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PACKINGS OF CUTS REALIZING DISTANCES BETWEEN CERTAIN VERTICES IN A PLANAR GRAPH

A.V. KARZANOV
Institute for System Studies, Prospect 60 let Oktyabrya, 9, 117312 Moscow, USSR

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Recently A. Schrijver proved the following theorem. Suppose that $G=(V, E)$ is a connected planar graph embedded in the euclidean plane, that O and I are two of its faces, and that the edges $e \in E$ have nonnegative integer-valued lengths $l(e)$ such that the length of each circuit in G is even. Then there exist cuts $B_1, \ldots, B_k$ in G weighted by nonnegative integer-valued weights $\lambda_1, \ldots, \lambda_k$ so that: (i) for each $e \in E$, the sum of the weights of the cuts containing e does not exceed $l(e)$, and (ii) for each two vertices s and t both in the boundary of O or in the boundary of I, the sum of the weights of the cuts 'separating' s and t is equal to the distance between s and t.

We given another proof of this theorem which provides a strongly polynomial-time algorithm for finding such cuts and weights.

1. Introduction

A. Schrijver proved the following theorem.

Theorem [5]. *Let $G=(V, E)$ be a connected undirected planar graph embedded in the euclidean plane. Let O and I be two faces in G. Let l be a nonnegative integer-valued function on E (regarded as a function of lengths of edges) such that*

$$\text{the length } l(C) \text{ of each circuit } C \text{ in } G \text{ is even.} \tag{1}$$

Then there exist cuts $\delta X_1, \ldots, \delta X_k$ in G and nonnegative integers $\lambda_1, \ldots, \lambda_k$ satisfying

$$\sum (\lambda_i : i=1, \ldots, k,\ e \in \delta X_i) \leq l(e) \quad \text{for all } e \in E; \tag{2}$$

$$\sum (\lambda_i : i=1, \ldots, k,\ \delta X_i \text{ separates } s \text{ and } t) = d_l(s, t) \quad \text{for all } (s, t) \in U, \tag{3}$$

where U is the set of pairs (s, t) of vertices of G such that both s and t belong to the boundary of O or belong to the boundary of I.

[Here and further the following conventions, terminology and notation are used. G can contain loops and multiple edges. $O=I$ is possible. For $X \subseteq V$, $\delta X = \delta_G X$ is the set of edges in G with one end in X and the other in $V-X$, called a *cut* of G; δX *separates* vertices x and y if $|X \cap \{x, y\}| = 1$. A *path* from x to y, or x–y

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path, in G is a nonempty sequence $P=(x=x_0, e_1, x_1, \ldots, e_m, x_m=y)$, where e_i is an edge of G with the ends x_i and x_{i+1}. P is a *circuit* if $x=y$ (we admit self-intersecting circuits and even *degenerate* circuits containing no edges); we identity all possible circuits obtained from P by cyclically shiffing. The *length* $l(P)$ of P is $\sum(l(e_i): i=1, \ldots, m)$. For $x, y \in V$, $d(x, y)=d_l(x, y)$ denotes the *distance* from x to y, i.e., the minimum of $l(P)$ over all x–y paths P in G.] A nonnegative integer-valued function l on E satisfying (1) will be called *cyclically even*.

It is easy to see that, for arbitrary G, l and $s, t \in V$, if δX_i and λ_i, $i=1, \ldots, k$, satisfy the packing condition (2), then the left-hand side in (3) does not exceed the right-hand one. Using the theorem of Okamura [4] on multicommodity flows in planar graphs and applying linear programming arguments one can show that, for G and U as in the hypotheses of the theorem, (2) and (3) hold for some δX_i's and *rational* λ_i's (a relation between cut packing problems and multicommodity flow problems is explained, for example, in [2] (see also [1, 5])). The essence of the theorem is that whenever l is cyclically even, λ_i's can be chosen *integer-valued* (for some δX_i's). Similar 'half-integrity' theorems for other cases of G and U occurred in [1, 3, 5].

Unfortunately, the 'decomposition' method developed in [4] for proving the theorem can be turned into an efficient algorithm only for l with bounded $\max\{l(e): e \in E\}$. In the present paper we give another proof of the theorem which provides a strongly polynomial-time algorithm (for arbitrary l). The construction of the algorithm is rather simple (in comparison with the proof of the theorem).

2. Proof of the theorem

We shall assume that O is the unbounded face. Let BF denote the circuit which follows the boundary of a face F and is oriented clockwise in the plane; the sets of distinct vertices and edges in BF are denoted by VF and EF, respectively. A path (circuit) $P=(x_0, e_1, x_1, \ldots, x_m)$ is *simple* if x_i's are distinct (resp., $x_i \neq x_j$ for $0 \leq i<j<k$ and e_i's are distinct). An x–y path P is *shortest* if $l(P)=d(x, y)$. An edge with ends x and y may be denoted by xy. For $e=xy \in E$, put

$$\varepsilon(e)=\varepsilon_l(e):=\min\{d(s, x)+l(e)+d(y, t)-d(s, t): (s, t) \in U\};$$

and, for $x, y, z \in V$, put

$$\Delta(x, y, z)=\Delta_l(x, y, z):=d(x, y)+d(y, z)-d(x, z).$$

Clearly $\varepsilon(e) \geq 0$ and $\Delta(x, y, z) \geq 0$. (1) easily implies the following.

2.1. *Let $v_1, v_2, \ldots, v_r=v_0$ be vertices of G, and let a_i be $l(P)$ for some v_i-v_{i+1} path P in G, $i=1, \ldots, r-1$. Let $k_1, \ldots, k_r \in \{1, -1\}$. Then the value $k_1a_1+k_2a_2+\cdots+k_ra_r$ is even.*

Thus, $\varepsilon(e)$ and $\Delta(x, y, z)$ are even. In order to prove the theorem we use induction on the number $\alpha = \alpha(G, O, I, l)$ to be

$$|V|^3\,|E| + |\{e \in E: l(e) > 0\}| + |\{e \in E: \varepsilon(e) > 0\}| \\ + |\{(s, x, t): (s, t) \in U, x \in V, \Delta(s, x, t) > 0\}|.$$

The theorem is obvious when G is a tree, i.e., $|V| = |E| + 1$; in this case $O = I$ is the unique face of G. Thus, one may assume that G is not a tree and $O \neq I$ (otherwise we replace I by an arbitrary inner (bounded) face of G). Also we shall assume that the properties (4)–(8) below hold. Otherwise the quadruple (G, O, I, l) can be reduced, each reduction yields one or more quadruples (G', O', I', l') with smaller α, and the result follows by induction.

$$l(e) > 0 \quad \textit{for all } e \in E. \tag{4}$$

For if $l(e) = 0$ for some $e = xy \in E$, then contract e, i.e., delete e and identify x and y.

G has no loops and multiple edges. (5)

For if e is a loop or e and e' are parallel edges with $l(e) \geq l(e')$, then delete e. If e is contained in BI, then I is replaced by a new face I' for which V$I \subseteq$ VI'.

The circuits BO *and* BI *are simple.* (6)

For if, for example, BO is not simple, then BO contains a vertex x removing of which make G disconnected. Let $G_1 = (V_1, E_1), \ldots, G_r = (V_r, E_r)$ be the set of maximal subgraphs of G such that: (i) the graph $G_i - \{x\}$ is connected, and (ii) $V_i - \{x\}$ meets V$O \cup$ VI. Clearly the problem for G, O, I, l is reduced to those for G_i, O_i, I_i, l_i, where l_i is the restriction of l to E_i, and I_i is chosen so that V$I_i \supseteq$ VI − VO_i.

For each $x \in V$, there is a shortest s–t path, $(s, t) \in U$,
passing through x or, equivalent, $\Delta(s, x, t) = 0$. (7)

For if $\Delta(s, x, t) > 0$ for all $(s, t) \in U$, then reduce l by setting $l(e) := l(e) - a$ if $e \in E$ is incident to x, where

$$a := \min\{\min\{l(e)\text{: } e \text{ incident to } x\}, \tfrac{1}{2}\min\{\Delta(s, x, t)\text{: } (s, t) \in U\}\}.$$

Clearly, a is an integer, the new l is cyclically even, $d(s, t)$ does not change for all $(s, t) \in U$, and α becomes smaller.

For any $e \in E$, at least one of the following is valid:
(i) $l(e) \leq 1$;
(ii) $\varepsilon(e) = 0$. (8)

For if it is not so for some edge $e = xy$, then reduce l by setting $l(e) := l(e) - a$, where $a := \min\{2\lfloor l(e)/2 \rfloor, \varepsilon(e)\}$ ($\lfloor b \rfloor$ is the largest integer less than or equal to b). Since a is even, the new l is cyclically even. Clearly $d(s, t)$ does not change for all $(s, t) \in U$, and α becomes smaller.

It follows from (4) that each shortest path in G is simple. Also we obtain from (8), (4) and (5) that

$$l(e) = d(u, v) \quad \textit{for each edge } e = uv \in \mathrm{E}O \cup \mathrm{E}I. \tag{9}$$

Indeed, this is so if $l(e) = 1$ (by (4) and (5)). And if $l(e) > 1$, then e belongs to some shortest s–t path, $(s, t) \in U$, by (8), and hence the path (u, e, v) is also shortest.

Now we need to give additional definitions and notation.

1) A path (circuit) $P = (x_0, e_1, x_1, \ldots, x_m)$ may be denoted as $x_0x_1 \cdots x_m$ (such a writing determines P uniquely, by (5)). $\mathrm{V}P$ and $\mathrm{E}P$ are the sets of distinct vertices and edges in P, respectively. P^{-1} is the opposite path $x_mx_{m-1} \cdots x_0$. For a path $Q = y_0y_1 \cdots y_r$ with $y_0 = x_m$, $P \cdot Q$ is the path $x_0x_1 \cdots x_my_1 \cdots y_m$. For $0 \leq i, j \leq m$, $P(x_i, x_j)$ denotes the path $x_ix_{i+1} \cdots x_j$ if $i \leq j$ and denotes the path $x_ix_{i+1} \cdots x_mx_1 \cdots x_j$ if P is a circuit and $i > j$; it is called the *part* of P from x_i to x_j. If P is a simple path or a simple circuit and $u, v \in V$, then $P(u, v)$ is the simple path $P(x_i, x_j)$ (if it exists), where $u = x_i$ and $v = x_j$; in particular, $P(v, v)$ is the trivial path v. Where P is a simple path (a simple circuit) and $v_0, v_1, \ldots, v_k \in V$, we write $v_0 \to v_1 \to \cdots \to v_k(P)$ if $v_i = x_{i(j)}$ for $0 \leq i(0) \leq i(1) \leq \cdots \leq i(k) \leq m$ (resp., $0 \leq i(r) \leq \cdots \leq i(k) \leq i(0) \leq \cdots \leq i(r-1) \leq m$ for some $0 \leq r \leq k$). For $F \in \{O, I\}$ and $s, t \in \mathrm{V}F$, the path $\mathrm{B}F(s, t)$ will be denoted by $F(s, t)$. Clearly if P and Q are shortest paths and $v_0, v_1, \ldots, v_k \in \mathrm{V}P \cap \mathrm{V}Q$, then $v_0 \to v_1 \to \cdots \to v_k(P)$ implies $v_0 \to v_1 \to \cdots \to v_k(Q)$ or $v_k \to v_{k-1} \to \cdots \to v_0(Q)$. We identify a simple path (circuit) and its image in the plane.

2) Let $\tilde{G} = (V, \tilde{E})$ be the directed graph obtained from G by replacing each edge $xy \in E$ by the directed edges (x, y) and (y, x). For a path $P = x_0x_1 \cdots x_m$ in G, define the function $\chi[P]: \tilde{E} \to \mathbb{Z}$ by

$$\chi[P](e) := |\{i: 0 \leq i < m,\ e = (x_i, x_{i+1})\}|,\ e \in \tilde{E}.$$

Circuits P and Q are called *homotopic* (denoted as $P \sim Q$) if

$$\chi[P] + \chi[Q^{-1}] = k_1\,\chi[C_1] + \cdots + k_n\,\chi[C_n],$$

where each k_i is an integer and each C_i is either the circuit $\mathrm{B}F'$ for some face $F' \neq O, I$ or the circuit xyx for some edge $xy \in E$ (when the space obtained from the euclidean plane by removing the interiors of the faces O and I is considered, such a 'homologic' definition is known to be equivalent to the usual definition of homotopness of two closed curves as the existence of a continuous deformation of one curve to the other). In particular, $\mathrm{B}I \sim \mathrm{B}O$ (since $\chi[\mathrm{B}I] + \chi[\mathrm{B}O^{-1}] = \Sigma\,(\chi[\mathrm{B}F']$: F' is a face $\neq O, I) - \Sigma\,(\chi[xyx]$: $e = xy \in E)$ taking into account the introduced orientations of the faces of G). A circuit is *null-homotopic* if it is homotopic to a degenerate circuit v. Two x–y paths P and Q are homotopic (denoted as $P \sim Q$) if the circuit $P \cdot Q^{-1}$ is *null-homotopic.* For a circuit C, define $\gamma(C)$ to be 0 if C is null-homotopic, k if $C \sim \mathrm{B}I \cdot \mathrm{B}I \cdot \cdots \cdot \mathrm{B}I$ (k times) and $-k$ if $C \sim \mathrm{B}I^{-1} \cdot \mathrm{B}I^{-1} \cdot \cdots \cdot \mathrm{B}I^{-1}$ (k times), $k \geq 1$.

In the sequel F will denote some face in $\{O, I\}$, and s and t will denote some vertices in VF (possibly $s = t$). A topological fact is that BI is not null-homotopic. This easily implies that:

(i) $\gamma(C)$ is well-defined for each circuit C, and circuits C and C' are homotopic if and only if $\gamma(C) = \gamma(C')$;

(ii) if $s \neq t$ and P is a simple s–t path, then either $P \sim F(s, t)$ or $P^{-1} \sim F(t, s)$.

For a simple circuit C, denote by $\text{int}(C)$ the set of points in the plane lying inside of C or on C. Let P be a simple s–t path, $\{v_0, v_1, \ldots, v_k\}$ be the set of vertices in P contained in $F(s, t)$, and let $v_0 \to v_1 \to \cdots \to v_k(P)$. Using the Jordan curve theorem one can show that $v_0 \to v_1 \to \cdots v_k(F(s, t))$. Let $\text{int}_F(P)$ denote the set of points of the plane between P and $F(s, t)$, i.e.,

$$P \cup F(s, t) \cup \bigcup (\text{int}(P(v_i, v_{i+1}) \cdot (F(v_i, v_{i+1})^{-1}) \colon i,\ v_i v_{i+1} \notin Es).$$

Obviously, for a simple circuit C, $|\gamma(C)| \leq 1$ and $|\gamma(C)| = 1$ if and only if $\text{int}(C)$ contains the interior of I. This and easy topological arguments imply the following.

2.2. (i) *A simple s–t path P is homotopic to $F(s, t)$ if and only if $\text{int}_F(P)$ does not contain the interior of I.*

(ii) *If $s \to u \to v \to t(BF)$, P is a simple s–t path, $P \sim F(s, t)$, Q is a simple u–v path and Q lies in $\text{int}_F(P)$, then $Q \sim F(u, v)$.*

(iii) *Let $s \to u \to v \to t(BF)$, P be a shortest s–t path, Q be a shortest u–v path, and let $P \sim F(s, t)$ and $Q \sim F(u, v)$. If $x, y \in VP \cap VQ$ and $x \to y(P)$, then $x \to y(Q)$, $P(s, x) \cdot Q(x, y) \cdot P(y, t)$ is a shortest path homotopic to P and $Q(u, x) \cdot P(x, y) \cdot Q(y, v)$ is a shortest path homotopic to Q.*

Let $\Gamma_F(s, t)$ denote the set of shortest s–t paths in G homotopic to $F(s, t)$. Note that if $s = t$ then $\Gamma_F(s, t)$ coincides with $\Gamma_F(t, s)$ and consists of the unique path s, and if $s \neq t$ then, by arguments above, for any shortest s–t path P, either $P \in \Gamma_F(s, t)$ or $P^{-1} \in \Gamma_F(t, s)$. It follows from (2.2) that if $\Gamma_F(s, t)$ is nonempty, then there exists the path P in $\Gamma_F(s, t)$ 'most remote' from $F(s, t)$ ('nearest' to $F(s, t)$, respectively), i.e., such that $\text{int}_F(P)$ includes (resp., is included in) $\text{int}_F(P')$ for each $P' \in \Gamma_F(s, t)$; P is denoted by $D_F(s, t)$ (resp., by $N_F(s, t)$), see Fig. 1.

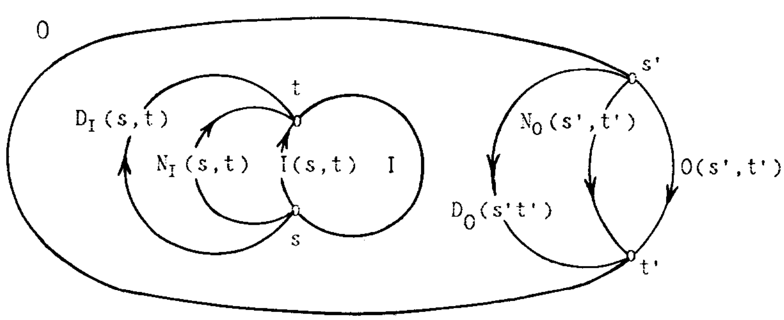

Fig. 1.

Let $X_F(s, t)$ be the set of vertices of G in $\mathrm{int}_F(D_F(s, t))$. The ordered pair (s, t) is called *essential* (with respect to F) if: (i) $\Gamma_F(s, t) \neq \emptyset$ and either $s = t$ or $\Gamma_F(t, s) = \emptyset$, and (ii) $\Delta(p, x, q) > 0$ for all $x \in X_F(s, t)$ and $(p, q) \in U$ so that $p, q \in V - X_F(s, t)$ (in other words, no shortest p–q path for such p and q meets $D_F(s, t)$).

The proof of the theorem will follow from the Lemmas 1 and 2. These lemmas will be proved in Sections 3 and 4.

Lemma 1. *Let (s, t) be an essential pair. Define*

$$\begin{aligned} a := \min\{&\min\{l(e)\colon e \in \delta X_F(s, t)\},\\ &\tfrac{1}{2}\min\{\Delta(s, x, t)\colon x \in V - X_F(s, t)\},\\ &\tfrac{1}{2}\min\{\Delta(p, x, q)\colon x \in X_F(s, t), (p, q) \in U, p, q \in V - X_F(s, t)\}\}\end{aligned}$$

and, for $e \in E$, define

$$\begin{aligned} l'(e) &:= l(e) - a \quad \text{if } e \in \delta X_F(s, t),\\ &:= l(e) \qquad \text{otherwise.}\end{aligned}$$

Then $l' \geqslant 0$, a is an integer $\geqslant 1$ and, for any $(u, v) \in U$,

$$\begin{aligned} d_{l'}(u, v) &= d_l(u, v) - a \quad \text{if } \delta X_F(s, t) \text{ separates } u \text{ and } v,\\ &= d_l(u, v) \qquad \text{otherwise.}\end{aligned} \tag{10}$$

A circuit C in G is called *shortest* if its length $l(C)$ is minimum among all circuits in G homotopic to C. Let $\mathscr{C}$ be the set of shortest circuits homotopic to BI.

Lemma 2. *Let G have no essential pair. Let b be the length of a circuit in $\mathscr{C}$. Then:*

(i) *for any $F \in \{O, I\}$ and $s, t \in$ VF, each path in $\Gamma_F(s, t)$ is a part of some circuit in $\mathscr{C}$;*

(ii) *there exist integers (a potential) $\pi(x)$, $0 \leqslant \pi(x) < b$, $x \in V$, such that:*

for each circuit $C = x_0x_1 \cdots x_m \in \mathscr{C}$, there is an index i, $0 < i \leqslant m$, such that

$$\begin{aligned} l(x_{j-1}x_j) &= \pi(x_j) - \pi(x_{j-1}) \quad \text{for } j = 1, \ldots, i-1, i+1, \ldots, m,\\ l(x_{i-1}x_i) &= \pi(x_i) - \pi(x_{i-1}) + b;\end{aligned} \tag{11}$$

if an edge $xy \in E$ belongs to no circuit in $\mathscr{C}$, then $\pi(x) = \pi(y)$. (12)

Assuming that Lemmas 1 and 2 are valid we prove the theorem as follows.

1) Let (s, t), a and l' be as in Lemma 1. Since a is an integer, l' is cyclically even. It follows from (10) and the definition of l' that

$$\begin{aligned} l(P) - l'(P) = a\,|\mathrm{E}P \cap \delta X_F(s, t)| &\geqslant a\,(|\{u, v\} \cap X_F(s, t)| \bmod 2)\\ &= d_l(u, v) - d_{l'}(u, v)\end{aligned}$$

for each u–v path P, $(u, v) \in U$, whence $\varepsilon_{l'}(e) \leq \varepsilon_l(e)$ for all $e \in E$ and $\Delta_{l'}(u, x, v) \leq \Delta_l(u, x, v)$ for all $x \in V$ and $(u, v) \in U$. Moreover, since $a > 0$, there is an e such that $0 = l'(e) < l(e)$ or there are u, x, v such that $0 = \Delta_{l'}(u, x, v) < \Delta_1(u, x, v)$. Thus, by induction, there exist required cuts $\delta X_1, \ldots, \delta X_k$ and integers $\lambda_1, \ldots, \lambda_k$ for (G, O, I, l'). Adding to them the cut $\delta X_{k+1} := \delta X_F(s, t)$ and the number $\lambda_{k+1} := a$ we obtain required objects for (G, O, I, l).

2) Let π be as in Lemma 2. For $x \in V$, define the number $\pi'(x)$ so that $0 \leq \pi'(x) < b$ and $|\pi(x) - \pi'(x)| = b/2$. Let $p_1 < p_2 < \cdots < p_n$ be all distinct numbers among $\pi(x)$ and $\pi'(x)$, $x \in V$. Put $k := n/2$. Obviously, k is an integer and $p_{i+k} = p_i + b/2$, $i = 1, \ldots, k$. For $i = 1, \ldots, k$, put

$$X_i := \{x \in V : p_i < \pi(x) \leq p_{i+k}\}, \qquad \lambda_i := p_{i+1} - p_i \; (= p_{i+k+1} - p_{i+k})$$

(assuming $p_{2k+1} := p_1 + b$). Since each $\pi(x)$ is an integer and b is even, all p_j and λ_j are integers. We assert that δX_i and λ_i, $i = 1, \ldots, k$, satisfy (2) and (3).

Indeed, it follows easily from the definitions of p_i, X_i and λ_j that, for any $x', y' \in V$,

$$\begin{aligned} \zeta(x', y') &:= \sum (\lambda_i \colon \delta X_i \text{ separates } x' \text{ and } y') \\ &= \min\{|\pi(x') - \pi(y')|, b - |\pi(x') - \pi(y')|\}. \end{aligned}$$

(11) and (12) imply that $\zeta(x, y) \leq l(e)$ for any edge $e = xy \in E$. Thus, (2) is true. Next, let $s, t \in VF$, $F \in \{O, I\}$, and let P be a shortest s–t path. Without loss of generality, one may assume that $P \in \Gamma_F(s, t)$. By (i) in Lemma 2, P is a part of some circuit $C \in \mathscr{C}$, i.e., $P = C(s, t)$. We have $l(P) \leq b/2$ (otherwise $d(s, t) = d(t, s) \leq l(C(t, s)) = b - l(P) < l(P)$). Furthermore, (11) implies that $l(P) = \pi(t) - \pi(s)$ if $\pi(s) \leq \pi(t)$ and that $l(P) = \pi(t) - \pi(s) + b$ if $\pi(s) > \pi(t)$. Thus, $l(P) = \zeta(s, t)$, and the equality in (3) holds for (s, t).

3. Proof of Lemma 1

Put $X := X_F(s, t)$. Obviously, $l' \geq 0$ and a is an integer ≥ 0. Let us show that $a > 0$. Let $x \in V - X$, Q' be a shortest s–x path, Q'' be a shortest x–t path, and $Q := Q' \cdot Q''$. Then $\Delta(s, x, t) = l(Q) - d(s, t)$. Suppose that $\Delta(s, x, t) = 0$; then Q is a shortest path. We have $Q \sim F(s, t)$ (otherwise $s \neq t$ and $Q^{-1} \in \Gamma_F(t, s) \neq \emptyset$, contrary to the property that (s, t) is essential). Hence $VQ \subseteq X$ (as $D_F(s, t)$ is the 'most remote' path in $\Gamma_F(s, t)$), contradicting $x \in V - X$. Thus, $\Delta(s, x, t) > 0$. We have also $l(e) > 0$ for $e \in \delta X$ (by (4)) and $\Delta(p, x, q) > 0$ for $x \in X$ and $(p, q) \in U$, $p, q \notin X$ (as (s, t) is essential). Therefore, $a > 0$.

Next, let $(u, v) \in U$. Clearly $d_{l'}(u, v) \leq d_l(u, v) - a\rho(u, v)$, where $\rho(u, v)$ is 1 if δX separates u and v, and 0 otherwise. In order to prove the converse inequality consider an arbitrary path $P = x_0 x_1 \cdots x_m$, $x_0 = u$, $x_m = v$. Put $k(P) := |\{i \colon x_i x_{i+1} \in \delta X\}|$. Obviously, $l'(P) = l(P) - ak(P)$. One must prove that

$$l(P) - ak(P) \geq d_l(u, v) - a\rho(u, v). \tag{13}$$

We proceed by induction on $k(P)$.

(i) $k(P)=0$ or 1. Then (13) is trival.

(ii) $k(P)=2$ and $u, v \in V-X$. Then $\rho(u, v)=0$. Choose $x \in VP \cap X$. Since $\Delta(u, x, v) \geqslant 2a$ (by the definition of a), we have

$$l(P)-2a \geqslant d(u, x)+d(x, v)-2a = d(u, v)+\Delta(u, x, v)-2a \geqslant d(u, v).$$

(iii) Suppose that P is not as in (i) or (ii). Put $D:=D_F(s, t)$. Then there are i and j, $0 \leqslant i<j \leqslant m$, $j-i \geqslant 2$, such that $x_i, x_j \in VD$ and $x_r \notin X$ for $r=i+1, \ldots, j-1$. One may assume that $x_i \rightarrow x_j(D)$ (otherwise it should to consider the pair (v, u) and the path P^{-1}). Form the paths $P':=P(u, x_i) \cdot D(x_i, x_j) \cdot P(x_j, v)$ and $Q:=D(s, x_i) \cdot P(x_i, x_j) \cdot D(x_j, t)$. Obviously, $k(P')=k(P)-2$, and using the induction hypothesis we have

$$\begin{aligned} l(P)-ak(P) &= l(P')-ak(P')+(l(P(x_i, x_j))-l(D(x_i, x_j))-2a) \\ &\geqslant d_l(u, v)-a\rho(u, v)+(l(Q)-l(D)-2a). \end{aligned}$$

Since D is a shortest path we have

$$l(Q)-l(D) \geqslant d(s, x_{i+1})+d(x_{i+1}, t)-d(s, t)=\Delta(s, x_{i+1}, t).$$

Finally, $x_{i+1} \notin X$ implies $\Delta(s, x_{i+1}, t) \geqslant 2a$, and hence (13) follows.

4. Proof of Lemma 2

The proof is divided into a number of claims. As before, F denotes some face in $\{O, I\}$, and s and t are some vertices in BF. For $s', t' \in VF$ and a simple s'–t' path P, $\Gamma_F(s', t')$ and $\mathrm{int}_F(P)$ will be denoted as $\Gamma(s', t')$ and $\mathrm{int}(P)$, respectively.

4.1. *Let $v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow v_4(\mathrm{B}F)$, $v_1 \neq v_4$ and $\Gamma(v_1, v_4) \neq \emptyset$. Then $\Gamma(v_2, v_3) \neq \emptyset$.*

Proof. Let Q be a shortest v_2–v_3 path and $P \in \Gamma(v_1, v_4)$. If Q lies in $\mathrm{int}(P)$, then $Q \sim F(v_2, v_3)$ (by 2.2(ii)), hence $\Gamma(v_2, v_3) \neq \emptyset$. Otherwise Q meets P; let x be the first vertex and y be the last one in Q occurring in P. The path $Q':=Q(v_2, x) \cdot P(x, y) \cdot Q(y, v_3)$ is shortest and lies in $\mathrm{int}(P)$, whence $Q' \sim F(v_2, v_3)$. □

4.2. *If $\Gamma(s, t) \neq \emptyset$, then the path $F(s, t)$ is shortest.*

Proof. We use induction on $r:=|VF(s, t)|$. The result is obvious when $r=1$; if $r=2$ it follows from (9). Assume that $r \geqslant 3$, and let u be the vertex in BF following s and v be the one preceding t. Then $s \rightarrow u \rightarrow v \rightarrow t(\mathrm{B}F)$. Put $D:=D_F(u, v)$ and $N:=N_F(s, t)$ (N is the path in $\Gamma(s, t)$ 'nearest' to $F(s, t)$). Two cases are possible.

1) D meets N. Let $x \in VD \cap VN$, $P:=N(s, x) \cdot D(x, v)$ and $Q:=D(u, x) \cdot N(x, t)$. By 4.1, each of $\Gamma(u, v)$, $\Gamma(s, v)$ and $\Gamma(u, t)$ is nonempty,

whence by induction $F(u, v)$, $F(s, v)$ and $F(u, t)$ are shortest paths. Then

$$d(s, t) = l(N) = l(P) + l(Q) - l(D)$$
$$\geq l(F(s, v)) + l(F(u, t)) - l(F(u, v)) = l(F(s, t)),$$

as required.

2) D does not meet N. We show that the pair (u, v) is essential, which implies impossibility of this case. First, suppose that $u \neq v$ and there exists a shortest v–u path Q homotopic to $F(v, u)$. Clearly Q meets N, and so there is a shortest u–v path Q' homotopic to $F(u, v)$ and meeting N (Q' is formed from Q^{-1} and N in a similar way as it is done in the proof of 4.1 for P and Q). But then the 'most remote' path D also meets N; a contradiction. Thus, if $u \neq v$ then $\Gamma(v, u) = \emptyset$.

Second, consider $x \in X_F(u, v)$ and $(p, q) \in U$, $p, q \in V - X_F(u, v)$, and suppose that there exists a shortest p–q path P' passing through x. Let Y be the set of vertices in int(N) not in N. Since D does not meet N we have $X_F(u, v) \subseteq Y$, and so $x \in Y$. Next, it is easy to see that $Y \cap (\mathrm{V}O \cup \mathrm{V}I) \subseteq \mathrm{V}F(u, v)$, which implies $p, q \notin Y$. Thus, there are vertices x' and y' such that $x' \to x \to y'(P')$, $x', y' \in \mathrm{V}N$ and all vertices in $P'(x', y')$ except x' and y' are in Y. One may assume that $x' \to y'(N)$. Then $P'' := N(s, x') \cdot P'(x', y') \cdot N(y', t)$ is a shortest path homotopic to $F(s, t)$, P'' is different from N and P'' lies in int(N), contrary to the definition of N. Therefore $\Delta(p, x, q) > 0$. Thus, (s, t) is essential. □

It follows from (4.2) that

$$d(x, y) = \min\{l(F(x, y)), l(F(y, x))\} \quad \text{for any } x, y \in \mathrm{B}F. \tag{14}$$

4.3. *Let* $v_1 \to v_2 \to v_3 \to v_4(\mathrm{B}F)$, $v_1 \neq v_4$, $v_2 \neq v_3$, $\{v_1, v_4\} \neq \{v_2, v_3\}$, *and let* $\Gamma(v_1, v_4) \neq \emptyset$. *Then* $\Gamma(v_3, v_2) = \emptyset$.

Proof. Suppose that it is not so. Put $l_{ij} := (F(v_i, v_j))$. We have from 4.2 and (14) that $l_{14} \leq l(\mathrm{B}F)/2$ and $l_{32} \leq l(\mathrm{B}F)/2$. But $l(\mathrm{B}F) = l_{14} + l_{32} - l_{12} - l_{34} < l_{14} + l_{32}$ (since $\{v_1, v_4\} \neq \{v_2, v_3\}$ implies $l_{12} + l_{34} > 0$); a contradiction. □

4.4. *Let* $v_1 \to v_2 \to v_3 \to v_4(\mathrm{B}F)$, $v_1 \neq v_4$, $v_2 \neq v_3$, *and let* $P \in \Gamma(v_1, v_2) \neq \emptyset$ *and* $Q \in \Gamma(v_3, v_4) \neq \emptyset$. *Then P does not meet Q.*

Proof. Suppose that it is not so, and let $z \in \mathrm{V}P \cap \mathrm{V}Q$. Put $l_{ij} := l(F(v_i, v_j))$. By 4.2 for (v_1, v_2) and for (v_3, v_4), we have

$$\begin{aligned} d(v_1, v_3) + d(v_2, v_4) &\leq l(P(v_1, z)) + l(Q^{-1}(z, v_3)) \\ &\quad + l(P^{-1}(v_2, z)) + l(Q(z, v_4)) \\ &= l(P) + l(Q) = l_{12} + l_{34}. \end{aligned} \tag{15}$$

Without loss of generality one may assume that $l_{13} \geq l_{31}$. Then $d(v_1, v_3) = l_{13}$, by (14). Two cases are possible.

1) $l_{24} \leq l_{42}$. Then $d(v_2, v_4) = l_{24}$ and $d(v_1, v_3) + d(v_2, v_4) = l_{13} + l_{24} > l_{12} + l_{34}$ (since $v_2 \neq v_3$ implies $l_{23} > 0$), contradicting (15).

2) $l_{24} > l_{42}$. Then

$$d(v_1, v_3) + d(v_2, v_4) = l_{13} + l_{42} = 2l_{12} + l_{41} + l_{23},$$

whence, in view of (15) and

$$l_{34} = d(v_3, v_4) \leq l_{41} + l_{12} + l_{23},$$

we obtain $l_{34} = l_{43} = l(\mathrm{B}F)/2$. Thus, the path $F(v_4, v_3)$ is shortest. Then the v_4–v_3 path $L := F(v_4, v_1) \cdot P \cdot F(v_2, v_3)$ is shortest, and therefore the v_4–v_3 paths $R := L(v_4, z) \cdot Q^{-1}(z, v_3)$ and $R' := Q^{-1}(v_4, z) \cdot L(z, v_3)$ are also shortest. The case $R^{-1} \sim F(v_3, v_4)$ is impossible because otherwise the shortest path $R^{-1}(v_3, v_1)$ would be homotopic to $F(v_3, v_1)$ and we would have $d(v_3, v_1) = l_{31} > l_{34} = l(\mathrm{B}F)/2$. Similarly $R'^{-1} \sim F(v_3, v_4)$ is impossible. Thus, $R \sim F(v_4, v_3) \sim R'$. But then both R and R' are in $J := \mathrm{int}(D_F(v_4, v_3))$, and hence Q^{-1} lies in J. This implies $Q^{-1} \sim F(v_4, v_3)$, contrary to $Q \sim F(v_3, v_4)$. □

We say that a family $\{C_1, \ldots, C_r\}$ of circuits (possible repeated) is a *decomposition* of a circuit C if $\chi[C_1] + \cdots + \chi[C_r] = \chi[C]$. It is easy to see that $l(C) = l(C_1) + \cdots + l(C_r)$ and $\gamma(C) = \gamma(C_1) + \cdots + \gamma(C_r)$.

4.5. *Let C be a shortest circuit. Then:*

(i) $l(C) = |\gamma(C)|\, b$ *(b is the length of a circuit in $\mathscr{C}$);*

(ii) *if $\{C_1, \ldots, C_r\}$ is a decomposition of C, then, for each i, $\gamma(C_i)\gamma(C) \geq 0$ and the circuit C_i is shortest; if $|\gamma(C)| = 1$, then the circuit C is simple.*

Proof. Without loss of generality one may assume that $k := \gamma(C) \geq 0$. (i) is obvious when $k = 0$. Let $k \geq 1$. If $C' \in \mathscr{C}$ and $C'' := C' \cdot \cdots \cdot C'$ (k times), then $\gamma(C'') = k$ and $l(C'') = kb$, which imply $l(C) \leq kb$ (since C is shortest). Obviously, C has a decomposition $\mathscr{D} = \{C'_1, \ldots, C'_{k'}\}$, where each C'_i is either a simple circuit or the circuit xyx for some $xy \in E$. Now $|\gamma(C'_i)| \leq 1$ and $\gamma(C'_1) + \cdots + \gamma(C'_{k'}) = k$ imply that there are at least k circuits C'_i with $\gamma(C'_i) = 1$, whence $l(C) \geq kb$. Thus, (i) is true. (ii) easily follows from (i). □

4.6. (i) *If $C, C' \in \mathscr{C}$, $x, y \in \mathrm{V}C \cap \mathrm{V}C'$ and $x \neq y$, then the circuits $C_1 := C(x, y) \cdot C'(y, x)$ and $C_2 := C'(x, y) \cdot C(y, x)$ are in $\mathscr{C}$.*

(ii) *If $C \in \mathscr{C}$, $P \in \Gamma(s, t)$, $x, y \in \mathrm{V}P \cap \mathrm{V}C$, $x \neq y$ and $x \to y(P)$, then $P(x, y) \sim C(x, y)$, $P(s, x) \cdot C(x, y) \cdot P(y, t) \in \Gamma(s, t)$ and $C(y, x) \cdot P(x, y) \in \mathscr{C}$.*

Proof. (i). Where $Q := C(x, y) \cdot C(y, x) \cdot C'(x, y) \cdot C'(y, x)$, we have $l(Q) = 2b$

and $\gamma(Q)=2$, therefore the circuit Q is shortest. Since $\{C_1, C_2\}$ is a decomposition of Q and $l(C_i)>0$, $i=1,2$, we have, by 4.5, that each C_i is shortest and $\gamma(C_i)\geq 1$, whence $\gamma(C_i)=1$. (ii). Let u be the first vertex and v be the last one in P contained in C. Then the path $P' := P(s,u)\cdot C(u,v)\cdot P(v,t)$ is simple, and so either $P'\sim F(s,t)$ or $P'\sim (F(t,s))^{-1}$. Similarly, either $P''\sim F(t,s)$ or $P''\sim (F(s,t))^{-1}$ is true for $P'' := P^{-1}(t,v)\cdot C(v,u)\cdot P^{-1}(u,s)$. Obviously, $P'\cdot P''\sim C\sim \mathrm{B}F$, which implies that only one combination is possible: $P'\sim F(s,t)$ and $P''\sim F(t,s)$. Thus, $P\sim P'$, whence $P(u,v)\sim C(u,v)$ and $l(P(u,v))=l(C(u,v))$ (since P and C are shortest). Now (ii) for x and y easily follows. □

It follows from (4.6) that there exists the circuit C in $\mathscr{C}$ nearest to $\mathrm{B}I$, i.e., $\mathrm{int}(C)\subseteq \mathrm{int}(C')$ for all $C'\in\mathscr{C}$. Now we come to the main claim in the proof of Lemma 2.

4.7. $\mathrm{B}I, \mathrm{B}O\in\mathscr{C}$.

Proof. We prove that $\mathrm{B}I\in\mathscr{C}$; the proof of $\mathrm{B}O\in\mathscr{C}$ is analogous. We say that a pair $(s,t)\in VI\times VI$ is *maximal* if the path $I(s,t)$ is shortest and no path in $\mathrm{B}I$ containing $I(s,t)$ as a proper part is shortest. Let $(s_0,t_0),\ldots,(s_{n-1},t_{n-1})$ be all maximal pairs, and let, for definiteness,

$$s_0\to s_1\to\cdots\to s_{n-1}(\mathrm{B}I). \tag{16}$$

Obviously, $n\geq 2$, all s_i are distinct and all t_i are distinct. Also one can see that

$$s_i\to s_{i+1}\to t_i\to t_{i+1}(\mathrm{B}I) \tag{17}$$

(here and later the indices are taken modulo n). Let C be the circuit in $\mathscr{C}$ nearest to $\mathrm{B}I$. Put $D_i := D_I(s_i,t_i)$, and let u_i be the first vertex and v_i be the last one in D_i contained in C whenever D_i meets C. It follows from 4.6(ii) and from the definitions of D_i and C that:

$$l(D_i(u_i,v_i))=l(C(u_i,v_i)); \tag{18}$$

$C(u_i,v_i)$ *lies in* $\mathrm{int}(D_i)$; (19)

for $p,q\in VO$, *each vertex of any shortest* p–q *path lies outside of* C *or on* C. (20)

We shall show that:

D_i *meets* C *and* $u_i\to u_{i+1}\to v_i(C)$ (see Fig. 2); (21)

$u_0\to u_1\to\cdots\to u_{n-1}(C)$ *and there are at least two distinct vertices among* $u_0,u_1,\ldots,u_{n-1}$; (22)

the path $L_i := D_{i+1}(s_{i+1},u_{i+1})\cdot C(u_{i+1},v_i)\cdot D(v_i,t_i)$ *is shortest and homotopic to* $I(s_{i+1},t_i)$. (23)

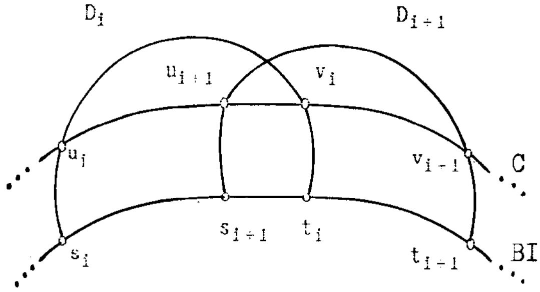

Fig. 2.

Then 4.7 can be obtained as follows. In view of (18), (21) and (22), we have (in each sum, i runs from 0 to $n-1$)

$$\sum l(D_i) = \sum (l(D_i(s_i, u_i)) + l(C(u_i, v_i)) + l(D_i(v_i, t_i)))$$

$$= \sum (l(D_{i+1}(s_{i+1}, u_{i+1})) + l(C(u_{i+1}, v_i)) + l(D_i(v_i, t_i)))$$

$$+ \sum l(C(u_i, u_{i+1})) = \sum l(L_i) + l(C).$$

Next, $l(D_i) = l(F(s_i, t_i))$ (by 4.2) and $l(L_i) = l(F(s_{i+1}, t_i))$ (by (23) and (4.2)), and now using (16) and (17) we obtain

$$\sum l(L_i) = \sum l(F(s_{i+1}, t_i)) = \sum l(F(s_i, t_i)) - \sum l(F(s_i, s_{i+1})$$

$$= \sum l(D_i) - l(\mathrm{B}I).$$

This and the above expression show that $l(\mathrm{B}I) = l(C)$. Thus, $\mathrm{B}I \in \mathscr{C}$, as required.

Now we prove (21)–(23). Let us fix a number i and denote s_i, t_i, D_i, s_{i+1}, t_{i+1}, d_{i+1}, D_{i+1} by s, t, D, s', t', D', respectively. Also denote u_i, v_i, u_{i+1}, v_{i+1} (if they exist) by u, v, u', v', respectively. It follows from (17) that D' meets D; let x be the first vertex in D' contained in D. Then $D'(s', x)$ lies in $\mathrm{int}_I(D)$. Put $\tilde{D} := \mathrm{D}_I(s', t)$ and $P := D'(s', x) \cdot D(x, t)$; obviously, the path P is simple. Easy arguments using the facts that the paths D and D' are 'most remote' show that $D(x, t)$ lies in $\mathrm{int}_I(D')$ and $\tilde{D}$ lies in $\mathrm{int}_I(P)$. Hence,

$$\tilde{D} \subseteq \mathrm{int}_I(P) \subseteq \mathrm{int}_I(D) \cap \mathrm{int}_I(D'). \tag{24}$$

Consider the pair (s', t). (17), 4.3 and $s' \neq s$ imply that if $s' \neq t$ then $\Gamma_I(t, s') = \emptyset$. Thus, there exists a shortest p–q path Q meeting $\tilde{D}$, where $(p, q) \in U$ and $p, q \notin X_I(s', t)$ (since (s', t) is not essential). Suppose that $p, q \in \mathrm{B}I$; one may assume that $Q \sim I(p, q)$. If $p \to s' \to t \to q(\mathrm{B}I)$, then the maximality of (s, t) and (s', t') implies $s \to p \to s' \to t \to q \to t'(\mathrm{B}I)$, $s \neq p$ and $q \neq t'$, contrary to that (s, t) and (s', t') are adjacent maximal pairs. Also the case $s' \to t \to p \to q(\mathrm{B}I)$ is impossible, by 4.4. Thus, $p, q \in \mathrm{V}O$. Then $\tilde{D}$ meets C because of (20) and $\tilde{D} \cap Q \neq \emptyset$, and Q, and so C meets each of D, D' and P (by

(24)). Let z be the first vertex and w be the last one in P contained in C. Form the paths $S := D(s, u) \cdot C(u, v) \cdot D(v, t)$ and $S' := D'(s', u') \cdot C(u', v') \cdot D'(v', t')$; S and S' are shortest, by 4.6. It follows from (19) that $C(z, w) \subseteq \text{int}_l(P)$ and at least one of the following holds: (i) $z = u'$, $w = v$; (ii) $z = u$, $w = v$; (iii) $z = u'$, $w = v'$. Suppose that $z = u \neq u'$. Then, in view of (24), $u' \notin \text{int}_l(D)$ and $x \to u' \to u(S')$, which implies that the shortest path $D(s, x) \cdot S'(x, u) \cdot D(u, t)$ does not lie entirely in $\text{int}_l(D)$; a contradiction. Similarly $w = v' \neq v$ is impossible. Therefore, $z = u'$ and $w = v$, which imply $u \to u' \to v(C)$ and $u' \to v \to v'(C)$. Thus, (21) is true. Next, the path $L := D'(s', u') \cdot C(u', v) \cdot D(v, t)$ lies in $\text{int}_l(P)$, whence $L \sim P \sim I(s', t)$. Let $y \in V\tilde{D} \cap VC$. We obviously have $z \to y \to w(C)$ and $L = S'(s', y) \cdot S(y, t)$. Now the facts that the paths S, S' and $\tilde{D}$ are shortest imply that L is also shortest. Thus, (23) is true.

It remains to prove (22). Let $C' := C(u_0, u_1) \cdot C(u_1, u_2) \cdot \cdots \cdot C(u_{n-1}, u_0)$. We show that $C' \sim BI$, whence (22) (or, in other words, $C' = C$) easily follows. Put $P_i := D_i(s_i, u_i)$, $T_i := P_i \cdot C(u_i, u_{i+1}) \cdot P_{i+1}^{-1}$ and $I_i := I(s_i, s_{i+1})$, and form the circuit $R := T_0 \cdot T_1 \cdot \cdots \cdot T_{n-1} \cdot BI^{-1}$. There are two decompositions of R, namely,

$$\mathscr{D}_1 := \{T_0 \cdot I_0^{-1}, T_1 \cdot I_1^{-1}, \ldots, T_{n-1} \cdot I_{n-1}^{-1}\}; \quad \text{and}$$

$$\mathscr{D}_2 := \{C', BI^{-1}, P_0 \cdot P_0^{-1}, P_1 \cdot P_1^{-1}, \ldots, P_{n-1} \cdot P_{n-1}^{-1}\}$$

(taking into account that $I_0^{-1} \cdot I_1^{-1} \cdot \cdots \cdot I_{n-1}^{-1} = BI^{-1}$, by (16)). As it was shown above, $C(u_i, u_{i+1})$ and P_{i+1} lie in $J := \text{int}_l(D_i)$, therefore T_i lies in J, whence $T_i \sim I_i$. Thus, $\gamma(T_i \cdot I_i^{-1}) = 0$. Now $0 = \sum (\gamma(C''): C'' \in \mathscr{D}_1) = \sum (\gamma(C''): C'' \in \mathscr{D}_2)$ and $\gamma(P_i \cdot P_i^{-1}) = 0$ imply $\gamma(C') + \gamma(BI^{-1}) = 0$, i.e., $C' \sim BI$, as required. Thus, 4.7 is proven. □

Now we finish to prove Lemma 2.

(a) Let $F \in \{O, I\}$, $s, t \in VF$ and $P \in \Gamma_F(s, t) \neq \emptyset$. If $s = t$, then $P = s$ and P is a part of the circuit BF. And if $s \neq t$, then the circuit $C := P \cdot F(t, s)$ is homotopic to BI, and $l(C) = l(BF) = b$, by 4.2 and 4.7. Thus, (i) of Lemma 2 is true.

(b) Form the directed graph $H = (V, A)$ in which $(x, y) \in A$ if and only if $x = x_i$ and $y = x_{i+1}$ for some circuit $C = x_0x_1 \cdots x_m$ in $\mathscr{C}$. Let $l(x, y) := l(xy)$ for $(x, y) \in A$. By a path (circuit) in H we mean a directed path (circuit); we identify a path (circuit) in H with the corresponding path (circuit) in G. We observe that H is strongly connected, i.e., for any $x, y \in V$, there is a path in H from x to y. Indeed, each vertex x is contained in some shortest s–t path, where $s, t \in VF$ and $F \in \{0, I\}$ (by (7)), hence x belongs to the strong component containing the circuit BF. Furthermore, as it was shown in the proof of 4.7, there exist a shortest s–t path with $s, t \in VI$ and a shortest p–q path with $p, q \in VO$ having a common vertex.

(c) Let us fix some vertex $s \in V$. Define the numbers $\pi(x)$, $x \in V$, by $\pi(x) := l(P) - \lfloor l(P)/b \rfloor b$, where P is some arbitrary s–x path in H. We assert that π satisfies (11) and (12).

First, we observe that each circuit C in H is shortest. Indeed, since H is strongly connected, there is a circuit D in G such that D has the decomposition $\{C': C' \in \mathscr{C}\}$ and it has a decomposition $\{C, Q_1, \ldots, Q_N\}$. As $l(D) = b\,|\mathscr{C}|$ and $\gamma(D) = |\mathscr{C}|$, the circuit D is shortest, and so C is shortest (by 4.5). In particular, the length of any circuit in H is a multiple of b.

Second, let $x \in V$, and let P and P' be two s–x paths in H. Choose an x–s path Q. Comparing the lengths of the circuits $P \cdot Q$ and $P' \cdot Q$ we obtain that $l(P) - l(P') = kb$ for some integer k. In particular, $\pi(x)$ does not depend on the choice of the s–x path P. Consider a circuit $C = x_0x_1 \cdots x_m \in \mathscr{C}$; one may assume that $\pi(x_0) \leq \pi(x_i)$, $i = 1, \ldots, m-1$. Choose an s–x_0 path P in H, and let $P_i := P \cdot (x_0, (x_0, x_1), x_1, \ldots, x_i)$. Then $l(x_{i-1}, x_i) = l(P_i) - l(P_{i-1})$ and $l(P_0) < l(P_1) < \cdots < l(P_{m-1}) < l(P_0) + b$, whence (11) follows.

Third, consider an edge $e = xy \in E$ contained in no circuit in $\mathscr{C}$. Let G' be the graph obtained from G by contracting e and identifying x and y with a new vertex z. Let $\mathscr{C}'$ be the set of shortest circuits in G' homotopic to BI. Suppose that $l(C') < b$ for $C' \in \mathscr{C}'$. Obviously, C' contains z and C' is obtained from a circuit C in G by contracting the edge xy. Since $l(xy) = 1$ (by (8)) we have $l(C) \leq b$. Clearly $C \sim \mathrm{B}I$, therefore $C \in \mathscr{C}$, contradicting the choice of xy. Thus, $l(C') = b$ and each member of $\mathscr{C}$ is in $\mathscr{C}'$. This implies that the potentials π and π' of corresponding vertices in G and in G', defined with respect to the same initial vertex s, coincide. Hence $\pi(x) = \pi'(z) = \pi(y)$, and (12) is true.

This completes the proof of Lemma 2.

5. Algorithm

In fact, the proof of the theorem given above yields an algorithm to find required δX_i's and λ_i's, which has a number of iterations bounded by a polynomial in $|V|$. Each iteration consists in either of a reduction of the current quadruple (G, O, I, l) to one or more quadruples satisfying (4)–(8) (an iteration of the *first* type), or of finding an essential pair (s, t), the cut $\delta X_i = \delta X_F(s, t)$ and the number $\lambda_i = a$ and then reducing l to l' according to Lemma 1 (an iteration of the *second* type), or of finding a potential π according to Lemma 2 and then producing some cuts $\delta X_1, \ldots, \delta X_i$ and numbers $\lambda_1, \ldots, \lambda_i$ from π (an iteration of the *third* type). Obviously, an iteration of the first type requires a polynomial in $|V|$ number of standard operations (mainly connected with calculations of distances). The only nontrivial procedure on an iteration of the second type is to find the 'most remote' shortest path $D_F(s, t)$ homotopic to $F(s, t)$ but one can see that it can be fulfilled in polynomial time by considering the faces of the graph of the shortest s–t paths and applying 2.2(i). Finally, in order to find a potential π on the iteration of the third type it is not necessary to construct the graph H (described in the proof of Lemma 2). It suffices to form the graph $H' = (V, A')$ containing only the edges (x, y) such that $x = x_i$ and $y = x_{i+1}$ for some path

$P = x_0 x_1 \cdots x_m \in \Gamma_F(s, t)$, $s, t \in VF$, $F \in \{O, I\}$. The graph H' is strongly connected (it can be shown in a similar way as for H) and $H' \subseteq H$, therefore potentials for H' and H coincide. These arguments show that the whole algorithm requires a polynomial in $|V|$ number of operations. Note also that the number of operations can be decreased by using the easy fact that the set of essential pairs does not increase in the process of the algorithm.

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