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The primitives and antipode in the Hopf algebra of symmetric functions in noncommuting variables

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ABSTRACT

We identify a collection of primitive elements generating the Hopf algebra *NCSym* of symmetric functions in noncommuting variables and give a combinatorial formula for the antipode.

1. Introduction

The Hopf algebra *NCSym* of symmetric functions in noncommuting variables was introduced in [5], following the work of Wolf, Rosas, and Sagan [16,14]. We refer the reader to these sources for details on its realization as formal sums of monomials invariant under the symmetric groups $\mathfrak{S}_n$ (for all $n > 0$). For our goals, it suffices to describe *NCSym* abstractly in terms of generators and relations, which we do in Section 3.

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The Hopf algebra *NCSym* is cocommutative (by definition) and freely generated as an algebra (essentially the main result of Wolf). Following [5], we choose the *atomic set partitions* $\dot{\boldsymbol{\Pi}}$ as a free generating set (see Section 3). The Cartier–Milnor–Moore theorem then guarantees that *NCSym* is isomorphic to $\mathfrak{U}(\mathfrak{L}(\dot{\boldsymbol{\Pi}}))$, the universal enveloping algebra of the free Lie algebra generated by $\dot{\boldsymbol{\Pi}}$. This note grew out of an attempt to realize this isomorphism explicitly.

In Section 2, we record some useful combinatorial machinery. Section 3 contains a precise definition of the Hopf algebra *NCSym* as well as the statements and proofs of our main results. The classical (commutative) ring of symmetric functions has an alternate realization within the character theory of $\mathfrak{S}_n$ $(n > 0)$. A satisfactory generalization of this realization has been difficult to find for *NCSym* (see [4] for one attempt). In Section 4, we comment briefly on a new connection to supercharacter theory that merits further study. We conclude with comments on key structural features of our proofs that will be further developed in [10].

2. Combinatorial preliminaries

We record some useful shorthand for manipulating set partitions and set compositions. Throughout, we let $\mathbb{N}$ and $\mathbb{P}$ denote the set of nonnegative integers and positive integers, respectively. Also, given $n \in \mathbb{P}$, we let $[n]$ denote the set $\{1, 2, \ldots, n\}$.

2.1. Set partitions

Fix $X \subseteq \mathbb{P}$ and let $\mathbf{A} = \{A_1, \ldots, A_r\}$ be a set of subsets of X. We say that $\mathbf{A}$ is a *set partition* of X, written $\mathbf{A} \vdash X$, if and only if $A_1 \cup \cdots \cup A_r = X$, $A_i \neq \emptyset$ $(\forall i)$, and $A_i \cap A_j = \emptyset$ $(\forall i \neq j)$. We order the parts in increasing order of their minimum elements. The *weight* $|\mathbf{A}|$ of $\mathbf{A}$ is the cardinality of X and the *length* $\ell(\mathbf{A})$ of $\mathbf{A}$ is its number of parts (r). In what follows, we lighten the heavy notation for set partitions, writing, e.g., the set partition $\{\{1,3\},\{2,8\},\{4\}\}$ as 13.28.4. We write $\boldsymbol{\Pi}(X)$ for the set partitions of X and $\boldsymbol{\Pi}(n)$ when $X = [n]$.

Given any $A \subseteq \mathbb{N}$ and $k \in \mathbb{N}$, we write A^{+k} for the set

$$A^{+k} := \{a + k \mid a \in A\}.$$

By extension, for any set partition $\mathbf{A} = \{A_1, A_2, \ldots, A_r\}$ we set $\mathbf{A}^{+k} := \{A_1^{+k}, A_2^{+k}, \ldots, A_r^{+k}\}$. The operator $(-)^{+k}$ has a complement $(-)^{\downarrow}$ called the *standardization* operator. It maps set partitions $\mathbf{A}$ of any cardinality n subset $X \subseteq \mathbb{P}$ to set partitions of $[n]$, by defining $\mathbf{A}^{\downarrow}$ as the pullback of $\mathbf{A}$ along the unique increasing bijection from $[n]$ to X. For example, $(18.4)^{\downarrow} = 13.2$ and $(18.4.67)^{\downarrow} = 15.2.34$. Given set partitions $\mathbf{B} \vdash [m]$ and $\mathbf{C} \vdash [n]$, we let $\mathbf{B}|\mathbf{C}$ denote the set partition $\mathbf{B} \cup \mathbf{C}^{+m}$ of $[m+n]$.

Definition 1. A set partition $\mathbf{A} = \{A_1, A_2, \ldots, A_r\}$ of $[n]$ is *atomic* ("connected" in [9]) if there does not exist a subset $\mathbf{B} \subseteq \mathbf{A}$ and an integer $m < n$ such that $\mathbf{B} \vdash [m]$. Conversely, $\mathbf{A}$ is not atomic if there are set partitions $\mathbf{B}$ of $[n']$ and $\mathbf{C}$ of $[n'']$ splitting $\mathbf{A}$ in two: $\mathbf{A} = \mathbf{B}|\mathbf{C}$.

For example, the partition 17.235.4.68 is atomic, while 12.346.57.8 is not. The maximal splitting of the latter would be 12|124.35|1. We denote the atomic set partitions by $\dot{\boldsymbol{\Pi}}$.

If $\mathbf{A}$ is a set partition with r parts, and $K \subseteq [r]$, we write A_K to denote the sub partition $\{A_k \mid k \in K\} \subseteq \mathbf{A}$. For example, if $\mathbf{A} = 17.235.4.68$, then $\mathbf{A}_{\{1,3,4\}} = 17.4.68$.

2.2. Set compositions

Fix $K \subseteq \mathbb{P}$ and let $\gamma = (\gamma_1, \ldots, \gamma_s)$ be a sequence of subsets of K. We say that γ is a *set composition* of K, written $\gamma \vDash K$, if and only if $\gamma_1 \cup \cdots \cup \gamma_s = K$, $\gamma_i \neq \emptyset$ $(\forall i)$, and $\gamma_i \cap \gamma_j = \emptyset$ $(\forall i \neq j)$. The *weight* $|\gamma|$ and *length* $\ell(\gamma)$ are defined as for set partitions. We use "**|**" in place of "." in our shorthand for set compositions, e.g., the set composition $(\{3,8\},\{1,2\},\{4\})$ is abbreviated as 38**|**12**|**4. We write $\boldsymbol{\Gamma}(K)$ for the set partitions of K and $\boldsymbol{\Gamma}(r)$ when $K = [r]$.

	15.28.346.7	13.28.456.7	24.39.58.7
13|2	14.235.67	12.345.67	12.34.56
2|34	12.345.6	12.345.6	12.35.4
1|234	12.38.456.7	12.38.456.7	12.37.46.5

Fig. 1. Set partitions $\gamma[\mathbf{A}]$ for several examples of γ and $\mathbf{A}$.

If γ is a set composition of X and $K \subseteq X$, we write $\gamma\rfloor_K$ for the induced set composition of K. For example, if $\gamma = 38|12|4$ and $K = \{3, 4, 8\}$, then $\gamma\rfloor_K = 38|4$. Similarly, $\gamma\rfloor_{\{1,3\}} = 3|1$. Note that $\gamma\rfloor_{\{1,3\}}$ is not the same as $\gamma_{\{1,3\}}$. Following the notation introduced for set partitions, we let $\gamma_{\{1,3\}}$ denote the subsequence $38|4$ of γ.

Given two set compositions $\gamma, \rho \vDash K$, we say that γ *refines* ρ, written $\gamma \succ \rho$, if each block of ρ is the union of a contiguous string of blocks of γ. For example, $2|4|3|17|9 \succ 234|179 \succ 123479$.

2.3. Set compositions as functions on set partitions

Let $\mathbf{A}$ be a set partition of X with r parts and suppose $\gamma = (\gamma_1, \ldots, \gamma_s)$ is a set composition of $K \subseteq [r]$. We define a new (standard) set partition $\gamma[\mathbf{A}]$ as follows:

$$\gamma[\mathbf{A}] := \mathbf{A}_{\gamma_1}{}^{\downarrow}|\mathbf{A}_{\gamma_2}{}^{\downarrow}|\cdots|\mathbf{A}_{\gamma_s}{}^{\downarrow}. \tag{1}$$

See Fig. 1 for several examples.

3. Structure of *NCSym*

Let $NCSym = \bigoplus_{n \geqslant 0} NCSym_n$ denote the graded $\mathbb{Q}$ vector space whose nth graded piece consists of formal (finite) sums of set partitions $\mathbf{A} \in \boldsymbol{\Pi}(n)$. We give *NCSym* the structure of graded Hopf algebra as follows. The algebra structure is given by

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{A}|\mathbf{B} \quad \text{and} \quad 1_{NCSym} = \emptyset, \tag{2}$$

where $\emptyset$ is the unique set partition of the empty set. The coalgebra structure is given by

$$\Delta(\mathbf{A}) = \sum_{K \dot{\cup} L = [r]} (\mathbf{A}_K)^{\downarrow} \otimes (\mathbf{A}_L)^{\downarrow} \quad \text{and} \quad \varepsilon(\mathbf{A}) = 0 \quad \text{for all } \mathbf{A} \neq \emptyset. \tag{3}$$

Here, $\mathbf{A}$ is a set partition with r parts, and $K \dot{\cup} L = [r]$ is understood as an ordered disjoint union, i.e., $K \cup L = [r]$, $K \cap L = \emptyset$, and $K \dot{\cup} L \neq L \dot{\cup} K$.

In terms of the symmetric function interpretation of *NCSym*, the formulas above correspond to working in the basis of *power sum symmetric functions*. See [5,14] for the definition of monomial symmetric functions $m_{\mathbf{A}}$. (These exhibit the same coproduct behavior as above but a more complicated product structure. The $m_{\mathbf{A}}$s have recently gained new relevance by virtue of their connection to supercharacters; see Section 4.1.) The product formula (2) is proven in [6], and the coproduct formula (3) is proven in [4]. A combinatorially meaningful formula for the antipode of *NCSym* (on any basis) has been missing until now (Theorem 6).

Evidently, *NCSym* is cocommutative and freely generated by $\dot{\boldsymbol{\Pi}}$. The Cartier–Milnor–Moore theorem guarantees algebraically independent primitive elements $p(\mathbf{A})$ of *NCSym* associated to each $\mathbf{A} \in \dot{\boldsymbol{\Pi}}$. We find them below (Theorem 2).

3.1. Primitive generators of NCSym

We aim to prove the following result.

Theorem 2. *Let $\boldsymbol{\Gamma}'(r)$ denote the set compositions γ of $[r]$ with $1 \in \gamma_1$. If $\mathbf{A}$ is a set partition with r parts, then*

$$p(\mathbf{A}) := \sum_{\gamma \in \boldsymbol{\Gamma}'(r)} (-1)^{\ell(\gamma)-1} \gamma[\mathbf{A}] \tag{4}$$

is a nonzero primitive if $\mathbf{A} \in \dot{\boldsymbol{\Pi}}$ and zero otherwise.

Remark. The choice in (4) to demand that the distinguished element 1 belong to the first block of γ was arbitrary. Different choices of distinguished elements (and blocks) give rise to different sets of primitives. Summing over these choices produces a projection operator onto the space of all primitives. This operator can be understood in the context of the work Patras and Reutenauer on Lie idempotents [11,12]. Fisher [8] uses similar idempotents to work out primitive formulas for several Hopf monoids in species. His formulas, which were discovered independently, also give rise to (4).

To prove Theorem 2, we need a lemma about *left quasi-shuffles.* Let $\mathbf{P} = \langle \mathrm{fin}(2^{\mathbb{P}}) \rangle$ denote the free monoid generated by the finite subsets of $\mathbb{P}$ and let $\mathbb{Q}\mathbf{P}$ denote the corresponding free algebra. We use $|$ to separate letters in the words $w \in \mathbf{P}$. Given a word $u = u_1 | \cdots | u_k$ on k letters and an index $i \leqslant k$, we write $u_{[i]}$ for the prefix $u_1 | \cdots | u_i$ and $u^{[i]}$ for the suffix $u_{i+1} | \cdots | u_k$.

We say that words $u = u_1 | \cdots | u_k$ and $v = v_1 | \cdots | v_l$ in $\mathbf{P}$ are *disjoint* if $(\bigcup_i u_i) \cap (\bigcup_j v_j) = \emptyset$. (We identify the set compositions $\boldsymbol{\Gamma}(K)$ with the words $w \in \mathbf{P}$ that further satisfy $w_i \cap w_j = \emptyset$ for $i \neq j$ and $\bigcup_j w_j = K$.)

The *quasi-shuffle* $u \sqcup\!\sqcup v$ of two disjoint words $u, v \in \mathbf{P}$ is defined recursively as follows:

- $u \sqcup\!\sqcup \emptyset = u$ and $\emptyset \sqcup\!\sqcup v = v$;
- if $u = a|u'$ and $v = b|v'$, then

$$u \sqcup\!\sqcup v = \{a|w\colon\ w \in u' \sqcup\!\sqcup v\} \cup \{ab|w\colon\ w \in u' \sqcup\!\sqcup v'\} \cup \{b|w\colon\ w \in u \sqcup\!\sqcup v'\}.$$

Here $\emptyset$ denotes the unique set composition of the empty set and "ab" denotes the union $a \cup b$, a single letter in $\mathbf{P}$. The quasi-shuffle governs the formula for the product of two monomial symmetric functions in noncommuting variables. We need a subset of these that we call the *left quasi-shuffles:*

$$u \mathbin{\tilde{\sqcup\!\sqcup}} v = \{a|w\colon\ w \in u' \sqcup\!\sqcup v\} \cup \{ab|w\colon\ w \in u' \sqcup\!\sqcup v'\}$$

for nonempty disjoint words $u = a|u'$ and $v = b|v'$ in $\mathbf{P}$. (Note that the recursive definition involves $\sqcup\!\sqcup$, not $\tilde{\sqcup\!\sqcup}$.)

Example 3. Consider the set compositions 1|3 and 24. We have

$$1|3 \mathbin{\tilde{\sqcup\!\sqcup}} 24 = \{1|3|24,\ 1|234,\ 1|24|3,\ 124|3\}.$$

Note that for each left quasi-shuffle γ above, $\gamma\rfloor_{\{1,3\}} = 1|3$ and $\gamma\rfloor_{\{2,4\}} = 24$.

Lemma 4. *Let $\boldsymbol{\Gamma}'(r)$ denote the set compositions γ of $[r]$ with $1 \in \gamma_1$. Given any partition $K \uplus L = [r]$ with $1 \in K \subsetneq [r]$, we have*

$$\sum_{\gamma \in \boldsymbol{\Gamma}'(r)} (-1)^{\ell(\gamma)} \gamma\rfloor_K \otimes \gamma\rfloor_L = 0,$$

as an element of $\mathbb{Q}\mathbf{P} \otimes \mathbb{Q}\mathbf{P}$.

Proof. Consider a term $u \otimes v$ in the sum. (The hypothesis $K \neq [r]$ guarantees that $v \neq \emptyset$.) The compositions γ that satisfy $\gamma\rfloor_K = u$ and $\gamma\rfloor_L = v$ are precisely the left quasi-shuffles $u \mathbin{\widetilde{\sqcup\!\sqcup}} v$. We now establish a bijection between those of even length and those of odd length. This will complete the proof.

A left quasi-shuffle $w \in u \mathbin{\widetilde{\sqcup\!\sqcup}} v$ falls into one of two types according to whether or not the first letter of v appears as a letter in w: after beginning with some (possibly empty) initial prefix of u, say up to the ith letter u_i, the rest of the word w looks like either $u_{i+1}|v_1|w'$ or $u_{i+1}v_1|w'$ for w' in $u^{[i+1]} \sqcup\!\sqcup v^{[1]}$. Here, again, $u_{i+1}v_1$ denotes $u_{i+1} \cup v_1$. The indicated pairing $w \mapsto \phi(w)$ $(u_{i+1}|v_1 \mapsto u_{i+1}v_1)$ decreases the number of letters by one. Thus w and $\phi(w)$ contribute opposite signs to the coefficient of $u \otimes v$. □

Proof of Theorem 2. Since Δ is cocommutative, we need only consider the terms $\gamma[\mathbf{A}]_K{}^{\downarrow} \otimes \gamma[\mathbf{A}]_L{}^{\downarrow}$ from each $\Delta(\gamma[\mathbf{A}])$ satisfying $1 \in K$. Notice that $\gamma[\mathbf{A}]_K = (\gamma\rfloor_{K'})[\mathbf{A}]$ for some $K' \subseteq [r]$ (with $1 \in K'$ if $1 \in \gamma_1$). Thus

$$\sum_{\gamma \in \boldsymbol{\Gamma}'(r)} (-1)^{\ell(\gamma)} \sum_{\substack{K \uplus L = [r] \\ 1 \in K}} \gamma[\mathbf{A}]_K{}^{\downarrow} \otimes \gamma[\mathbf{A}]_L{}^{\downarrow}$$

is the same sum as

$$\sum_{\substack{K \uplus L = [r] \\ 1 \in K}} \sum_{\gamma \in \boldsymbol{\Gamma}'(r)} (-1)^{\ell(\gamma)} (\gamma\rfloor_K)[\mathbf{A}]^{\downarrow} \otimes (\gamma\rfloor_L)[\mathbf{A}]^{\downarrow},$$

or even

$$\left(\sum_{\substack{K \uplus L = [r] \\ 1 \in K}} \sum_{\gamma \in \boldsymbol{\Gamma}'(r)} (-1)^{\ell(\gamma)} \gamma\rfloor_K \otimes \gamma\rfloor_L \right) [\mathbf{A} \otimes \mathbf{A}]. \tag{5}$$

Lemma 4 reduces the sum in expression (5) to $(0)[\mathbf{A} \otimes \mathbf{A}] = 0$ when $K \neq [r]$. When $K = [r]$, the expression is precisely $p(\mathbf{A}) \otimes 1$. Conclude that $p(\mathbf{A})$ is a primitive element when $\mathbf{A}$ is atomic.

To show that $p(\mathbf{A})$ is nonzero when $\mathbf{A}$ is atomic, we merely remark that $\gamma[\mathbf{A}]$ has at least as many atoms as γ has parts. It is only when $\mathbf{A} \in \dot{\boldsymbol{\Pi}}$ and $\gamma = [r]$ that a single atom is obtained: $\gamma[\mathbf{A}]$ contributes $\mathbf{A}$ to the sum $p(\mathbf{A})$ only when $\gamma = [r]$. Thus $p(\mathbf{A}) \neq 0$.

We now show that $p(\mathbf{A}) = 0$ when $\mathbf{A}$ is not atomic. Suppose $\mathbf{A} = \mathbf{A}'|\mathbf{A}''$ and put $\ell(\mathbf{A}) = r, \ell(\mathbf{A}') = r'$. (Note that $r' < r$ by assumption.) Divide the set compositions $\gamma \in \boldsymbol{\Gamma}'(r)$ into two types according to the following dichotomy. If γ_i is the first letter, from the left, satisfying $\gamma_i \cap \{r'+1, \ldots, r\} \neq \emptyset$, then either $\gamma_i \cap \{1, \ldots, r'\}$ is empty or it is not. The pairing $\gamma \mapsto \phi(\gamma)$ $(\gamma_{i-1}|\gamma_i \mapsto \gamma_{i-1}\gamma_i)$ decreases the number of letters by one. However, the difference is not visible at the level of functions on $\mathbf{A}$. That is, $\gamma[\mathbf{A}] = \phi(\gamma)[\mathbf{A}]$. This completes the proof. □

Corollary 5. *The set $\{p(\mathbf{A}) \mid \mathbf{A} \in \dot{\boldsymbol{\Pi}}\}$ comprises irredundant (algebraically independent) generators of the Lie algebra of primitive elements of NCSym.*

Proof. First we show algebraic independence of $\{p(\mathbf{A}) \mid \mathbf{A} \in \dot{\boldsymbol{\Pi}}\}$. Let $<$ be a total order on the set $\dot{\boldsymbol{\Pi}}$ satisfying $\mathbf{A} < \mathbf{A}'$ when $|\mathbf{A}| > |\mathbf{A}'|$ (e.g., $13.2 < 12$). Extend $<$ to a lexicographic ordering of $\boldsymbol{\Pi}$ (e.g., $13.2 < 13.2|12 < 12$). With this choice, the leading (minimum) term of $p(\mathbf{A})$ is $\mathbf{A}$. Indeed, as remarked in the proof above, the only term in $p(\mathbf{A})$ with one atom is $\mathbf{A}$. Conclude that any polynomial in the $p(\mathbf{A})$s ($\mathbf{A} \in \dot{\boldsymbol{\Pi}}$) has the same leading term as the corresponding polynomial in the $\mathbf{A}$s. Hence the $p(\mathbf{A})$s freely generate *NCSym*.

Turning to the Lie algebra of primitive elements in *NCSym*, we recall the construction of Hall polynomials. Given an ordered alphabet X and a word $w = x_1 \cdots x_t$ over X, we say that w is a *Lyndon word* if w is lexicographically smaller than all its proper suffixes $x_i \cdots x_t$ $(i > 1)$. The *standard factorization* of a Lyndon word w is the factorization $w = uv$ with v lexicographically smallest among proper suffixes of w. A classical result on the standard factorization has that both u and v are Lyndon. See [13, (5.1.1)]. We define the Hall polynomial $[[w]]$ by forming the Lie bracket $[u, v]$ at successively smaller Lyndon standard factorizations. For example, if $w = aabb$, then the Lyndon factorization of w is (a, abb); next, abb is further factored as (ab, b) and ab is factored as (a, b). The resulting Hall polynomial $[[w]]$ is the Lie bracket $[a, [[a, b], b]]$.

If w is Lyndon, then the leading (minimal) term of $[[w]]$ is w. A consequence is the classical result that the Hall polynomials $\{[[w]] \mid w \text{ is Lyndon}\}$ form a basis of the free Lie algebra generated by X. See [13, Thms. 4.9, 5.1]. Turning to *NCSym*, the first paragraph of the proof shows that we may replace the alphabet $\dot{\boldsymbol{\Pi}}$ with the alphabet $\{p(\mathbf{A}) \mid \mathbf{A} \in \dot{\boldsymbol{\Pi}}\}$. Conclude that the Hall polynomials $[[p(\mathbf{A}')p(\mathbf{A}'') \cdots p(\mathbf{A}^{(t)})]]$, with $\mathbf{A}'|\mathbf{A}''| \cdots |\mathbf{A}^{(t)}$ Lyndon in the atoms $\mathbf{A}^{(i)}$, form a basis of the Lie algebra of primitive elements in *NCSym*. □

3.2. The antipode of NCSym

We aim to prove the following result.

Theorem 6. *Suppose* $\mathbf{A}$ *is a set partition with* r *parts and let* $\boldsymbol{\Gamma}(r)$ *denote the set compositions of* $[r]$. *The antipode* S *of NCSym acts on* $\mathbf{A}$ *by*

$$S(\mathbf{A}) = \sum_{\gamma \in \boldsymbol{\Gamma}(r)} (-1)^{\ell(\gamma)} \gamma[\mathbf{A}]. \tag{6}$$

Remarks. 1. The above is a combinatorial description of the Takeuchi's formula for the antipode. We include a direct proof as it is short and illustrates nicely the idea of the transfer of structure discussed in Section 4.2 below.

2. This description of the antipode typically contains many cancellations. For example, it says that $S(12.3) = -(12.3) + (12.3) + (1.23)$. There may even be cancellations when $\mathbf{A}$ is atomic. We invite the reader to compute $S(14.2.3)$, which *a priori* has 13 terms, but in fact has only nine. On the other hand, (6) is irredundant for many atomic $\mathbf{A}$.

3. In case $\mathbf{A}$ is not atomic, we can do better. Let $\mathbf{A} = \mathbf{A}'|\mathbf{A}''| \cdots |\mathbf{A}^{(t)}$ be a splitting of $\mathbf{A}$ into atomic pieces and put $\ell(\mathbf{A}^{(i)}) = r_i$ with $\sum_i r_i = r$. Finally, let $\boldsymbol{\Gamma}(\overleftarrow{r})$ denote all refinements of the set composition

$$\overleftarrow{r} = \big(\{r{-}r_t{+}1, \ldots, r\}, \ldots, \{r_1{+}1, \ldots, r_1{+}r_2\}, \{1, \ldots, r_1\}\big).$$

Then

$$S(\mathbf{A}) = \sum_{\gamma \in \boldsymbol{\Gamma}(\overleftarrow{r})} (-1)^{\ell(\gamma)} \gamma[\mathbf{A}]. \tag{7}$$

This follows immediately from the fact that S is an algebra antimorphism and (6) holds on atomic set partitions. Using this formula, we may express $S(13.2.4)$ using three terms, $S(13.2.4) = (1.24.3) - (1.23.4) - (1.2.34)$. Using (6) would have required us to write down 13 terms, 10 of which would cancel.

Proof of Theorem 6. Since the antipode is guaranteed to exist in graded connected bialgebras, we need only check that (6) provides a left convolution inverse of id. Writing m for multiplication in *NCSym*, we have

$$\begin{aligned} m(S\otimes \mathrm{id})\Delta(\mathbf{A}) &= m(S\otimes \mathrm{id})\Bigl(\sum_{K\dot\cup L=[r]} \mathbf{A}_K{}^{\downarrow}\otimes \mathbf{A}_L{}^{\downarrow}\Bigr) \\ &= m(S\otimes \mathrm{id})\Bigl(\sum_{K\dot\cup L=[r]} K[\mathbf{A}]\otimes L[\mathbf{A}]\Bigr) \end{aligned}$$

(viewing K and L as set compositions with one part)

$$\begin{aligned} &= m\sum_{K\dot\cup L=[r]}\Bigl(\sum_{\gamma\in\boldsymbol{\Gamma}_{|K|}} (-1)^{\ell(\gamma)}\gamma\bigl[K[\mathbf{A}]\bigr]\Bigr)\otimes L[\mathbf{A}] \\ &= \Bigl(\sum_{K\dot\cup L=[r]}\sum_{\gamma\in\boldsymbol{\Gamma}_K} (-1)^{\ell(\gamma)}(\gamma|L)\Bigr)[\mathbf{A}]. \end{aligned} \tag{8}$$

We show that the sum in (8) is the zero function on **A**. We have

$$\sum_{K\dot\cup L=[r]}\sum_{\gamma\in\boldsymbol{\Gamma}_K}(-1)^{\ell(\gamma)}(\gamma|L) = \sum_{\gamma\in\boldsymbol{\Gamma}_{[r]}}(-1)^{\ell(\gamma)}\gamma + \sum_{\substack{K\dot\cup L=[r]\\ K\neq[r]}}\sum_{\gamma\in\boldsymbol{\Gamma}_K}(-1)^{\ell(\gamma)}(\gamma|L)$$

or

$$\sum_{\emptyset\subsetneq L\subseteq[r]}\ \sum_{\substack{\gamma\in\boldsymbol{\Gamma}_{[r]}\\ \gamma=\gamma'|L}}(-1)^{\ell(\gamma'|L)}\bigl(\gamma'|L\bigr) - \sum_{\substack{K\dot\cup L=[r]\\ K\neq[r]}}\sum_{\gamma\in\boldsymbol{\Gamma}_K}(-1)^{\ell(\gamma|L)}(\gamma|L) = 0,$$

as claimed. We conclude that $m(S\otimes\mathrm{id})\Delta(\mathbf{A}) = 0$ for $|\mathbf{A}| > 0$, which completes the proof. □

4. Summary remarks

4.1. Supercharacter theory

Let $\mathbb{F}_q$ be a finite field with q elements. The classification problem for irreducible representations of the upper triangular groups $U_n(\mathbb{F}_q)$ is known to be of wild type. After the work of André and Yan [3,17], it seems that a good first step at understanding the character theory of $U_n(\mathbb{F}_q)$ is to study its *supercharacters* and *superclasses*. Superclasses are formed by clumping together certain conjugacy classes of $U_n(\mathbb{F}_q)$. To each superclass is associated a supercharacter, which is the corresponding sum of characters. See [7] for an excellent exposition by Diaconis and Isaacs.

Following this approach to the problem, and mimicking the classical constructions in the representation ring of the symmetric groups, Thiem [15] gave the space of supercharacters $\mathcal{SC}_q$ a Hopf algebra structure, with product and coproduct coming from superinflation and restriction.[3] The result was something that bore a strong resemblance to *NCSym* for $q = 2$. One of the main topics of a recent AIM workshop was to explore this connection and see what could be said for q arbitrary. Quite a lot of progress was made in several directions. Relevant to the present discussion is that $\mathcal{SC}_2$ is indeed isomorphic to *NCSym* as Hopf algebras [1].

The isomorphism above is straightforward, simply taking superclass functions $\kappa_{\mathbf{A}}$ to monomial symmetric functions $m_{\mathbf{A}}$. (See [15] or [1] for notation.) In light of this, it would be very interesting to

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[3] More precisely, the coproduct is not explicitly defined there, nor is the compatibility between product and coproduct checked, but these were subsequently verified by Thiem (private communication). The details of his Hopf algebra construction were hammered out at a recent AIM workshop and appear in [1].

have analogs of (4) and (6) for the monomial basis. Some work in this direction was begun the last day of the workshop [2].

4.2. *Transfer of structure*

In Section 3, we used the idea of set compositions as functions on set partitions to formulate our results. As the proofs of these results indicate, this idea can be mined further. There is a graded connected cocommutative Hopf algebra structure on $\mathbb{Q}\mathbf{P}$ which is freely generated as an algebra by $\operatorname{fin}(2^{\mathbb{P}})$. The coproduct is defined on finite sets X and extended multiplicatively to $\mathbb{Q}\mathbf{P}$. Given X, it is simply the sum over decompositions $K \uplus L = X$, e.g.,

$$\Delta(134) = \emptyset \otimes 134 + 1 \otimes 34 + \cdots + 14 \otimes 3 + \cdots + 134 \otimes \emptyset.$$

Formulas for primitives and the antipode in $\mathbb{Q}\mathbf{P}$ mimic (4) and (6).

Corollary 7. *Let $X \neq \emptyset$ be a finite set with distinguished element x. Let $\boldsymbol{\Gamma}(X)$ denote the set compositions γ of X and let $\boldsymbol{\Gamma}'(X)$ denote those γ with $x \in \gamma_1$. (View $\boldsymbol{\Gamma}(X)$ as elements of $\mathbb{Q}\mathbf{P}$ in the obvious way.) We have that*

$$p(X) := \sum_{\gamma \in \boldsymbol{\Gamma}'(X)} (-1)^{\ell(\gamma)-1}\gamma \tag{9}$$

is a primitive (free) generator of $\mathbb{Q}\mathbf{P}$, and

$$S(X) = \sum_{\gamma \in \boldsymbol{\Gamma}(X)} (-1)^{\ell(\gamma)}\gamma. \tag{10}$$

More is true. In fact, the $\mathbb{Q}\mathbf{P}$ formulas above engender (4) and (6) via a transfer of structure (see (5) and (8) and the surrounding discussions). The structure is transfered by means of a *measuring* of Hopf algebras.

Theorem 8. *(See [10].) Let A, B be Hopf algebras and let C be a coalgebra. Suppose $\theta\colon B \otimes C \to A$ is a* covering *(a surjective coalgebra map that* measures B to A*). Let $\iota : A \to B \otimes C$ be any linear section of θ, that is, $\theta \circ \iota = \mathrm{id}$. Then the following hold.*

1. *If $p \in B$ is primitive, then the element $\theta(p, c) \in A$ is primitive $\forall c \in C$.*
2. $S_A = \theta \circ (S_B \otimes \mathrm{id}) \circ \iota$.

Here, $\mathbb{Q}\mathbf{P}$ covers *NCSym* by taking C to be the free pointed coalgebra on set partitions; the covering $\theta(\gamma \otimes \mathbf{A})$ is the evaluation $\gamma[\mathbf{A}]$ discussed in Section 2.3. We leave the details to [10], where further examples of transfer of structure will be worked out.

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