $f \in C(n)$, $(\theta_{n+n^2})^{-1}(\hat{f}\Delta(Y)) = (\theta_n)^{-1}(\hat{f})\Delta_0$.

Any $f_0 \in KS_n$ is equal to $(\theta_n)^{-1}(f)$ for some $f \in C(n)$ since θ_n is an isomorphism. Hence, by using (1) and (2), Lemma 6 can be translated into

LEMMA 7. *Suppose $f_0 \in KS_n$, and let $\beta \in \overline{KS_{n+n^2}}$ and $\Delta_0 \in \overline{KS_{n^2}}$ be as defined above. Then there is a unique $\hat{f}_0 \in KS_n$ such that*

$$f_0\,\beta = \Gamma_0(f_0) = \hat{f}_0 \Delta_0. \quad ∎$$

It is clear that $\hat{f}_0 = (\theta_n)^{-1}\,\Phi\theta_n(f_0)$, and we define Φ_0 to be $(\theta_n)^{-1}\,\Phi\theta_n$. The significance of Lemma 7 is that it implies, in a purely formal way, that $f_0 \to \hat{f}_0$ is a left S_n-homomorphism, and hence that Φ_0 and Φ are left S_n-homomorphisms. For suppose that $\sigma \in S_n$. Then

$$(\sigma f_0)\beta = \sigma(f_0\,\beta) = \sigma(\hat{f}_0\Delta_0) = (\sigma\hat{f}_0)\Delta_0$$

in $\overline{KS_{n+n^2}}$. But Lemma 7 asserts that $\widehat{(\sigma f_0)}$ is the *unique* element of KS_n such that $(\sigma f_0)\beta = \widehat{(\sigma f_0)}\Delta_0$. Thus $\sigma\hat{f}_0 = \widehat{(\sigma f_0)}$, or

THEOREM 8. *The maps Φ: $C(n) \to C(n)$ and Φ_0: $KS_n \to KS_n$ are left S_n-module homomorphisms.* ∎

5. A FORMULA FOR $\Phi_0(1)$

Since Φ_0: $KS_n \to KS_n$ is a left S_n-module homomorphism, it is completely determined by $\Phi_0(1)$. In this section we will compute $\Phi_0(1)$ and find that it is a central unit of KS_n. The method consists of evaluating $T_\sigma(X_1,..., X_n)$ and $\Phi(T_\sigma)(X_1,..., X_n)$ under the specialization $X_1 = X_2 = \cdots = X_n = I$, the $n \times n$ identity matrix. This gives rise to a system of $n!$ linear equations whose solution determines $\Phi_0(1)$.

The next lemma follows immediately from the definition of T_σ and the fact that the trace of the $n \times n$ identity matrix is n.

LEMMA 9. *Let $\sigma \in S_n$, and let $z(\sigma)$ denote the number of cycles in the decomposition of σ into disjoint cycles. Then $T_\sigma(I,..., I) = n^{z(\sigma)}$.* ∎

LEMMA 10. *Let $\sigma \in S_n$, and let C_n be the nonzero constant of Theorem 4. Then*

$$\Phi(T_\sigma)(I,..., I) = \begin{cases} C_n & \text{if } \sigma = 1 \\ 0 & \text{if } \sigma \neq 1. \end{cases}$$

Proof. For λ a partition of n^2, let $A_\lambda(Y) = A_\lambda(Y_1,..., Y_{n^2}) \in C(n^2)$ be as in Theorem 4, and recall the auxiliary function Γ: $C(n) \to C(n+n^2)$ defined before Lemma 6. By the definition of Φ: $C(n) \to C(n)$ via Lemma 6.

$$\Phi(T_1)(I,..., I)\, \Delta(Y) = \Gamma(T_1)(I,..., I, Y_1,..., Y_{n^2}) = A_{\lambda_0}(Y_1,..., Y_{n^2}),$$

where $\lambda_0 = (2n-1, 2n-3,..., 5, 3, 1)$, and

$$\Phi(T_\sigma)(I,..., I)\, \Delta(Y) = \Gamma(T_\sigma)(I,..., I, Y_1,..., Y_{n^2}) = A_{\lambda(\sigma)}(Y_1,..., Y_{n^2}),$$

where $\lambda(\sigma)$ is some partition of n^2 with $z(\sigma)$ parts. If $\sigma \neq 1$, $z(\sigma) < n$, so that $\lambda(\sigma) \neq \lambda_0$.

By Theorem 4, $A_{\lambda_0}(Y) = C_n \Delta(Y)$ and $A_\lambda(Y) = 0$ if $\lambda \neq \lambda_0$, which proves the lemma. ∎

Let $\Phi_0(1) = \sum a_\sigma \sigma \in KS_n$, so that $\Phi(T_1) = \sum a_\sigma T_\sigma$. By Theorem 8, Φ is a left S_n-homomorphism, so for each $\rho \in S_n$

$$\sum a_\sigma T_{\rho\sigma}(X_1,..., X_n) = \Phi(T_\rho)(X_1,..., X_n). \tag{$*$}$$

Using Lemmas 9 and 10 to evaluate $(*)$ under the specialization $X_1 = \cdots = X_n = I$ gives rise to a system of $n!$ equations indexed by the elements ρ of S_n, namely

$$\sum a_\sigma n^{z(\sigma)} = C_n \qquad (\rho = 1),$$

$$\sum a_\sigma n^{z(\rho\sigma)} = 0 \qquad (\rho \neq 1).$$

Noting that $z(\rho) = z(\rho^{-1})$ since ρ and ρ^{-1} have the same cycle structure leads to the following calculation in KS_n:

$$\begin{aligned}\left(\sum_\sigma a_\sigma \sigma\right)\left(\sum_\rho n^{z(\rho)}\rho\right) &= \left(\sum_\sigma a_\sigma \sigma\right)\left(\sum_\rho n^{z(\rho)}\rho^{-1}\right) \\ &= \sum_{\sigma,\rho} a_\sigma n^{z(\rho)} \sigma\rho^{-1} = \sum_{\sigma,\rho} a_\sigma n^{z(\rho\sigma)}\rho^{-1} \\ &= \sum_\rho \left(\sum_\sigma a_\sigma n^{z(\rho\sigma)}\right)\rho^{-1} = C_n.\end{aligned}$$

Thus $\Phi_0(1) = \sum a_\sigma \sigma = C_n(\sum n^{z(\rho)}\rho)^{-1}$, a central unit in KS_n, and so

THEOREM 11. *Let C_n be the nonzero constant of Theorem 4, and let $z(\rho)$ denote the number of disjoint cycles in $\rho \in S_n$. Then*

(1) $\Phi_0(1) = C_n(\sum n^{z(\rho)}\rho)^{-1}$, *a central unit in KS_n.*

(2) *Since $\Phi_0(1)$ is central, Φ and Φ_0 are S_n-bimodule homomorphisms and $\Phi(T_1)(X_1,..., X_n)$ is a symmetric function of $X_1,..., X_n$.*

(3) *Since $\Phi_0(1)$ is a unit, Φ and Φ_0 are isomorphisms.* ∎

6. THE COEFFICIENT OF AN n-CYCLE IN $\Phi_0(1)$

This section is devoted to computing the coefficient of an n-cycle in $\Phi_0(1)$. We will see in the next section that the validity of Regev's conjecture is equivalent to the fact that this coefficient is nonzero. It is fortunate that this particular coefficient of $\Phi_0(1)$ is rather easy to compute even though a simple formula for an arbitrary coefficient of $\Phi_0(1)$ is not apparent.

Let $\chi_1,..., \chi_t$ be the irreducible characters of S_n, and let

$$e_i = \frac{1}{n!}\chi_i(1)\sum_{\sigma}\chi_i(\sigma) \tag{1}$$

be the corresponding minimal central idempotents of KS_n. Since $\sum n^{z(\sigma)}\sigma$ is central in KS_n,

$$\sum n^{z(\sigma)}\sigma = r_1e_1 + \cdots + r_te_t \quad \text{and} \quad \left(\sum n^{z(\sigma)}\sigma\right)^{-1} = r_1^{-1}e_1 + \cdots + r_t^{-1}e_t \tag{2}$$

for certain $r_i \in K$, where

$$r_i = \frac{\chi_i(\sum n^{z(\sigma)}\sigma)}{\chi_i(1)} = \frac{\sum n^{z(\sigma)}\chi_i(\sigma)}{\chi_i(1)}.$$

Let V be a K-vector space of dimension n, and let S_n act on $V^{\otimes n}$ by place permutation:

$$\sigma(v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}.$$

This representation of S_n is called P_n^n in [3], where it is introduced on p. 150.

Let P: $S_n \to K$ be the character of this representation. Since P is a permutation character it is easy to verify that $P(\sigma) = n^{z(\sigma)}$ for all $\sigma \in S_n$ (or see [3, 4.3.12, p. 150]). Hence

$$P = \langle P, \chi_1\rangle\chi_1 + \cdots + \langle P, \chi_t\rangle\chi_t,$$

where $\langle\,,\,\rangle$ denotes the inner product of characters and

$$\langle P, \chi_i\rangle = \frac{1}{n!}\sum_\sigma P(\sigma)\,\chi_i(\sigma) = \frac{1}{n!}\sum_\sigma n^{z(\sigma)}\chi_i(\sigma) = \frac{\chi_i(1)\,r_i}{n!},$$

$$r_i^{-1} = \frac{\chi_i(1)}{n!\,\langle P, \chi_i\rangle}. \tag{3}$$

By Eqs. (1)–(3),

$$\begin{aligned}\left(\sum n^{z(\sigma)}\sigma\right)^{-1} &= r_1^{-1}e_1 + \cdots + r_t^{-1}e_t\\ &= \sum_{i=1}^{t}\left(\frac{\chi_i(1)}{n!\,\langle P, \chi_i\rangle}\right)\left(\frac{1}{n!}\chi_i(1)\sum_\sigma \chi_i(\sigma)\sigma\right)\\ &= \frac{1}{(n!)^2}\sum_\sigma\left(\sum_{i=1}^{t}\frac{\chi_i(1)^2\,\chi_i(\sigma)}{\langle P, \chi_i\rangle}\right)\sigma,\end{aligned}$$

and Theorem 11 (1) can be restated as

THEOREM 12. *In terms of the above notation,*

$$\Phi_0(1) = \frac{C_n}{(n!)^2}\sum_{\sigma\in S_n}\left(\sum_{i=1}^{t}\frac{\chi_i(1)^2\chi_i(\sigma)}{\langle P, \chi_i\rangle}\right)\sigma,$$

where C_n is the nonzero constant of Theorem 4. ∎

Although Theorem 12 gives a formula for the coefficient of an arbitrary element of S_n in $\Phi_0(1)$, it $\Phi_0(1)$, it does not in general seem to simplify. In particular, it is not apparent which coefficients are nonzero. However, if μ is an n-cycle, two strokes of good fortune come into play. First, $\chi_i(\mu)$ is zero for nearly all irreducible characters χ_i. Second, $\chi_i(1)$, $\chi_i(\mu)$, and $\langle P, \chi_i\rangle$ all have a simple form when $\chi_i(\mu)\neq 0$.

The irreducible characters $\chi_1,\ldots,\chi_t$ of S_n have a natural indexing by the partitions of n [3, 2.1.11, p. 37],

$$\{\chi_1,\ldots,\chi_t\} = \{\chi(\lambda)\mid \lambda \text{ a partition of } n\}.$$

Among the partitions of n are the n hooks, $\varepsilon_0=(n)$, $\varepsilon_1=(n-1, 1)$, $\varepsilon_2=(n-2, 1, 1),\ldots, \varepsilon_{n-1}=(1,\ldots, 1)$.

LEMMA 13. *Suppose that μ is an n-cycle and λ is a partition of n. Let $\varepsilon_0,\ldots,\varepsilon_{n-1}$ and P be as above.*

(1) [3, 2.3.17, p. 54]. $\chi(\varepsilon_i)(\mu) = (-1)^i$, $i = 0, ..., n-1$. $\chi(\lambda)(\mu) = 0$ *if* $\lambda \neq \varepsilon_0, ..., \varepsilon_{n-1}$.

(2) (*Hook formula*) [3, 2.3.21, p. 56]. $\chi(\varepsilon_i)(1) = \binom{n-1}{i}$.

(3) [3, 5.2.20, p. 192]. $\langle P, \chi(\varepsilon_i)\rangle = \binom{2n-i-1}{n}\chi(\varepsilon_i)(1)$.

Combining Lemma 13 with the elementary identity

$$\sum_{i=0}^{n-1} (-1)^i \binom{2n-1}{i} = (-1)^{n+1} \binom{2n-2}{n-1}$$

gives the following computation for an n-cycle μ

$$\begin{aligned}
\sum_{i=1}^{t} \frac{\chi_i(1)^2\,\chi_i(\mu)}{\langle P, \chi_i\rangle} &= \sum_{i=0}^{n-1} \frac{[\chi(\varepsilon_i)(1)]^2\,\chi(\varepsilon_i)(\mu)}{\langle P, \chi(\varepsilon_i)\rangle} \\
&= \sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i}\binom{2n-i-1}{n}^{-1} = \binom{2n-1}{n}^{-1} \sum_{i=0}^{n-1} (-1)^i \binom{2n-1}{i} \\
&= \binom{2n-1}{n}^{-1} (-1)^{n+1} \binom{2n-2}{n-1} = (-1)^{n+1} \frac{n}{2n-1}.
\end{aligned}$$

In conjunction with Theorem 2 we therefore have

THEOREM 14. *The coefficient of an n-cycle in* $\Phi_0(1) \in KS_n$ *is* $(-1)^{n+1}(C_n/(n-1)!\,n!\,(2n-1))$, *where* C_n *is the nonzero constant of Theorem* 4. *In particular, it is nonzero.* ∎

7. Proof of Regev's Conjecture

Regev conjectured [7, p. 1429] that the matrix polynomial

$$F(X, Y) = \sum_{\pi, \rho \in S_{n^2}} (\operatorname{sign} \pi\rho)\, X_{\pi(1)} Y_{\rho(1)} X_{\pi(2)} X_{\pi(3)} X_{\pi(4)} Y_{\rho(2)} Y_{\rho(3)} Y_{\rho(4)} \cdots X_{\pi(n^2-2n+2)} \cdots X_{\pi(n^2)} Y_{\rho(n^2-2n+2)} \cdots Y_{\rho(n^2)}$$

is not zero. As Regev observed, $F(X, Y)$ is a central polynomial, so $F(X, Y) = (1/n)\, T(F(X, Y))$.

LEMMA 15. *Let* $\sigma \in S_n$, $T_\sigma(X_1, ..., X_n) \in C(n)$. *Then*

$$\sum_{\pi \in S_{n^2}} (\operatorname{sign} \pi)\, T_\sigma(X_{\pi(1)}, X_{\pi(2)} X_{\pi(3)} X_{\pi(4)}, ..., X_{\pi(n^2-2n+2)} \cdots X_{\pi(n^2)}) = \begin{cases} C_n\, \Delta(X) & (\sigma = 1) \\ 0 & (\sigma \neq 1). \end{cases}$$

Proof. If $\sigma=1$, the left-hand summation is $A_{\lambda_0}(X_1,\ldots,X_{n^2})$. If $\sigma\neq 1$, it is $A_{\lambda(\sigma)}(X_1,\ldots,X_{n^2})$, where $\lambda(\sigma)$ is a partition of n^2 not equal to λ_0. The conclusion then follows from Theorem 4. ∎

Let $\Phi_0(1)=\sum a_\sigma\sigma$, and let $\mu=(1\cdots n)$. Then $\Phi_0(\mu)=\sum a_\sigma\,\mu\sigma$ and

$$\Phi(T_\mu)(X_1,\ldots,X_n)=\sum a_\sigma T_{\mu\sigma}(X_1,\ldots,X_n),$$

since Φ_0 and Φ are left S_n-homomorphisms. Using the definition of Φ in Lemma 6,

$$\sum_{\rho\in S_{n^2}}(\operatorname{sign}\rho)\,T(X_1Y_{\rho(1)}X_2Y_{\rho(2)}Y_{\rho(3)}Y_{\rho(4)}\cdots X_nY_{\rho(n^2-2n+2)}\cdots Y_{\rho(n^2)})$$
$$=\Phi(T_\mu)(X_1,\ldots,X_n)\,\Delta(Y)=\sum_{\sigma\in S_n}a_\sigma T_{\mu\sigma}(X_1,\ldots,X_n)\,\Delta(Y). \qquad (*)$$

Make the substitution $X_1\mapsto X_1$, $X_2\mapsto X_2X_3X_4,\ldots,X_n\mapsto X_{n^2-2n+2}\cdots X_{n^2}$ in $(*)$, and take the alternating sum over all permutations of $X_1,\ldots,X_{n^2}$. The left hand side then becomes the trace of Regev's polynomial, while the right hand side can be evaluated using Lemma 15. Thus

$$T(F(X,Y))=a_{\mu^{-1}}C_n\,\Delta(X)\,\Delta(Y).$$

Taking the value of $a_{\mu^{-1}}$ given by Theorem 14 and recalling that $F(X,Y)=(1/n)\,T(F(X,Y))$ finally gives

THEOREM 16. *Let* $F(X,Y)=F(X_1,\ldots,X_{n^2},Y_1,\ldots,Y_{n^2})$ *be Regev's polynomial. Then*

$$F(X,Y)=\frac{(-1)^{n+1}(C_n)^2}{(n!)^2(2n-1)}\,\Delta(X)\,\Delta(Y)I,$$

where $\Delta(X)$ *is the discriminant of* $X_1,\ldots,X_{n^2}$, $\Delta(Y)$ *is the discriminant of* $Y_1,\ldots,Y_{n^2}$, C_n *is the nonzero constant of Theorem* 4, *and* I *is the* $n\times n$ *identity matrix. In particular,* $F(X,Y)\neq 0$. ∎

8. EVALUATING THE CAPELLI POLYNOMIAL

The construction of the map Φ: $C(n)\to C(n)$ in Lemma 6 can be generalized to associate a map Φ^λ: $C(m)\to C(m)$ with any unordered partition λ of n^2 into m parts. The maps so obtained are left S_m-module homomorphisms, but in general they are neither right S_m-module homomorphisms nor isomorphisms. There is also a corresponding map

Φ_0^λ: $\overline{KS_m} \to \overline{KS_m}$ which is completely determined by $\Phi_0^\lambda(1)$, and as in Theorem 11:

(1) Φ^λ and Φ_0^λ are right S_m-module homomorphisms if and only if $\Phi_0^\lambda(1)$ is central in $\overline{KS_m}$.

(2) Φ^λ and Φ_0^λ are isomorphisms if and only if $\Phi_0^\lambda(1)$ is a unit in $\overline{KS_m}$.

The case $m=n^2$, $\gamma=(1,\dots,1)=(1^{n^2})$ is an example where Φ^γ: $C(n^2)\to C(n^2)$ is not a right S_{n^2}-module homomorphism. We will compute $\Phi_0^\gamma(1)$ in order to give an expression for the trace of the Capelli polynomial

$$C_{n^2}(X_1,\dots,X_{n^2},Y_1,\dots,Y_{n^2})=\sum_{\pi\in S_{n^2}}(\operatorname{sign}\pi)\,X_1Y_{\pi(1)}X_2Y_{\pi(2)}\cdots X_{n^2}Y_{\pi(n^2)}$$

as a product $g(X_1,\dots,X_{n^2})\,\Delta(Y)$, where $g\in C(n^2)$. Unfortunately, the expression obtained for g does not appear to be very useful.

To begin, the arguments of Section 4 can be repeated to prove

LEMMA 17. *Let $f(X_1,\dots,X_{n^2})\in C(n^2)$, and let $Y_1,\dots,Y_{n^2}$ be generic $n\times n$ matrices, with discriminant $\Delta(Y)$. Then there is a unique $\hat f(X_1,\dots,X_{n^2})\in C(n^2)$ such that*

$$\hat f(X_1,\dots,X_{n^2})\,\Delta(Y)=\sum_{\pi\in S_{n^2}}(\operatorname{sign}\pi)\,f(X_1Y_{\pi(1)},\dots,X_{n^2}Y_{\pi(n^2)}).$$

Moreover, the map Φ^γ: $C(n^2)\to C(n^2)$ defined by $\Phi^\gamma(f)=\hat f$ is a homomorphism of left S_{n^2}-modules. ∎

As in Section 4, we let Φ_0^γ denote the corresponding map $\overline{KS_{n^2}}\to\overline{KS_{n^2}}$.

LEMMA 18. $\Phi^\gamma(T_1)(X_1,\dots,X_{n^2})=(-1)^{\binom{n}{2}}\,\Delta(X)$.

Proof. By definition, $\Delta(X)$ is the determinant of the $n^2\times n^2$ matrix U whose rth row is

$$(x_{11}(r),x_{12}(r),\dots,x_{1n}(r),x_{21}(r),\dots,x_{nn}(r)).$$

It is easy to see that $(-1)^{\binom{n}{2}}\,\Delta(Y)$ is the determinant of the $n^2\times n^2$ matrix V whose rth column is the transpose of

$$(y_{11}(r),y_{21}(r),\dots,y_{n1}(r),y_{12}(r),\dots,y_{nn}(r)).$$

Since the (ij)th entry of UV is $T(X_iY_j)$,

$$\begin{aligned}
&\Phi^\gamma(T_1)(X_1,\dots,X_{n^2})\,\Delta(Y)\\
&\quad=\sum_{\pi\in S_{n^2}}(\operatorname{sign}\pi)\,T(X_1Y_{\pi(1)})\,T(X_2Y_{\pi(2)})\cdots T(X_{n^2}Y_{\pi(n^2)})\\
&\quad=\det(UV)=(-1)^{\binom{n}{2}}\,\Delta(X)\,\Delta(Y).
\end{aligned}$$

Thus $\Phi^\gamma(T_1)(X_1,\dots,X_{n^2})=(-1)^{\binom{n}{2}}\,\Delta(X)$. ∎

Set $\eta=(1)(234)(56789)\cdots((n^2-2n+2)\cdots n^2)$. Theorem 4 asserts that

$$\sum_{\pi\in S_{n^2}}(\operatorname{sign}\pi)\,T_{\pi\eta\pi^{-1}}(X_1,\dots,X_{n^2})=C_n\,\Delta(X)$$

or

$$\Phi_0^\gamma(1)=(-1)^{\binom{n}{2}}\,C_n^{-1}\sum_{\pi\in S_{n^2}}(\operatorname{sign}\pi)\,\pi\eta\pi^{-1}. \tag{1}$$

Note that (1) implies that

$$\Phi_0^\gamma(\alpha\beta)=(\operatorname{sign}\beta)\,\Phi_0^\gamma(\alpha)\,\beta. \tag{2}$$

Hence Φ_0^γ and Φ^γ are not right S_{n^2}-module homomorphisms.

Set $\mu=(1\cdots n^2)$. Then the trace of the Capelli polynomial is

$$\begin{aligned}
T(C_{n^2}(X,Y))&=\sum_{\pi\in S_{n^2}}(\operatorname{sign}\pi)\,T(X_1Y_{\pi(1)}X_2Y_{\pi(2)}\cdots X_{n^2}Y_{\pi(n^2)})\\
&=\Phi^\gamma(T_\mu)(X_1,\dots,X_{n^2})\,\Delta(Y).
\end{aligned}\tag{3}$$

Using (1) and (3) and the fact that Φ^γ is left S_{n^2}-module homomorphism yields

THEOREM 19. *Let $C_{n^2}(X,Y)=C_{n^2}(X_1,\dots,X_{n^2},\ Y_1,\dots,Y_{n^2})$ be the Capelli polynomial. Then*

$$T(C_{n^2}(X,Y))=(-1)^{\binom{n}{2}}\,C_n^{-1}\sum_{\pi\in S_{n^2}}(\operatorname{sign}\pi)\,T_{\mu\pi\eta\pi^{-1}}(X_1,\dots,X_{n^2})\,\Delta(Y),$$

where $\mu=(1\cdots n^2)$, $\eta=(1)(234)\cdots((n^2-2n+2)\cdots n^2)$, and C_n is the non-zero constant of Theorem 4. ∎

9. THE CONSTANT C_n

By Theorem 4(2),

$$\begin{aligned}
&\sum_{\pi\in S_{n^2}}(\operatorname{sign}\pi)\,T(X_{\pi(1)})\,T(X_{\pi(2)}X_{\pi(3)}X_{\pi(4)})\cdots T(X_{\pi(n^2-2n+2)}\cdots X_{\pi(n^2)})\\
&\qquad=C_n\,\Delta(X,\dots,X_{n^2}),
\end{aligned}$$

where C_n is a nonzero constant which subsequently made repeated appearances. We will now determine it up to a sign.

We adopt a notation similar to that introduced in Section 4, but with $2n^2$ generic $n \times n$ matrices $X_1, ..., X_{n^2}$, $Y_1, ..., Y_{n^2}$. Set

$$S_{n^2} = \text{permutations of } \{1, ..., n^2\},$$
$$S^*_{n^2} = \text{permutations of } \{1^*, ..., (n^2)^*\},$$
$$S_{2n^2} = \text{permutations of } \{1, ..., n^2, 1^*, ..., (n^2)^*\}.$$

The corresponding spaces of multilinear invariants are $C(n^2)$, $C^*(n^2)$, $C(2n^2)$, and the corresponding isomorphisms are $\theta_{n^2}\colon \overline{KS_{n^2}} \to C(n^2)$, $\theta^*_{n^2}\colon \overline{KS^*_{n^2}} \to C^*(n^2)$, $\theta_{2n^2}\colon \overline{KS_{2n^2}} \to C(2n^2)$. Set

$$\eta = (1)(234) \cdots ((n^2 - 2n + 2) \cdots n^2),$$
$$\tau = (11^*)(22^*) \cdots (n^2(n^2)^*),$$
$$\eta^* = \tau\eta\tau^{-1} = (1^*)(2^*3^*4^*) \cdots ((n^2 - 2n + 2)^* \cdots (n^2)^*).$$

By Theorem 4(2),

$$\begin{aligned} C_n \Delta(X) &= A_{\lambda_0}(X_1, ..., X_{n^2}) = \sum_{\pi \in S_{n^2}} (\operatorname{sign} \pi)\, T_{\pi\eta\pi^{-1}}(X_1, ..., X_{n^2}), \\ C_n \Delta(Y) &= A_{\lambda_0}(Y_1, ..., Y_{n^2}) = \sum_{\pi^* \in S^*_{n^2}} (\operatorname{sign} \pi^*)\, T_{\pi^*\eta^*(\pi^*)^{-1}}(Y_1, ..., Y_{n^2}). \end{aligned} \tag{1}$$

By Lemma 18 and its proof (reversing the roles of the X's and Y's),

$$\begin{aligned} (-1)^{\binom{n}{2}} \Delta(X)\, \Delta(Y) &= \sum_{\pi \in S_{n^2}} (\operatorname{sign} \pi)\, T(X_{\pi(1)} Y_1) \cdots T(X_{\pi(n^2)} Y_{n^2}) \\ &= \sum_{\pi \in S_{n^2}} (\operatorname{sign} \pi)\, T_{\pi\tau\pi^{-1}}(X_1, ..., X_{n^2}, Y_1, ..., Y_{n^2}). \end{aligned} \tag{2}$$

Using (1) and (2) and the θ-isomorphisms gives the following equation in $\overline{KS_{2n^2}}$:

$$\begin{aligned} (-1)^{\binom{n}{2}} C_n^2 \sum_{\pi \in S_{n^2}} (\operatorname{sign} \pi)\, \pi\tau\pi^{-1} &= C_n^2 (\theta_{2n^2})^{-1} [\Delta(X)\, \Delta(Y)] \\ &= C_n^2 [(\theta_{n^2})^{-1}\, \Delta(X)] [(\theta^*_{n^2})^{-1}\, \Delta(Y)] \\ &= \sum_{\pi \in S_{n^2}} (\operatorname{sign} \pi)\, \pi\eta\pi^{-1} \sum_{\pi \in S^*_{n^2}} (\operatorname{sign} \pi^*)\, \pi^*\eta^*(\pi^*)^{-1} \\ &= \left[\sum_{\pi \in S_{n^2}} (\operatorname{sign} \pi)\, \pi\eta\pi^{-1} \right] \tau \left[\sum_{\pi \in S_{n^2}} (\operatorname{sign} \pi)\, \pi\eta\pi^{-1} \right] \tau^{-1}. \end{aligned}$$

Multiply on the right by τ $(\tau=\tau^{-1})$ to get

$$(-1)^{\binom{n}{2}} C_n^2 \sum_{\pi \in S_{n^2}} (\operatorname{sign} \pi)\, \pi\tau\pi^{-1}\tau^{-1}$$
$$=\left[\sum_{\pi \in S_{n^2}} (\operatorname{sign} \pi)\, \pi\eta\pi^{-1}\right]\tau\left[\sum_{\pi \in S_{n^2}} (\operatorname{sign} \pi)\, \pi\eta\pi^{-1})\pi\eta\pi^{-1}\right], \quad (3)$$

an equation in $\overline{KS_{2n^2}}$. We will determine C_n^2 by comparing the values of a character $P_{2n^2}^n\colon \overline{KS_{2n^2}} \to K$ which assumes computable nonzero values on both sides of (3).

Following [3] with some changes in notation, let P_r^m denote the character of the representation of KS_r on $(V_m)^{\otimes r}$ by place permutation, where V_m is a K-vector space of dimension m. Note that if $\sigma \in S_r$, $P_r^m(\sigma)=m^{z(\sigma)}$, where $z(\sigma)$ is the number of disjoint cycles in σ [3, 4.3.12, p. 150]. Since $\sigma \in S_{n^2}$ has a different number of disjoint cycles if regarded as an element of S_{2n^2} and we count both, we adopt the notation

$$z(\sigma)=\text{number of disjoint cycles in } \sigma \in S_{2n^2},$$
$$z_0(\sigma)=\text{number of disjoint cycles in } \sigma \in S_{n^2} \text{ or } S_{n^2}^*.$$

If λ is a partition of r, let $\chi(\lambda)$ denote the character of S_r corresponding to λ.

LEMMA 20. *Let $\lambda=(\lambda_1,\dots,\lambda_k)$ be a partition of r of length k. Then*

(1) [3, 5.2.20, p. 192]. *The multiplicity of $\chi(\lambda)$ in P_r^n is nonzero if and only if $k \leqslant n$.*

(2) *Thus $J(n,r)$ is the kernel of the action of KS_n on $V_n^{\otimes r}$ by place permutation, and P_r^n induces a well-defined character $P_r^n\colon \overline{KS_r} \to K$.* ▌

Recall that $\tau=(11^*)(22^*)\cdots(n^2(n^2)^*)$.

LEMMA 21. (1) *If $\pi \in S_{n^2}$, then $z(\pi\tau\pi^{-1}\tau^{-1})=2z_0(\pi)$.*

(2) *If $\pi \in S_{n^2}$, then $P_{2n^2}^n(\pi\tau\pi^{-1}\tau^{-1})=P_{n^2}^{n^2}(\pi)$.*

(3) *If $\alpha, \beta \in S_{n^2}$, then $z(\alpha\tau\beta)=z_0(\alpha\beta)$.*

(4) *If $\alpha, \beta \in KS_{n^2}$, then $P_{2n^2}^n(\alpha\tau\beta)=P_{n^2}^n(\alpha\beta)$.*

Proof. (1) If $\pi \in S_{n^2}$, $\pi^*=\tau\pi\tau^{-1} \in S_{n^2}^*$, and π, π^* and $(\pi^*)^{-1}$ have the same cycle structure. The disjoint cycle structure of $\pi\tau\pi^{-1}\tau^{-1}$ is obtained by juxtaposing those those of π and $(\pi^*)^{-1}$. Hence $z(\pi\tau\pi^{-1}\tau^{-1})=z_0(\pi)+z_0((\pi^*)^{-1})=2z_0(\pi)$.

(2) By (1), $P_{2n^2}^n(\pi\tau\pi^{-1}\tau^{-1})=n^{z(\pi\tau\pi^{-1}\tau^{-1})}=n^{2z_0(\pi)}=(n^2)^{z_0(\pi)}=P_{n^2}^{n^2}(\pi)$.

(3) Note that $\alpha\tau\beta = (\alpha\beta)(\beta^{-1}\tau\beta)$, and

$$\beta^{-1}\tau\beta = (1\beta(1)^*)(2\beta(2)^*)\cdots(n^2, \beta(n^2)^*).$$

The disjoint cycle structure of $\alpha\tau\beta$ in S_{2n^2} is therefore obtained from the disjoint cycle structure of $\alpha\beta$ in S_{n^2} by replacing each $i = 1,..., n^2$ by the string $[i\beta(i)^*]$. Hence $z(\alpha\tau\beta) = z_0(\alpha\beta)$.

(4) For group elements $\alpha, \beta \in S_{n^2}$, $P^n_{2n^2}(\alpha\tau\beta) = n^{z(\alpha\tau\beta)} = n^{z_0(\alpha\beta)} = P^n_{n^2}(\alpha\beta)$, For $\alpha, \beta \in KS_{n^2}$, the conclusion follows by K-linearity. ∎

By [3, 5.2.20, p. 192], the sign character $\chi(1^{n^2})$ of S_{n^2} has multiplicity one as a component of $P^{n^2}_{n^2}$, or

$$1 = \langle \chi(1^{n^2}), P^{n^2}_{n^2} \rangle = \frac{1}{(n^2)!} \sum_{\pi \in S_{n^2}} (\operatorname{sign} \pi)\, P^{n^2}_{n^2}(\pi).$$

Hence by Lemma 21(2),

$$P^n_{2n^2}\left[\sum_{\pi \in S_{n^2}} (\operatorname{sign} \pi)\, \pi\tau\pi^{-1}\tau^{-1}\right] = P^{n^2}_{n^2}\left[\sum_{\pi \in S_{n^2}} (\operatorname{sign} \pi)\, \pi\right]$$

$$= \sum_{\pi \in S_{n^2}} (\operatorname{sign} \pi)\, P^{n^2}_{n^2}(\pi) = (n^2)!,$$

and the value of $P^n_{2n^2}$ applied to the left-hand side of Eq. (3) is

$$\begin{aligned} P^n_{2n^2}(\mathrm{LHS}(3)) &= P^n_{2n^2}\left[(-1)^{\binom{n}{2}} C_n^2 \sum_{\pi \in S_{n^2}} (\operatorname{sign} \pi)\, \pi\tau\pi^{-1}\tau^{-1}\right] \\ &= (-1)^{\binom{n}{2}} C_n^2 (n^2)!. \end{aligned} \tag{4}$$

The evaluation of $P^n_{2n^2}$ on the right-hand side of (3) is more elaborate and requires working with KA_{n^2}, the group ring of the alternating group. The starting point is that by Lemma 21(4),

$$\begin{aligned} P^n_{2n^2}(\mathrm{RHS}(3)) &= P^n_{2n^2}\left[\sum_{\pi \in S_{n^2}} (\operatorname{sign} \pi)\, \pi\eta\pi^{-1}\right] \tau \left[\sum_{\pi \in S_{n^2}} (\operatorname{sign} \pi)\, \pi\eta\pi^{-1}\right] \\ &= P^n_{n^2}\left[\sum_{\pi \in S_{n^2}} (\operatorname{sign} \pi)\, \pi\eta\pi^{-1}\right]^2, \end{aligned} \tag{5}$$

where $\sum_{\pi \in S_{n^2}} (\operatorname{sign} \pi)\, \pi\eta\pi^{-1}$ not only lies in KA_{n^2}, but is central there.

We assume henceforth that K is algebraically closed. This of course does not affect the value of C_n.

LEMMA 22. *Suppose G is a finite group, U is a central element of KG, and χ is an irreducible character of KG. Then $\chi(U^2)=[\chi(U)]^2/\chi(1)$.*

Proof. Let $\rho\colon KG \to M_{\chi(1)}(K)$ be a representation whose character is χ, and let I be the $\chi(1)\times\chi(1)$ identity matrix. Since U is central, $\rho(U)=(\chi(U)/\chi(1))I$, $\rho(U^2)=[\chi(U)/\chi(1)]^2 I$, and $\chi(U^2)=[\chi(U)]^2/\chi(1)$. ∎

Set $U=\sum_{\pi\in S_{n^2}}(\operatorname{sign}\pi)\,\pi\eta\pi^{-1}$. In order to evaluate $P^n_{n^2}(U^2)$ using Lemma 22, we have to do two things

(A) Find $\chi(U)$ for each irreducible character χ of A_{n^2}.

(B) Find the decomposition of $P^n_{n^2}|_{n^2}$ into irreducible characters of A_{n^2}.

Consider $\eta=(1)(234)\cdots((n^2-2n+2)\cdots n^2)$, which has cycle type $(2n-1, 2n-3,\ldots, 5, 3, 1)$. Since its cycles have distinct odd lengths, $\eta_1=\eta$ and $\eta_2=(12)\,\eta(12)^{-1}$ are not conjugate in A_{n^2} if $n>1$ [3, 1.2.10, p. 12].

There is a unique *self-associated* (or *self-conjugate*) partition (see [3, p. 22] for the definition) whose *main hooks* (see [3, p. 67] for the definition) are $2n-1, 2n-3,\ldots, 5, 3, 1$, namely $\delta=(n,\ldots, n)=(n^n)$.

THEOREM 23 [3, pp. 66–67]. *Let $\chi(\lambda)$ be the irreducible character of S_{n^2} $(n>1)$ associated to the partition λ of n^2.*

(1) *If λ is not self-associated, then $\chi(\lambda)\,|\,A_{n^2}$ is irreducible, and*

$$[\chi(\lambda)\,|\,A_{n^2}](\eta_1)=\chi(\lambda)(\eta)=[\chi(\lambda)\,|\,A_{n^2}](\eta_2).$$

(2) *If λ is self-associated, but $\lambda\neq(n^n)$, then $\chi(\lambda)\,|\,A_{n^2}=\chi^+(\lambda)+\chi^-(\lambda)$, the sum of two distinct irreducible characters, and*

$$\chi^{\pm}(\lambda)(\eta_1)=\tfrac{1}{2}\chi(\lambda)(\eta)=\chi^{\pm}(\lambda)(\eta_2).$$

(3) *If $\delta=(n^n)$, then $\chi(\delta)\,|\,A_{n^2}=\chi^+(\delta)+\chi^-(\delta)$, the sum of two distinct irreducible characters, and*

$$\chi(\delta)(\eta)=(-1)^{\binom{n}{2}}$$

$$\chi^+(\delta)(\eta_1)=\chi^-(\delta)(\eta_2)=\tfrac{1}{2}\Big[(-1)^{\binom{n}{2}}+\sqrt{(-1)^{\binom{n}{2}}1\cdot 3\cdots(2n-1)}\Big],$$

$$\chi^-(\delta)(\eta_1)=\chi^+(\delta)(\eta_2)=\tfrac{1}{2}\Big[(-1)^{\binom{n}{2}}-\sqrt{(-1)^{\binom{n}{2}}1\cdot 3\cdots(2n-1)}\Big]. \quad ∎$$

By [3, 5.2.20, p. 192], the multiplicity of the S_{n^2}-character $\chi(\delta)$ in $P^n_{n^2}$ is one. Hence by Theorem 23,

$$P^n_{n^2}\,|\,A_{n^2}=\chi^+(\delta)+\chi^-(\delta)+\chi, \tag{6}$$

where χ is a linear combination of irreducible characters of A_{n^2} which agree on η_1 and η_2 and hence vanish on

$$U = \sum_{\pi \in S_{n^2}} (\operatorname{sign} \pi)\, \pi \eta \pi^{-1} = \sum_{\pi \in A_{n^2}} \pi \eta_1 \pi^{-1} - \sum_{\pi \in A_{n^2}} \pi \eta_2 \pi^{-1}.$$

By Theorem 23(3),

$$\begin{aligned} \chi^{\pm}(\delta)(U) &= \pm |A_{n^2}| \sqrt{(-1)^{\binom{n}{2}} 1 \cdot 3 \cdot 5 \cdots (2n-1)}, \\ [\chi^{\pm}(\delta)(U)]^2 &= \tfrac{1}{4}[(n^2)!]^2 (-1)^{\binom{n}{2}} 1 \cdot 3 \cdot 5 \cdots (2n-1). \end{aligned} \tag{7}$$

By the hook formula [3, 2.3.21, p. 56],

$$\chi^{\pm}(\delta)(1) = \tfrac{1}{2}\chi(\delta)(1) = \frac{(n^2)!}{2[1 \cdot 2^2 \cdot 3^3 \cdots n^n (n+1)^{n-1} \cdots (2n-2)^2 (2n-1)]}. \tag{8}$$

Finally, by (5), (6), (7), (8), and Lemma 22,

$$\begin{aligned} P^{n}_{2n^2}(\mathrm{RHS}(3)) &= P^{n}_{n^2}(U^2) = \chi^{+}(\delta)(U^2) + \chi^{-}(\delta)(U^2) \\ &= \frac{[\chi^{+}(\delta)(U)]^2}{\chi^{+}(\delta)(1)} + \frac{[\chi^{-}(\delta)(U)]^2}{\chi^{-}(\delta)(1)} \\ &= (-1)^{\binom{n}{2}} (n^2)!\, 1 \cdot 2^2 \cdots n^n \cdots (2n-2)^2 (2n-1)\, 1 \cdot 3 \cdots (2n-1) \\ &= (-1)^{\binom{n}{2}} (n^2)! \left[\frac{1!\,3!\,5! \cdots (2n-1)!}{1!\,2! \cdots (n-1)!} \right]^2. \end{aligned} \tag{9}$$

Comparing (4) and (9) gives

THEOREM 24. *The constant C_n of Theorem 4, which is defined implicitly by*

$$\sum_{\pi \in S_{n^2}} (\operatorname{sign} n)\, T(X_{\pi(1)})\, T(X_{\pi(2)} X_{\pi(3)} X_{\pi(4)}) \cdots T(X_{\pi(n^2-2n+2)} \cdots X_{\pi(n^2)})$$

$$= C_n \Delta(X_1, \ldots, X_{n^2}),$$

is equal to $\pm 1!\,3!\,5! \cdots (2n-1)!/1!\,2! \cdots (n-1)!$. ∎

$C_1 = +1$ and $C_2 = -6$, but otherwise I do not know the sign of C_n. Almost certainly it is periodic of period two or four.

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