On the optimization of the initial boundary value problem for a conservation law

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Abstract

In the present note, the theory of shift differentiability for the Cauchy problem is extended to the case of an initial boundary value problem for a conservation law. This result allows to exhibit an Euler–Lagrange equation to be satisfied by the extrema of integral functionals defined on the solutions of initial boundary value problems of this kind.

Keywords: Conservation laws; Initial boundary value problem; Euler–Lagrange equation

1. Introduction

Aim of this note is the study of optimization problems related to the scalar initial boundary value problem (IBVP) for a conservation law

\[
\begin{aligned}
\partial_t u + \partial_x [f(u)] &= 0, & (t, x) & \in [0, +\infty[ \times [0, +\infty[, \\
\left. u \right|_{t=0} &= \tilde{u}(x), & x & \in [0, +\infty[, \\
\left. u \right|_{x=0} &= \bar{u}(t), & t & \in [0, +\infty[,
\end{aligned}
\]

where the flow \( f : \mathbb{R} \mapsto \mathbb{R} \) is smooth while \( \tilde{u} \) and \( \bar{u} \) are in \( X = \mathbf{L}^1([0, +\infty[) \cap \mathbf{BV}([0, +\infty[) \).

It is well known (see, e.g., [3,15]) that (1.1) generates a process \( \mathcal{P} : [0, +\infty[ \times X \times X \mapsto X \), i.e., \( u(t, x) = \mathcal{P}_t(\tilde{u}, \bar{u})(x) \) is the (weak entropic) solution to (1.1). The process \( \mathcal{P}_t \) is Lipschitzian, but it is not differentiable with respect to \( \tilde{u} \) and \( \bar{u} \); see [7]. This lack of

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regularity is a major obstacle toward the solution of optimization problems defined on the solutions to (1.1). Indeed, while the basic well posedness theory of scalar conservation laws dates back to the classical work by Kružkov [14] and was recently extended to the case of systems [6], the literature on the control and optimization of (1.1) is still very limited; see [1,2,10] or, in slightly different contexts, also [16–18].

To bypass the lack of differentiability, in the case of the Cauchy problem a new approach [7] recently appeared in the literature; see also [4,8,9]. This construction amounts to introduce a differential structure [5] on a metric space $\mathcal{M}$ by means of a suitable equivalence relation (first order contact) in the space of continuous curves on $\mathcal{M}$. A definition of differentiability can then be given for maps $\Phi : \mathcal{M} \mapsto \mathcal{M}$. In the case $\mathcal{M} = L^1 \cap BV$, the tangent space to $\mathcal{M}$ at some $u$ in $\mathcal{M}$ is identified with $L^1(Du)$, the space of absolutely integrable functions with respect to the Radon measure $Du$, $Du$ being the weak derivative of $u$.

In the present note, as a first step, we suitably extend the theory of shift differentiability to the IBVP (1.1). As a consequence, the process $P_t$ turns out to be shift differentiable, under suitable assumptions on the initial and boundary data. These conditions are essentially equivalent to the ones needed in the case of the Cauchy problem.

Secondly, we consider an integral functional $J$ defined on the solutions to (1.1). By means of the first order shift expansion obtained above, we deduce a necessary condition to be satisfied by the stationary points of $J$. Indeed, for fixed times $t_2 > t_1 \geq 0$, consider integral functionals of the type

$$J(\bar{u}, \tilde{u}) = \int_{t_1}^{t_2} \int_{0}^{+\infty} \psi\left(P_t(\bar{u}, \tilde{u})(x)\right) dx \, dt \quad \text{or}$$

$$J_\psi(\bar{u}, \tilde{u}) = \int_{0}^{+\infty} \int_{0}^{+\infty} \psi(t,x)\psi\left(P_t(\bar{u}, \tilde{u})(x)\right) dx \, dt, \quad (1.2)$$

$\psi$ being a suitable locally Lipschitzian cost function and $\psi$ a compactly supported smooth weight. Integral functionals of the type (1.2) naturally arise in problems related to the optimizations of systems modeled through scalar conservation laws, as, for example, in the management of traffic flows; see [2,10]. Once the shift differentiability of $P_t$ is proved together with a bound on the operator norm of the shift differential, it is straightforward to derive a necessary condition satisfied by the stationary points of $J$ or $J_\psi$. More precisely, we obtain below an Euler–Lagrange type equation, in the same spirit of [10].

This note is organized as follows. In Section 2 we slightly extend the theory of shift differentiability introduced in [7] and state the main results. The technical proofs are deferred to the final Section 3.

2. Notation and results

The Lipschitz constant and the $L^\infty$ norm of a function $v$ are denoted by $\text{Lip}(v)$ and $\|v\|_{L^\infty}$, respectively. Consider the space
\[ X = \{ u \in L^1([0, +\infty]) \cap BV([0, +\infty]) : u \text{ is right continuous} \} \]  
(2.1)
equipped with the \( L^1 \)-norm \( \| u \|_{L^1} = \int_0^{+\infty} |u| \, ds \) with respect the usual Lebesgue measure.

Motivated by the similar situation considered in [7], for \( u \in X \), define \( T_u X = L^1(Du) \)
equipped with the \( L^1 \)-norm with respect the total variation \( |Du| \) of the signed Radon measure \( Du \),
\[ \| v \|_{T_u X} = \int_{[0, +\infty]} |v| |Du|. \]

Let \( u \in X \) and \( v \in T_u X \). Consider a map \([0, \vartheta^*] \mapsto T_u X, \vartheta \mapsto v^\vartheta \) such that \( v^\vartheta \) is Lipschitzian on \([0, +\infty)\) for all \( \vartheta \in [0, \vartheta^*] \). We say that \( v^\vartheta \) has convergence to \( v \), and write \( v^\vartheta \rightharpoonup v \), whenever
\[ \forall \vartheta, \quad v^\vartheta(0) = 0, \quad \lim_{\vartheta \to 0} v^\vartheta = v \quad \text{in} \ T_u X, \]
(2.2)
\[ \limsup_{\vartheta \to 0} \text{Lip}(\vartheta v^\vartheta) < 1, \quad \lim_{\vartheta \to 0} \| \vartheta v^\vartheta \|_{L^\infty} = 0. \]
(2.3)
Similarly to [7, Definition 2], for \( u \in X \) and \( v^\vartheta \) as in (2.2)–(2.3), with moreover \( \text{Lip}(\vartheta v^\vartheta) \lesssim \vartheta < 1 \), the curve \( \vartheta \mapsto v^\vartheta \ast u \) in \( X \) is implicitly defined by
\[ (v^\vartheta \ast u)(x) = u(x). \]
(2.4)

The above definitions are motivated by the introduction of the following differential structure on \( X \); see [5]. For any \( u \in X \), we identify \( T_u X \) with a class of continuous curves \( \vartheta \mapsto u^\vartheta \) exiting \( u \) modulo the equivalence relation of first order contact. More precisely, for \( u \in X \) we say that the curve in \( X \) given by \( \vartheta \mapsto u^\vartheta \) for \( \vartheta \in [0, \vartheta^*] \), generates the shift tangent vector \( v \in T_u X \) if there exist functions \( v^\vartheta \) satisfying (2.2)–(2.3) with \( v^\vartheta \rightharpoonup v \) and such that
\[ \lim_{\vartheta \to 0} \frac{1}{\vartheta} \left\| u^\vartheta - (v^\vartheta \ast u) \right\|_{L^1([0, +\infty))} = 0 \]
(2.5)
holds. By (2.1), a slight modification of [7, Theorem 1] ensures that for any \( u \in X \), \( v_1, v_2 \in T_u X \) and \( v_1^\vartheta \rightharpoonup v_1, v_2^\vartheta \rightharpoonup v_2 \), it holds that
\[ \lim_{\vartheta \to 0} \frac{1}{\vartheta} \left\| v_1^\vartheta \ast u - v_2^\vartheta \ast u \right\|_{L^1([0, +\infty))} = \| v_1 - v_2 \|_{T_u X}. \]
(2.6)
Hence, shift tangent vectors are well defined and do not depend on the choice of \( v^\vartheta \) in (2.5).

Below, we will be concerned with the solution operator \( P : [0, +\infty] \times X \times X \mapsto X \) of the IBVP (1.1). As a general reference about IBVPs for scalar conservation laws, we refer to [3,15]. Introduce on \( X \times X \) the norm \( \|(\tilde{u}, \tilde{v})\|_{X \times X} = \|\tilde{u}\|_{L^1} + \|\tilde{v}\|_{L^1} \). Correspondingly, for \((\tilde{u}, \tilde{v}) \in X \times X \), let \( T_{(\tilde{u}, \tilde{v})}(X \times X) = L^1(D\tilde{u}) \times L^1(D\tilde{v}) \) with \( \|(\tilde{v}, \tilde{u})\|_{T_{(\tilde{u}, \tilde{v})}(X \times X)} = \|\tilde{v}\|_{T_{\tilde{u}} X} + \|\tilde{u}\|_{T_{\tilde{v}} X} \).

A map \( \Phi : X \times X \mapsto X \) is directionally shift differentiable at \((\tilde{u}, \tilde{v}) \in X \times X \) in the direction \((\tilde{v}, \tilde{u}) \in T_{(\tilde{u}, \tilde{v})}(X \times X) \) if there exists \( v \in T_{\Phi(\tilde{u}, \tilde{v})} X \) such that
\[ \lim_{\vartheta \to 0} \frac{1}{\vartheta} \left\| w^\vartheta \ast \Phi(\tilde{u}, \tilde{v}) - \Phi(v^\vartheta \ast \tilde{u}, v^\vartheta \ast \tilde{v}) \right\|_{L^1([0, +\infty))} = 0 \]
(2.7)
for all \( \bar{v}^0, \tilde{v}^0 \) that hat converge, respectively, to \( \bar{v}, \tilde{v} \) and \( w^0 \) hat converging to \( w \). 

\( \Phi \) is (totally) shift differentiable if there exists a linear map (the shift differential) \( \Lambda_{(\hat{t}, \bar{u}, \tilde{u})} : T_{(\bar{u}, \tilde{u})}X \to T_{(\bar{u}, \tilde{u})}X \) such that (2.7) holds for all \( (\bar{v}, \tilde{v}) \in T_{(\bar{u}, \tilde{u})}X \). When no misunderstanding arises, we simply denote \( \Lambda_{(\hat{t}, \bar{u}, \tilde{u})} \) by \( \Lambda \).

In the sequel, for a fixed positive \( \hat{t} \), we make often use of the following assumption:

\[
\begin{align*}
\text{(H)} & \quad \begin{cases}
U > 0, & \bar{u}, \tilde{u} \in X, \\
\bar{u}(0, \hat{t}) \subseteq [-U, U], & \tilde{u}(0, \hat{t}) \subseteq [-U, U], \\
f \in C^2([\bar{u}, \tilde{u}]), & \\
\inf_{w \in [-U, U]} f''(w) > c_1 > 0, & \text{if } f'(\bar{u}) > c_2 > 0,
\end{cases}
\end{align*}
\]

for suitable constants \( U \) and \( c_1, c_2 \). The latter condition above ensures that generalized characteristics [11] may not be tangent to the boundary \( x = 0 \).

We are now ready to state the main result of this note.

**Theorem 2.1.** Fix a positive \( \hat{t} \) and let (H) hold. Assume that at time \( \hat{t} \) the solution to (1.1) contains neither centers of compression waves nor interactions of shocks. Then the map \( \mathcal{P}_1 : X \times X \mapsto X \) is shift differentiable at \( (\bar{u}, \tilde{u}) \) along all directions \( (\bar{v}, \tilde{v}) \) in \( T_{(\bar{u}, \tilde{u})}X \). Moreover, the shift differential \( \Lambda_{(\hat{t}, \bar{u}, \tilde{u})} \) satisfies

\[
\|\Lambda_{(\hat{t}, \bar{u}, \tilde{u})}\| \leq L
\]

in the operator norm, \( L \) being the Lipschitz constant of \( \mathcal{P}_1 \).

The condition on the absence of interactions among shocks and of compression waves can be stated, as in [7], also in terms of a suitable set \( S_{(\hat{t}, \bar{u}, \tilde{u})}(x) \) constructed by means of the solution to (1.1); see also [15]. Essentially, \( S_{(\hat{t}, \bar{u}, \tilde{u})}(x) \) is the set of the limit points \( \lim_{n \to +\infty} \mathcal{P}_n(\bar{u}, \tilde{u})(\xi_n) \), \( (\tau_n, \xi_n) \in \mathbb{R}^2 \) being a sequence converging to \( (\hat{t}, x) \). The absence of shock interactions and of centers of compression waves assumed in Theorem 2.1 amounts to require that \( S_{(\hat{t}, \bar{u}, \tilde{u})}(x) \) contains at most 2 points. This assumption leads to the exclusion of an at most countable set of times; see [7].

The above regularity result allows to shift differentiate integral functionals defined on the solution to the IBVP (1.1).

**Corollary 2.2.** Fix a positive \( \hat{t} \) and let (H) hold. Let \( t_1, t_2 \in [0, \hat{t}] \) with \( t_1 < t_2 \) be fixed, \( \psi : \mathbb{R} \mapsto \mathbb{R} \) be locally Lipschitzian and \( \varphi : [0, +\infty] \times \mathbb{R} \mapsto [0, +\infty] \) be \( C^1 \) with compact support. Consider the functionals \( J, J_{\psi} : X \times X \mapsto \mathbb{R} \) defined by

\[
J(\bar{u}, \tilde{u}) = \int_{t_1}^{t_2} \int_0^{+\infty} \psi(\mathcal{P}_t(\bar{u}, \tilde{u})(x)) \, dx \, dt
\]

and

\[
J_{\psi}(\bar{u}, \tilde{u}) = \int_0^{+\infty} \int_0^{+\infty} \varphi(t, x) \psi(\mathcal{P}_t(\bar{u}, \tilde{u})(x)) \, dx \, dt,
\]

where \( \mathcal{P}_t \) is the solution to the IBVP (1.1).
$P_t$ being the process generated by (1.1). Then, for all directions $(\tilde{v}, \tilde{v}) \in T_{(\hat{u}, \hat{u})}(X \times X)$ and for all curves $(\tilde{v}, \tilde{v})$ such that $(\tilde{v}, \tilde{v}) \rightarrow (\hat{u}, \hat{v})$, the following first order “shift expansions” hold:

$$J(\tilde{v} \ast \tilde{u}, \tilde{v} \ast \tilde{u}) = J(\hat{u}, \hat{u}) + \vartheta \int_0^{t_1} \int_{x_1}^{x_2} \int_{x_3}^{x_4} f(\hat{u}, \hat{v})(x) D\left(P_t(\hat{u}, \hat{v})\right) dt + o(\vartheta),$$

$$J\vartheta(\tilde{v} \ast \tilde{u}, \tilde{v} \ast \tilde{u}) = J\vartheta(\hat{u}, \hat{u}) + \vartheta \int_0^{t_1} \int_{x_1}^{x_2} \int_{x_3}^{x_4} \psi(t, x) A(\hat{u}, \hat{v})(x) D\left(P_t(\hat{u}, \hat{v})\right) dt$$

$$+ o(\vartheta)$$

for $\vartheta \rightarrow 0$.

Above, by “$\psi$ locally Lipschitzian” we mean that on every compact $K \subset \mathbb{R}$, the restriction of $\psi$ to $K$ is Lipschitzian. Note that the integral on the right-hand side in the expansion above is well defined. In fact, the map $x \mapsto \psi(P_t(\hat{u}, \hat{v})(x))$ is of bounded variation.

Thanks to Theorem 2.1, the proof of Corollary 2.2 is as in [10, Theorem 2.10]. The main ingredients are the shift differentiability of $P_t$, which holds for all but countably many times $t \in [0, \check{t}]$, and (2.8). Indeed, the latter uniform estimate allows to apply Lebesgue theorem, exactly as in [10, Theorem 2.10].

3. Proofs

Throughout this section, $\check{t}$ is a given positive time and hypothesis (H) is assumed for fixed $f, \check{u}$ and $\check{u}$. To simplify the notation, when no misunderstanding may arise we let $u(t, x) = P_t(\check{u}, \check{u})(x)$ and omit the dependence on the various fixed quantities. The following bounded function is of use:

$$\sigma(t, \check{u}, \check{u})(x) = \begin{cases} f'(u(t, x)) & \text{if } u(t, x^+) = u(t, x^-), \\ \frac{f(u(t, x^+)) - f(u(t, x^-))}{u(t, x^+) - u(t, x^-)} & \text{if } u(t, x^+) \neq u(t, x^-), \end{cases}$$

(3.11)

where $u(t, x^\pm) = \lim_{y \to x^\pm} u(t, y)$. Below, we exploit the theory of generalized characteristics, as introduced and developed by Dafermos [11,12]. For $t \in [0, \check{t}]$, call $t \mapsto \chi^+(t; \check{t}, \check{u}, \check{u})$ (respectively, $t \mapsto \chi^-(t; \check{t}, \check{u}, \check{u})$) the maximal (respectively, minimal) Filippov solution [13, Section 4] to

$$\begin{cases} \dot{x} = f'(u(t, x)) \\ \dot{\check{u}} = x. \end{cases}$$

In other words, $\chi^+$ and $\chi^-$ are the extremal backward characteristics through $(\check{t}, \check{u})$. They can either intersect the $x$-axis $t = 0$ at, say, $\xi^-(\check{t}, \check{u}, \check{u})$ (respectively, $\xi^+(\check{t}, \check{u}, \check{u})$), or the $t$-axis $x = 0$ at, say, $\tau^- (\check{t}, \check{u}, \check{u})$ (respectively, $\tau^+(\check{t}, \check{u}, \check{u})$). If $\check{x}$ is a point of continuity of the map $x \mapsto u(t, x)$, then it is well known that $\chi^+(t; \check{t}, \check{u}, \check{u}) = \chi^-(t; \check{t}, \check{u}, \check{u})$ and, hence, the coinciding characteristics will be denoted simply by $\chi$. The same notation
will be used both for $\xi^{\pm}$ and $\tau^{\pm}$; see Fig. 1. For the sake of simplicity, when no ambiguity arises, the explicit dependence of $\chi^{\pm}$, $\xi^{\pm}$ and $\tau^{\pm}$ from the initial and boundary data will be omitted.

Let $\mathcal{X}$ be the set of those points $\hat{x}$ such that both the extremal backward characteristics from $(\hat{t}, \hat{x})$ intersect the space axis $t = 0$. Similarly, let $\mathcal{T}$ be the set of those points $\hat{x}$ such that both the extremal backward characteristics from $(\hat{t}, \hat{x})$ intersect the time axis $x = 0$. Finally, $\mathcal{M}$ is the set of points $(\hat{t}, \hat{x})$ having the minimal characteristic intersecting the time axis and the maximal one impinging on the space axis; see Fig. 1. If both the maximal characteristics coming from $(\hat{t}, \hat{x})$ reach the origin, we let $\hat{x}$ belong to $\mathcal{M}$.

Exactly as in [7], one can prove the following lemma.

**Lemma 3.1.** Fix a positive $\hat{t}$ and let (H) hold. Assume that at time $\hat{t}$ the solution to (1.1) contains neither centers of compression waves nor interactions of shocks. Let $\hat{x}$ be a point of discontinuity of $x \mapsto u(\hat{t}, x)$, with $\omega^{\pm} = u(\hat{t}, \hat{x}^{\pm})$ and $\Delta \omega = \omega^{+} - \omega^{-}$. Assume $\hat{x} \in \mathcal{X}$. Choose a suitable $\bar{v}$ in $\mathcal{T}$ such that there exists a curve $\vartheta \mapsto \bar{v}^{\vartheta}$ with $\bar{v}^{\vartheta} \wedge \bar{v}$ and moreover

\[
\begin{align*}
\bar{v}^{\vartheta}(x) &= \alpha \quad \text{if } x \leq a', \\
\bar{v}^{\vartheta}(x) &= \beta \quad \text{if } x \geq b',
\end{align*}
\] (3.12)

where $a := \xi^{-}(\hat{t}, \hat{x}) < a' < b' < \xi^{+}(\hat{t}, \hat{x}) := b$. Define $u^{\vartheta}(t, x) := \mathcal{T}_{\vartheta}(\bar{u}, \bar{v}^{\vartheta} \star \bar{u})(x)$. Then, the following facts hold:

1. $x \mapsto u^{\vartheta}(\hat{t}, x)$ has a jump at $\hat{x}^{\vartheta}$ and this point satisfies

\[
\lim_{\vartheta \to 0} \frac{\hat{x}^{\vartheta} - \hat{x}}{\Delta \omega} = \frac{1}{\int_{a,b}} \left\{ \int_{[a,b]} \bar{v}D\bar{u} + \alpha \left[ \bar{u}(a^{+}) - \omega^{-} \right] + \beta \left[ \omega^{+} - \bar{u}(b^{-}) \right] \right\}.
\] (3.13)

2. There exists a positive $M$ such that for $\vartheta$ small, $|\hat{x}^{\vartheta} - \hat{x}| \leq M \vartheta$.

3. The quantity

\[
\Delta_{\vartheta,M} := \sup_{x \in [\hat{t} - M \vartheta, \hat{t} \vartheta]} |\omega^{-} - u^{\vartheta}(\hat{t}, x)| + \sup_{[\hat{t}^{+}, \hat{t} + M \vartheta]} |\omega^{+} - u^{\vartheta}(\hat{t}, x)|
\]

\[
+ \sup_{x \in [\hat{t} - M \vartheta, \hat{t} \vartheta]} |\omega^{-} - u(\hat{t}, x)| + \sup_{[\hat{t}^{+}, \hat{t} + M \vartheta]} |\omega^{+} - u(\hat{t}, x)|
\]

approaches zero as $\vartheta \to 0$.

A slightly different procedure leads to the following result, analogous to the previous one but referred to points of jump in $\mathcal{T}$. 
Lemma 3.2. Fix a positive \( \hat{t} \) and let (H) hold. Assume that at time \( \hat{t} \) the solution to (1.1) contains neither centers of compression waves nor interactions of shocks. Let \( \hat{x} \) be a point of discontinuity of \( x \mapsto u(\hat{t}, x) \), with \( \omega^\pm = u(\hat{t}, \hat{x}^\pm) \) and \( \Delta \omega = \omega^+ - \omega^- \). Assume \( \hat{x} \in T \).

Choose a suitable \( \tilde{v} \) in \( T \) such that there exists a curve \( \tilde{v}^\vartheta \) with \( \tilde{v}^\vartheta \nearrow \tilde{v} \) and moreover

\[
\begin{align*}
\tilde{v}^\vartheta(t) &= \alpha \quad \text{if } t \leq a', \\
\tilde{v}^\vartheta(t) &= \beta \quad \text{if } t \geq b',
\end{align*}
\]

where we define \( a := \tau^+(\hat{t}, \hat{x}) < a' < b' < \tau^-(\hat{t}, \hat{x}) := b \). Define \( u^\vartheta(t, x) = T_t(\tilde{v}^\vartheta \ast \bar{u})(x) \). Then, the following facts hold:

1. \( x \mapsto u^\vartheta(\hat{t}, x) \) has a jump at \( \hat{x}^\vartheta \) and this point satisfies

\[
\lim_{\vartheta \to 0} \hat{x}^\vartheta - \hat{x} = z_0 = \frac{1}{\Delta \omega} \left\{ \int_{[a,b]} \tilde{v} D(f \circ \bar{u}) + \alpha \left[ f(\bar{u}(a^+)) - f(\omega^+) \right] \ight.
\]

\[
+ \beta \left[ f(\omega^-) - f(\bar{u}(b^-)) \right] \}.
\]

2. There exists a positive \( M \) such that for \( \vartheta \) small, \( |\hat{x}^\vartheta - \hat{x}| \leq M \vartheta \).

3. The quantity

\[
\Delta \vartheta, M := \sup_{x \in [\hat{x} - M \vartheta, \hat{x} + M \vartheta]} |\omega^- - u^\vartheta(\hat{t}, x)| + \sup_{[\hat{x}^\vartheta, \hat{x} + M \vartheta]} |\omega^+ - u^\vartheta(\hat{t}, x)|
\]

\[
+ \sup_{x \in [\hat{x} - M \vartheta, \hat{x}]} |\omega^- - u(\hat{t}, x)| + \sup_{[\hat{x}, \hat{x} + M \vartheta]} |\omega^+ - u(\hat{t}, x)|
\]

approaches zero as \( \vartheta \to 0 \).

Proof. The statements (2) and (3) are proved as in Lemma 3.1, with \( M \) depending on the constant \( c_2 \) introduced in (H).

Consider (1). Exchanging the roles of \( t \) and \( x \), an application of Lemma 3.1 ensures that the position of the discontinuity shifts vertically to a point \( \hat{t}^\vartheta \) such that

\[
\lim_{\vartheta \to 0} \frac{\hat{t}^\vartheta - \hat{t}}{\vartheta} = \frac{1}{\Delta f} \left\{ \int_{[a,b]} \tilde{v} D(f \circ \bar{u}) + \alpha \left[ f(\bar{u}(a^+)) - f(\omega^-) \right] \ight.
\]

\[
+ \beta \left[ f(\omega^+) - f(\bar{u}(b^-)) \right] \}.
\]

where \( \Delta f = f(u(\hat{t}^-, \hat{x})) - f(u(\hat{t}^+, \hat{x})) \). Note that, for \( \vartheta \) small,

\[
\hat{x}^\vartheta - \hat{x} = (\hat{t}^\vartheta - \hat{t}) \cdot \alpha(\hat{t}^\vartheta, \hat{x}^\vartheta \ast \bar{u}(\hat{x}^\vartheta))
\]

so that

\[
\lim_{\vartheta \to 0} \frac{\hat{x}^\vartheta - \hat{x}}{\vartheta} = \lim_{\vartheta \to 0} \frac{\hat{t}^\vartheta - \hat{t}}{\vartheta} \alpha(\hat{t}^\vartheta, \hat{x}^\vartheta \ast \bar{u})(\hat{x}^\vartheta)
\]
and (1) follows from (3) and (3.11) which, due to the smoothness of $f$, ensure that
$$\lim_{\sigma \to 0} \sigma(\hat{\tau}, \hat{\psi}^{\sigma}(\hat{u}, \hat{v})) = \sigma(\hat{\tau}, \hat{u}, \hat{v}).$$

The next result reduces the set of directions that need to be considered to show that $\mathcal{P}_i$ is shift differentiable. We omit the proof since it is a simple modification of [7, Theorem 3].

**Lemma 3.3.** Fix $(\hat{u}, \hat{v})$ in $X \times X$ and choose a dense subset $Y$ of $T_u(\hat{u}, \hat{v})(X \times X)$. Let $\Phi : X \times X \mapsto X$ be locally Lipschitz continuous and shift differentiable at $(\hat{u}, \hat{v})$ along all directions $(\hat{\psi}, \hat{v}) \in Y$. Then, $\Phi$ is shift differentiable at $(\hat{u}, \hat{v})$ along all directions $(\hat{\psi}, \hat{v})$ in $T_u(\hat{u}, \hat{v})(X \times X)$.

A careful choice of the space $Y$ considerably simplifies the computations. To this aim, define the sets

$$B_1 := \{ x \in [0, +\infty[ : u(\hat{t}, x^+) \neq u(\hat{t}, x^-) \},$$

$$\tilde{A}_1 := \{ t \in [0, \hat{t}]: \hat{u}(t^+) \neq \hat{u}(t^-) \},$$

$$\tilde{A}_2 := \{ t \in [0, \hat{t}]: t = \tau^-(\hat{t}, y) \text{ or } t = \tau^+(\hat{t}, y) \text{ for some } y \in B_1 \cap T \},$$

$$A := \{ x \in [0, +\infty[ : x = \xi^-(\hat{t}, y) \text{ or } x = \xi^+(\hat{t}, y) \text{ for some } y \in B_1 \cap X \},$$

$$Y := \{ (\hat{\psi}, \hat{v}) \in T_u(\hat{u}, \hat{v})(X \times X): \hat{v} = \sum_{i=1}^{n} \tilde{\alpha}_i \chi_{[\tilde{y}_i, \tilde{y}_i]} \hat{\psi} = \sum_{i=1}^{n} \tilde{\alpha}_i \chi_{[\tilde{y}_i, \tilde{y}_i]} \},$$

where $n \geq 1$ and for $i = 1, \ldots, n$ we set $\tilde{\alpha}_i, \tilde{\alpha}_i \in \mathbb{R}$, $\tilde{y}_i, \tilde{y}_i \in [0, +\infty[$, $\tilde{y}_i \notin (\tilde{A}_1 \cup \tilde{A}_2)$, $\tilde{y}_i \notin (\hat{A}_1 \cup \hat{A}_2)$ and $\tilde{y}_0 = \hat{y}_0 = 0$. $Y$ is dense in $T_u(\hat{u}, \hat{v})(X \times X)$.

The shift differential $A(\hat{\tau}, \hat{u}, \hat{v})$ is defined as follows:

$$A(\hat{\tau}, \hat{u}, \hat{v})(\hat{\psi}, \hat{v})(x) := \begin{cases} \psi_1(x), & x \in X \cap B_1, \\ \psi_2(x), & x \in T \cap B_1, \\ \psi_3(x), & x \in M \cap B_1, \\ 0, & x \in M \setminus B_1, \end{cases}$$

where, setting $\Delta u(\hat{t}, x) = u(\hat{t}, x^+) - u(\hat{t}, x^-)$,

$$\psi_1(x) := \frac{1}{\Delta u(\hat{t}, x)} \left\{ \int_{[\xi^-(\hat{t}, x), \xi^+(\hat{t}, x)]} \hat{v} \hat{D}\hat{u} \\ + \hat{v}(\xi^-)(\hat{t}, x)[\hat{u}(\xi^-(\hat{t}, x)^+) - u(\hat{t}, x^-)] \\ + \hat{v}(\xi^+(\hat{t}, x))[u(\hat{t}, x^+) - \hat{u}(\xi^+(\hat{t}, x)^+)] \right\}.$$
Lemma 3.4. Fix a positive \( \hat{t} \) and let (H) hold. Then, the function \( x \mapsto \Lambda(\hat{v}, \hat{\nu})(x) \) is piecewise constant and \( \Lambda(\hat{v}, \hat{\nu}) \) is in \( T_u(X) \). Moreover, for any fixed \( \hat{x} \in [0, +\infty[ \), if \( \hat{v}(x) = \alpha \) for all \( x \in [\hat{\xi}(-\hat{t}, \hat{x}), \hat{\xi}^+(\hat{t}, \hat{x})] \), then \( \Lambda(\hat{v}, \hat{\nu})(\hat{x}) = \alpha \). Similarly, if \( \hat{v}(t) = \alpha \) for \( t \in [\hat{\tau}^+(-\hat{t}, \hat{x}), \hat{\tau}^-(\hat{t}, \hat{x})] \), then \( \Lambda(\hat{v}, \hat{\nu})(\hat{x}) = \alpha \cdot \sigma_{(\hat{t}, \hat{\nu}, \hat{\nu})}(\hat{x}) \).

Proof. Compute

\[
\int_{\mathbb{R}} |\Lambda(\hat{v}, \hat{\nu})| \, |Du| = \int_{B_1 \cap X} |\psi_1| \, |Du| + \int_{X \setminus B_1} |\hat{v}(\hat{\xi}(\hat{t}, \hat{x}))| \, |Du| \\
+ \int_{B_1 \cap T} |\psi_2| \, |Du| + \int_{T \setminus B_1} |\hat{v}(\hat{\tau}(\hat{t}, \hat{x}))| \, |Du| + \int_{B_1 \cap X} |\psi_3| \, |Du|.
\]

Consider the various summands separately:

\[
\int_{B_1 \cap X} |\psi_1| \, |Du| \leq \|\hat{v}\|_{L^1(B_\delta)} + \|\hat{v}\|_\infty \sum_{x \in B_1 \cap X} |D\hat{u}([\xi^-(\hat{t}, x))]) \\
+ \|\hat{v}\|_\infty \sum_{x \in B_1 \cap X} |D\hat{u}([\xi^+(\hat{t}, x))]) \\
\leq \|\hat{v}\|_{L^1(B_\delta)} + 2\|\hat{v}\|_\infty |D\hat{u}|([0, +\infty[) < +\infty,
\]

\[
\int_{X \setminus B_1} |\hat{v}(\hat{\xi}(\hat{t}, x))| \, |Du| \leq \|\hat{v}\|_\infty |Du|(|\mathbb{R}|) < +\infty,
\]
\[
\int_{B_1 \cap T} |\psi_2| |Du| \leq \|\tilde{\psi}\|_{L^1(D(\tilde{f}(\tilde{u})))} + \|\tilde{\psi}\|_\infty \sum_{x \in B_1 \cap T} |D(f \circ \tilde{u})(\{(\tau^+(\hat{t}, \hat{x})\})
+ \|\tilde{\psi}\|_\infty \sum_{x \in B_1 \cap T} |D(f \circ \tilde{u})(\{(\tau^-(\hat{t}, \hat{x})\})
\leq \|\tilde{\psi}\|_{L^1(D(\tilde{f}(\tilde{u})))} + 2\|\tilde{\psi}\|_\infty |D(f \circ \tilde{u})|[0, \hat{t}) < +\infty,
\int_{T \setminus B_1} |\tilde{\psi}(\tau(\hat{t}, \hat{x}))| |Du| \leq \|\tilde{\psi}\|_\infty |Du|[0, \infty) < +\infty,
\]
and with similar computations we have
\[
\int_{B_1 \cap \mathcal{M}} |\psi_3| |Du| < +\infty.
\]

Suppose now that \( \hat{x} \in B_1 \), the other case being immediate. If \( \tilde{v}(x) = \alpha \) for all \( x \in [\tilde{\xi}^-(\hat{t}, \hat{x}), \tilde{\xi}^+(\hat{t}, \hat{x})] \), then \( A(\tilde{v}, \tilde{v})(\hat{x}) = \psi_1(\hat{x}) = \alpha \). On the other hand, if \( \tilde{v}(t) = \alpha \) for all \( t \in [\tau^+(\hat{t}, \hat{x}), \tau^-(\hat{t}, \hat{x})] \), then

\[
A(\tilde{v}, \tilde{v})(\hat{x}) = \frac{\alpha}{\Delta v(\hat{t}, \hat{x})} \{D(f \circ \tilde{u})([\tau^+(\hat{t}, \hat{x}), \tau^-(\hat{t}, \hat{x})]) - f(\tilde{u}(\tau^-(\hat{t}, \hat{x})))
+ f(u(\hat{t}, \hat{x}^-)) - f(u(\hat{t}, \hat{x}^+)) + f(\tilde{u}(\tau^+(\hat{t}, \hat{x}^+)))
\]
\[
= \alpha \cdot \sigma_{\tilde{v}(\hat{t}, \hat{x})}(\hat{x}). \quad \square
\]

Fix \( (\tilde{v}, \tilde{v}) \in Y \), with \( Y \) as in (3.16), and introduce the intervals
\[
\hat{I}_1 := [\hat{y}_1 - \vartheta (4\|\tilde{v}\|_\infty + |\tilde{a}_1|), \hat{y}_1 - 4\vartheta \|\tilde{v}\|_\infty],
\hat{I}_i := [\hat{y}_1 - \vartheta |\tilde{a}_i - \tilde{a}_{i+1}|, \hat{y}_1 + \vartheta |\tilde{a}_i - \tilde{a}_{i+1}|], \quad i \geq 1,
\hat{I}_0 := [\hat{y}_1 - \vartheta (4\|\tilde{v}\|_\infty + |\tilde{a}_1|), \hat{y}_1 - 4\vartheta \|\tilde{v}\|_\infty],
\hat{I}_i := [\hat{y}_1 - \vartheta |\tilde{a}_i - \tilde{a}_{i+1}|, \hat{y}_1 + \vartheta |\tilde{a}_i - \tilde{a}_{i+1}|], \quad i \geq 1,
\]
and define the family of functions
\[
\tilde{v}^0(x) := \begin{cases}
0 & \text{if } x \in [0, \hat{y}_1 - \vartheta (4\|\tilde{v}\|_\infty + |\tilde{a}_1|)],
\hat{a}_1 + \frac{\operatorname{sgn} \vartheta}{2\vartheta} (x - \hat{y}_1 + 4\vartheta \|\tilde{v}\|_\infty) & \text{if } x \in \hat{I}_0, \\
\hat{v}(x) & \text{if } x \in \hat{I}_1, \quad \text{otherwise},
\end{cases}
\]
\[
\tilde{v}^0(t) := \begin{cases}
0 & \text{if } t \in [0, \hat{y}_1 - \vartheta (4\|\tilde{v}\|_\infty + |\tilde{a}_1|)],
\hat{a}_1 + \frac{\operatorname{sgn} \vartheta}{2\vartheta} (t - \hat{y}_1 + 4\vartheta \|\tilde{v}\|_\infty) & \text{if } t \in \hat{I}_0, \\
\hat{v}(t) & \text{if } t \in \hat{I}_i, \quad \text{otherwise},
\end{cases}
\]
(3.17)
(3.18)

From the above definitions, it follows that \( (\tilde{v}^0, \tilde{v}^0) \xrightarrow{\hat{\omega}} (\tilde{v}, \tilde{v}) \), \( \tilde{v}^0(0) = 0 \), \( \tilde{v}^0(0) = 0 \) and, for small \( \vartheta \), that.
Proof. We consider separately the discontinuity points and two types of continuity points of $u$. For $(\tilde{v}, \tilde{v})$ in $Y$, let $B_1$ as in (3.15) and

$$B_2(\tilde{v}, \tilde{v}) := \{x \in [0, +\infty[ : x \notin B_1, \, \xi(\hat{t}, x) \neq \hat{y}_t \text{ and } \tau(\hat{t}, x) \neq \hat{y}_i, \, \forall i \geq 0\},$$

$$B_3(\tilde{v}, \tilde{v}) := [0, +\infty[ \setminus (B_1 \cup B_2(\tilde{v}, \tilde{v})),$$

Lemma 3.5. There exists $C > 0$ such that if $\hat{x} \in B_2(\tilde{v}, \tilde{v})$, then there exists $\delta > 0$ such that for any path $\vartheta \mapsto (\tilde{u}^\vartheta, \tilde{u}^\vartheta)$ and $\vartheta \mapsto z^\vartheta$ which generate the shift tangent vectors $(\tilde{v}, \tilde{v})$ and $\Lambda(\tilde{v}, \tilde{v})$, setting $u^\vartheta(\hat{t}, x) = \mathcal{P}_t(\tilde{u}^\vartheta, \tilde{u}^\vartheta)(x)$ we have

$$\limsup_{\vartheta \to 0} \frac{1}{\delta} \int_{\hat{t}-\delta}^{\hat{t}+\delta} |u^\vartheta(\hat{t}, x) - z^\vartheta(\hat{t}, x)| \, dx \leq C \cdot \sum_{x \in [\hat{t}-2\delta, \hat{t}+2\delta]} |\Delta u(\hat{t}, x)|.$$

Proof. If $\hat{x} \in B_2(\tilde{v}, \tilde{v}) \cap \mathcal{X}$ then the proof is in [7, Lemma 10]. Since $B_2(\tilde{v}, \tilde{v}) \cap \mathcal{M} = \emptyset$, we are left to consider the case $\hat{x} \in B_2(\tilde{v}, \tilde{v}) \cap T$. Let $j$ be such that $\tau(\hat{t}, \hat{x}) \in [\hat{y}_j, \hat{y}_{j+1}[. \text{ The map } x \mapsto \tau(\hat{t}, x) \text{ is continuous at } \hat{x}, \text{ hence there exists } \delta \in (0, 1) \text{ such that}$

$$\hat{y}_j - \tau^-(\hat{t}, \hat{x} - 2\delta) < \tau^+(\hat{t}, \hat{x} + 2\delta) < \hat{y}_{j+1}.$$

From Lemma 3.4, it follows that $\Lambda(\tilde{v}, \tilde{v})(x) = \tilde{a}_j \cdot \sigma(\vartheta, \hat{t}, \hat{x})(x)$ for all $x \in [\hat{t} - 2\delta, \hat{t} + 2\delta]$. By [10, Lemma 2.2] we may choose $w^\vartheta$ such that

$$w^\vartheta \rightharpoonup \Lambda(\tilde{v}, \tilde{v}) \quad \text{and} \quad \|w^\vartheta\|_\infty \leq \|\tilde{v}\|_\infty + \|\tilde{v}\|_\infty.$$

The proof is completed considering the following two cases.

Case 1. If $\tilde{a}_j < 0$, then we claim that

$$\lim_{\vartheta \to 0} \frac{1}{\delta} \int_{\hat{t}-\delta}^{\hat{t}+\delta} \mathcal{P}_t(\tilde{u}^\vartheta, \tilde{u}^\vartheta)(x) - w^\vartheta \ast u(\hat{t}, x) \, dx = 0.$$

Indeed we have $\mathcal{P}_t(\tilde{u}^\vartheta, \tilde{u}^\vartheta)(y) = u(\hat{t} - \vartheta \tilde{a}_j, y) = u(\hat{t}, y + \vartheta \tilde{a}_j \cdot \sigma(\vartheta, \hat{t}, \hat{x})(y))$, while $w^\vartheta \ast u(\hat{t}, y) = u(\hat{t}, y + \vartheta \chi_\vartheta(y))$ for all $y \in [\hat{t} - \delta, \hat{t} + \delta]$, $\vartheta$ sufficiently small and where $y + \vartheta \chi_\vartheta(y)$ is the inverse function of $x \mapsto x + \vartheta u^\vartheta(x)$. Hence we have

$$\frac{1}{\vartheta} \int_{\hat{t}-\delta}^{\hat{t}+\delta} \mathcal{P}_t(\tilde{u}^\vartheta, \tilde{u}^\vartheta)(y) - w^\vartheta \ast u(\hat{t}, y) \, dy \quad \text{and} \quad \frac{1}{\vartheta} \int_{\hat{t}-\delta}^{\hat{t}+\delta} u(\hat{t}, y + \vartheta \tilde{a}_j \cdot \sigma(\vartheta, \hat{t}, \hat{x})(y)) - u(\hat{t}, y + \vartheta \chi_\vartheta(y)) \, dy \quad \text{and} \quad \frac{1}{\vartheta} \int_{\hat{t}-\delta}^{\hat{t}+\delta} u(\hat{t}, y + \vartheta \chi_\vartheta(y)) \, dy \quad \text{and} \quad$$
Arguing as in Case 1, one can deduce that

\[ \lim_{\vartheta \to 0} \int_{\hat{\tau} - \delta}^{\hat{\tau} + \delta} \left| \tilde{a}_j \sigma_i (t, \hat{\tau}, \hat{\alpha}) (y) - x_{\vartheta} (y) \right| \frac{\left| u (\hat{\tau}, y + \vartheta \alpha_j \cdot \sigma_i (\hat{\tau}, \hat{\alpha}) (y)) - u (\hat{\tau}, y + \vartheta x_{\vartheta} (y)) \right|}{\vartheta \left| \tilde{a}_j \sigma_i (t, \hat{\tau}, \hat{\alpha}) (y) - x_{\vartheta} (y) \right|} \, dy \]

\[ \leq \int_{\hat{\tau} - \delta}^{\hat{\tau} + \delta} \left| \tilde{a}_j \sigma_i (t, \hat{\tau}, \hat{\alpha}) (y) - x_{\vartheta} (y) \right| |Du| ([\hat{\tau} - \delta, \hat{\tau} + \delta]) \, dy. \]

Recalling that the total variation of \( u \) is bounded and using (3.19), we deduce that

\[ \lim_{\vartheta \to 0} \int_{\hat{\tau} - \delta}^{\hat{\tau} + \delta} \left| P_t (\hat{\theta} \cdot \hat{\alpha}) (y) - w^{\vartheta} \star u (\hat{\tau}, y) \right| \, dy = 0. \]

This yields

\[ \lim_{\vartheta \to 0} \sup_{\hat{\tau} - \delta}^{\hat{\tau} + \delta} \int_{\hat{\tau} - \delta}^{\hat{\tau} + \delta} \left| u^{\vartheta} (\hat{\tau}, x) - z^{\vartheta} (\hat{\tau}, x) \right| \, dx \]

\[ \leq \lim_{\vartheta \to 0} \sup_{\hat{\tau} - \delta}^{\hat{\tau} + \delta} \frac{1}{\vartheta} \left\| u^{\vartheta} - P_t (\hat{\theta} \cdot \hat{\alpha}) (y) \right\| \left. \| L^1 ([0, +\infty]) \right. \]

\[ + \lim_{\vartheta \to 0} \sup_{\hat{\tau} - \delta}^{\hat{\tau} + \delta} \frac{1}{\vartheta} \left\| P_t (\hat{\theta} \cdot \hat{\alpha}) (y) - w^{\vartheta} \star u \right\| \left. \| L^1 ([0, +\infty]) \right. \]

\[ + \lim_{\vartheta \to 0} \sup_{\hat{\tau} - \delta}^{\hat{\tau} + \delta} \frac{1}{\vartheta} \left\| w^{\vartheta} \star u - w^{\vartheta} \right\| \left. \| L^1 ([0, +\infty]) \right. \]

\[ = 0. \]

Case 2. Consider \( \hat{a}_j > 0 \). Since \( \hat{a} \) is small, the interval \([\hat{\tau} - 2\delta, \hat{\tau} + 2\delta]\) at time \( \hat{\tau} \) contains the dependence domain of the interval \([\hat{\tau} - \delta, \hat{\tau} + \delta]\) at time \( \hat{\tau} - \vartheta \hat{a}_j \). We divide the interval \([\hat{\tau} - \delta, \hat{\tau} + \delta]\) into two subsets

\[ A = \{ x \in [\hat{\tau} - \delta, \hat{\tau} + \delta]: \exists y \in [\hat{\tau} - 2\delta, \hat{\tau} + 2\delta] \text{ with } x = P^+ (\hat{\tau}, \hat{\tau} - \vartheta \hat{a}_j, y), x = P^- (\hat{\tau}, \hat{\tau} - \vartheta \hat{a}_j, y) \}, \]

\[ B = [\hat{\tau} - \delta, \hat{\tau} + \delta] \setminus A, \]

where \( P^+ (\hat{\tau}, \hat{\tau} - \vartheta \hat{a}_j, y) \) (respectively, \( P^- (\hat{\tau}, \hat{\tau} - \vartheta \hat{a}_j, y) \)) is the intersection of the maximal (respectively, minimal) backward characteristic from the point \((\hat{\tau}, y)\) with the line \( \tau = \hat{\tau} - \vartheta \hat{a}_j \). We want to prove that

\[ \lim_{\vartheta \to 0} \sup_{\hat{\tau} - \delta}^{\hat{\tau} + \delta} \frac{1}{\vartheta} \int_{\hat{\tau} - \delta}^{\hat{\tau} + \delta} \left| P_t (\hat{\theta} \cdot \hat{\alpha}) (x) - w^{\vartheta} \star u (\hat{\tau}, x) \right\| \, dx \leq C \cdot \sum_{x \in [\hat{\tau} - \delta, \hat{\tau} + \delta]} |\Delta u (\hat{\tau}, x)|. \]

Arguing as in Case 1, one can deduce that

\[ \lim_{\vartheta \to 0} \frac{1}{\vartheta} \int_{[\hat{\tau} - \delta, \hat{\tau} + \delta] \setminus A} \left| P_t (\hat{\theta} \cdot \hat{\alpha}) (x) - w^{\vartheta} \star u (\hat{\tau}, x) \right\| \, dx = 0 \]
while concerning the integral over the set \( B \), note that
\[
\text{meas}(B) \leq 2\vartheta \cdot \max_{x \in [-\vartheta, \vartheta]} |f'(u(\hat{t}, x))| \cdot \sum_{x \in [\hat{\delta}, \hat{\delta}]} |\Delta u(\hat{t}, x)|
\]
hence
\[
\lim_{\vartheta \to 0} \frac{1}{\vartheta} \int_{[\hat{\delta} - \delta, \hat{\delta} + \delta] \cap B} \left| \mathcal{P}_{\vartheta}(v^\vartheta \ast \tilde{u}, \tilde{u})(x) - w^\vartheta \ast u(\hat{t}, x) \right| dx \leq C \cdot \sum_{x \in [\hat{\delta} - \delta, \hat{\delta} + \delta]} |\Delta u(\hat{t}, x)|
\]
and the proof is complete. \( \square \)

**Lemma 3.6.** If \( \hat{x} \in B_1(\hat{v}, \bar{v}) \), then for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for any path \( \vartheta \mapsto (\tilde{u}^\vartheta, \tilde{v}^\vartheta) \) and \( \vartheta \mapsto z^\vartheta \) which generate the shift tangent vectors \((\hat{v}, \bar{v}) \) and \( A(\hat{v}, \bar{v}) \), setting \( u^\vartheta(\hat{t}, x) = \mathcal{P}_{\vartheta}(\tilde{v}^\vartheta \ast \tilde{u}, \tilde{u})(x) \) we have
\[
\limsup_{\vartheta \to 0} \frac{1}{\vartheta} \int_{\hat{\delta} - \delta}^{\hat{\delta} + \delta} \left| u^\vartheta(\hat{t}, x) - z^\vartheta(\hat{t}, x) \right| dx \leq \varepsilon.
\]

**Proof.** If \( y \in B_1(\hat{v}, \bar{v}) \cap X \) then the proof is in [7, Lemma 11].

Suppose that \( \hat{x} \in B_3(\hat{v}, \bar{v}) \cap T \). By assumptions, \( \tau(\hat{t}, \hat{x}) = \tilde{y}_j \) for some \( j \). Since \( \hat{x}, \tilde{y}_j \) are continuity points for \( u \) and \( \bar{u} \), respectively, there exists \( \delta' > 0 \) such that \( \hat{x} - \delta' \) is a continuity point for \( u \) and there holds
\[
|Du((\hat{x} - \delta', \hat{x}))| < \frac{\varepsilon}{7\|\tilde{v}\|_\infty}, \quad |D\bar{u}|((\tilde{y}_j, \tau(\hat{t}, \hat{x} - \delta'))| \leq \frac{\varepsilon}{7\|\tilde{v}\|_\infty}.
\]
Moreover, we can choose \( \delta' \) small enough so that \( \tilde{y}_j < \tau(\hat{t}, \hat{x} - \delta') < \tilde{y}_{j + 1} \). Now take \( y' \in [\tilde{y}_j, \tau(\hat{t}, \hat{x} - \delta') \setminus (\tilde{A}_1 \cup \tilde{A}_2) \) and define
\[
\hat{v}(t) = \begin{cases} 
\tilde{y}_{j - 1} & \text{if } t \in [\tilde{y}_j, y'], \\
\tilde{v}(t) & \text{otherwise}.
\end{cases}
\]
It is possible to verify that \( (\hat{v}, \bar{v}) \in Y \),
\[
A(\hat{v}, \bar{v})(x) = A(\tilde{v}, \bar{v})(x) \quad \text{if } x \notin [\hat{x} - \delta', \hat{x}]
\]
and \( \hat{x} \in B_2(\hat{v}, \bar{v}) \). If \( z^\vartheta \rightharpoonup A(\hat{v}, \bar{v}) \), we can thus apply Lemma 3.5 and using (2.6) we obtain the existence of some \( \delta \in [0, \delta'] \) such that
\[
\limsup_{\vartheta \to 0} \frac{1}{\vartheta} \int_{\hat{\delta} - \delta}^{\hat{\delta} + \delta} \left| u^\vartheta(\hat{t}, x) - z^\vartheta(\hat{t}, x) \right| dx
\]
\[
\leq \limsup_{\vartheta \to 0} \frac{1}{\vartheta} \int_{\hat{\delta} - \delta}^{\hat{\delta} + \delta} \left| \mathcal{P}_{\vartheta}(\tilde{v}^\vartheta \ast \tilde{u}, \tilde{u})(x) - z^\vartheta \ast u(\hat{t}, x) \right| dx
\]
\[
+ \limsup_{\vartheta \to 0} \frac{1}{\vartheta} \left\| u^\vartheta - \mathcal{P}_{\vartheta}(\tilde{v}^\vartheta \ast \tilde{u}, \tilde{u}) \right\|_{L^1([0, +\infty])} + \left\| z^\vartheta \ast u - z^\vartheta \right\|_{L^1([0, +\infty])}
\]
\[ \leq \| \tilde{v} - \hat{v} \|_{L^1(Du)} + \| A(\tilde{v}, \tilde{v}) - A(\hat{v}, \hat{v}) \|_{L^1(Du)} \]
\[ \leq 2\| \tilde{v} \|_{\infty} |D\hat{u}|([\bar{y}_j, \tau(\hat{t}, \hat{x} - \delta')]) + \int_{[\hat{x} - \delta', \hat{x}]} \left( |A(\tilde{v}, \tilde{v})(x)| + |A(\hat{v}, \hat{v})(x)| \right) |Du| \]
\[ \leq \frac{2\varepsilon}{7} + 2\| \tilde{v} \|_{\infty} |D\hat{u}|([\hat{x} - \delta', \hat{x}]) + \sum_{x \in B_i(\hat{t}, \hat{x} - \delta')} |A(\hat{v}, \hat{v})(x)| |Du|([x]) \]
\[ \leq \frac{4\varepsilon}{7} + 3\| \tilde{v} \|_{\infty} |D\hat{u}|([\bar{y}_j, \tau(\hat{t}, \hat{x} - \delta')]) \leq \varepsilon. \]

Suppose now that \( \hat{x} \in B_2(\hat{v}, \tilde{v}) \cap M. \) By assumptions, \( \tau(\hat{t}, \hat{x}) = \xi(\hat{t}, \hat{x}) = \bar{y}_1 = \bar{y}_1 = 0. \) Since \( \hat{x}, \bar{y}_1, \bar{y}_1 \) are continuity points for \( u, \tilde{u} \) and \( \hat{u}, \) respectively, there exists \( \delta' > 0 \) such that \( \hat{x} - \delta' \) and \( \hat{x} + \delta' \) are continuity points for \( u \) and there holds

\[ |Du|([\hat{x} - \delta', \hat{x}]) < \frac{\varepsilon}{14\| \tilde{v} \|_{\infty}}, \quad |Du|([\hat{x}, \hat{x} + \delta']) < \frac{\varepsilon}{14\| \tilde{v} \|_{\infty}}, \]
\[ |D\hat{u}|([0, \tau(\hat{t}, \hat{x} - \delta')]) < \frac{\varepsilon}{14\| \tilde{v} \|_{\infty}}, \quad |D\hat{u}|([0, \xi(\hat{t}, \hat{x} + \delta')]) < \frac{\varepsilon}{14\| \tilde{v} \|_{\infty}}. \]

Moreover, we can choose \( \delta' \) small enough so that \( 0 < \tau(\hat{t}, \hat{x} - \delta') < \bar{y}_2 \) and \( 0 < \xi(\hat{t}, \hat{x} + \delta') < \bar{y}_2. \) We now take \( y' \in ]0, \tau(\hat{t}, \hat{x} - \delta')[ \setminus (A_1 \cup A_2), y'' \in ]0, \xi(\hat{t}, \hat{x} + \delta')[ \setminus (A_1 \cup \hat{A}_2) \) and define

\[ w(t) = \begin{cases} 0 & \text{if } t \in [0, y'], \\ \tilde{v}(t) & \text{otherwise,} \end{cases} \quad \hat{w}(x) = \begin{cases} 0 & \text{if } t \in [0, y''], \\ \tilde{v}(x) & \text{otherwise.} \end{cases} \]

One easily checks that \((w, \hat{w}) \in Y, A(w, \hat{w})(x) = A(\tilde{v}, \tilde{w})(x) \) if \( x \notin [\hat{x} - \delta', \hat{x} + \delta'] \) and \( \hat{x} \in B_2(w, \hat{w}). \) If \( z^0 \to A(\tilde{v}, \tilde{w}), \) we can thus apply Lemma 3.5 and using (2.6) we obtain the existence of some \( \delta \in ]0, \delta'[ \) such that

\[ \limsup_{\theta \to 0} \frac{1}{\theta} \int_{\hat{x} - \delta}^{\hat{x} + \delta} |u^\theta(\hat{t}, x) - z^\theta(\hat{t}, x)| \, dx \]
\[ \leq \limsup_{\theta \to 0} \frac{1}{\theta} \int_{\hat{x} - \delta}^{\hat{x} + \delta} |P_j(w^\theta \ast \tilde{u}, \tilde{w}^\theta \ast \tilde{u})(x) - z^\theta \ast u(\hat{t}, x)| \, dx \]
\[ + \limsup_{\theta \to 0} \frac{1}{\theta} \left( \| u^\theta - P_j(w^\theta \ast \tilde{u}, \tilde{w}^\theta \ast \tilde{u}) \|_{L^1(0, \infty)} + \| z^\theta \ast u - z^\theta \|_{L^1(0, \infty)} \right) \]
\[ \leq \| (\tilde{v}, \tilde{v}) - (w, \hat{w}) \|_{L^1(D\tilde{u}) \times L^1(D\hat{u})} + \| A(\tilde{v}, \tilde{v}) - A(w, \hat{w}) \|_{L^1(Du)} \]
\[ \leq 2\| \tilde{v} \|_{\infty} |D\hat{u}|([\hat{x} - \delta', \hat{x}]) + 2\| \tilde{v} \|_{\infty} |D\hat{u}|([0, \xi(\hat{t}, \hat{x} + \delta')]) \]
\[ + \int_{[\hat{x} - \delta', \hat{x} + \delta']} \left( |A(\tilde{v}, \tilde{v})(x)| + |A(w, \hat{w})(x)| \right) |Du| \]
\[ \leq \frac{2\varepsilon}{7} + 2\| \tilde{v} \|_{\infty} |D\hat{u}|([\hat{x} - \delta', \hat{x}]) + 2\| \tilde{v} \|_{\infty} |D\hat{u}|([\hat{x}, \hat{x} + \delta']) \]
+ \sum_{x \in \tilde{B}([\hat{x} - \delta', \hat{x} + \delta'])} |A(\hat{v}, \tilde{v})(x)| |Du|(x) | \\
\leq \frac{4\varepsilon}{7} + 3\|\tilde{v}\|_{\infty} |D\tilde{u}|(\{(0, \tau(\hat{x}, \hat{x} - \delta'))\}) + 3\|\tilde{v}\|_{\infty} |D\tilde{u}|(\{(0, \hat{x}, \hat{x} + \delta')\}) \leq \varepsilon. \quad \Box

Lemma 3.7. There exists C > 0 such that if \( \hat{x} \in B_1 \), then for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for any path \( \vartheta \mapsto (\tilde{u}^\vartheta, \tilde{u}^\vartheta) \) and \( \vartheta \mapsto z^\vartheta \) which generate the shift tangent vectors \( (\tilde{v}, \tilde{v}) \) and \( A(\tilde{v}, \tilde{v}) \), setting \( u^\vartheta(\hat{x}, x) = \mathcal{P}_j(\tilde{u}^\vartheta, \tilde{u}^\vartheta)(x) \) we have
\[
\limsup_{\vartheta \to 0} \frac{1}{\vartheta} \int_{\hat{x} - \delta}^{\hat{x} + \delta} |u^\vartheta(\hat{x}, x) - z^\vartheta(\hat{x}, x)| dx \leq C \cdot \sum_{x \in [\hat{x} - \delta, \hat{x} + \delta]} |\Delta u(\hat{x}, x)|.
\]

Proof. The set \( B_1 \cap \mathcal{M} \) is negligible, since it is at most a singleton. On the other hand if \( \hat{x} \in B_1 \cap \mathcal{T} \) the proof is in [7, Lemma 12]. So we consider only the case \( \hat{x} \in B_1 \cap \mathcal{T} \).

Choose \( i \leq j \) such that
\[
\tau^+(\hat{i}, \hat{x}) \in ]\hat{y}_i, \hat{y}_{i+1}[ \quad \text{and} \quad \tau^-(\hat{i}, \hat{x}) \in ]\hat{y}_j, \hat{y}_{j+1}[.
\]

Introduce the function
\[
w(t) = \begin{cases} 
\tilde{u}_i & \text{if } t < \hat{y}_{i+1}, \\
\tilde{v}(t) & \text{if } t \in [\hat{y}_{i+1}, \hat{y}_j], \\
\tilde{u}_j & \text{if } t > \hat{y}_j.
\end{cases}
\]

We now define \( u^\vartheta \) and \( \tilde{v}^\vartheta \), respectively, as in (3.17) and (3.18), and choose \( \eta > 0, \delta > 0 \) small enough so that
\[
\hat{y}_i < \tau^+(\hat{i}, \hat{x} - 2\delta) - \eta < \tau^-(\hat{i}, \hat{x} + 2\delta) + \eta < \hat{y}_{j+1}. \quad (3.20)
\]

Therefore, for \( \vartheta \) sufficiently small, there holds
\[
(u^\vartheta \ast \tilde{u})(y) = (\tilde{u}^\vartheta \ast \tilde{v})(y), \quad \forall y \in [\tau^+(\hat{i}, \hat{x} - 2\delta) - \eta, \tau^-(\hat{i}, \hat{x} + 2\delta) + \eta],
\]

and hence, by standard results on the dependency domain in an initial boundary value problem for a scalar conservation law,
\[
\mathcal{P}_j(u^\vartheta \ast \tilde{u}, \tilde{u})(y) = \mathcal{P}_j(\tilde{u}^\vartheta \ast \tilde{v}, \tilde{v})(y), \quad \forall y \in [\hat{x} - 2\delta, \hat{x} + 2\delta].
\]

If \( z^\vartheta \xrightarrow{\vartheta \to 0} A(\tilde{v}, \tilde{v}) \), to prove the lemma it thus suffices to show that
\[
\limsup_{\vartheta \to 0} \frac{1}{\vartheta} \int_{\hat{x} - \delta}^{\hat{x} + \delta} |\mathcal{P}_j(u^\vartheta \ast \tilde{u}, \tilde{u})(x) - z^\vartheta \ast u(\hat{x}, x)| dx \leq C \cdot \sum_{x \in [\hat{x} - \delta, \hat{x} + \delta]} |\Delta u(\hat{x}, x)|. \quad (3.21)
\]

By (3.20) and the definition of \( A(\tilde{v}, \tilde{v}) \), it follows that
\[
A(\tilde{v}, \tilde{v})(x) = \begin{cases} 
\tilde{a}_i \cdot \sigma(\tilde{i}, \tilde{a}, \tilde{a})(x) & \text{if } x \in [\hat{x} - 2\delta, \hat{x}[, \\
\tilde{z}_0 & \text{if } x = \hat{x}, \\
\tilde{a}_j \cdot \sigma(\tilde{j}, \tilde{a}, \tilde{a})(x) & \text{if } x \in [\hat{x}, \hat{x} + 2\delta[.}
\end{cases}
\]
where \( z_0 \) is defined as in (1) of Lemma 3.2, with \( \tilde{\alpha}_i = \alpha \) and \( \tilde{\alpha}_j = \beta \). Now to the functions \( w^{\theta} \ast \tilde{u} \), we can apply Lemma 3.2. In analogy with (2) of Lemma 3.2, we thus take
\[
M = 1 + \max \left\{ \frac{2}{\Delta \theta}, \| \tilde{v} \| \tau, |\tilde{\alpha}_j| + |\tilde{\alpha}_i| + |z_0| \right\}
\]

and choose \( z^{\theta} \xrightarrow{\theta} A(\tilde{v}, \bar{v}) \) such that if \( \Delta_{\theta, M} \) is defined as in Lemma 3.2, calling \( I^{\theta} \) the interval enclosed by the points \( \hat{x}_\theta \) and \( \hat{x} + \theta z_0 \), there holds
\[
\left| \mathcal{P}_f^{\theta}(w^{\theta} \ast \tilde{u}, \bar{v})(x) - z^{\theta} \ast u(\hat{t}, x) \right| \leq \begin{cases} 
2(\| \tilde{u} \| \infty + \| \bar{u} \| \infty) & \text{if } x \in I^{\theta}, \\
\Delta_{\theta, M} & \text{if } x \in [\hat{x} - \theta M, \hat{x} + M \theta] \setminus I^{\theta}, \\
C \theta \sum_{x \in [\hat{x} - \theta M, \hat{x} + M \theta]} |\Delta u(\hat{t}, x)| & \text{if } x \in [\hat{x} - \delta, \hat{x} - M \theta] \cup [\hat{x} + \theta M, \hat{x} + \delta].
\end{cases}
\]

Applying Lemma 3.1 we obtain (3.21). Indeed
\[
\limsup_{\theta \to 0} \frac{1}{\theta} \int_{\hat{x} - \delta}^{\hat{x} + \delta} \left| \mathcal{P}_f^{\theta}(w^{\theta} \ast \tilde{u}, \bar{v})(x) - z^{\theta} \ast u(\hat{t}, x) \right| dx \leq \limsup_{\theta \to 0} \frac{1}{\theta} \left( 2(M \theta \Delta_{\theta, M} + 2|\hat{x}_\theta - \hat{x} - z_0 \theta | (\| \tilde{u} \| \infty + \| \bar{u} \| \infty) \\
+ C \theta \sum_{x \in [\hat{x} - \theta M, \hat{x} + M \theta]} |\Delta u(\hat{t}, x)| \right) \leq C \cdot \sum_{x \in [\hat{x} - \theta M, \hat{x} + M \theta]} |\Delta u(\hat{t}, x)|.
\]

The proof is complete. \( \square \)

We now complete the proof of Theorem 2.1. Let \((\tilde{v}, \bar{v}) \in Y \) and choose \((\tilde{v}^{\theta}, \bar{v}^{\theta}) \) as in (3.17) and (3.18), so that \((\tilde{v}^{\theta}, \bar{v}^{\theta}) \xrightarrow{\theta} (\tilde{v}, \bar{v}) \). Since the supports of the function \( \tilde{v}^{\theta}, \bar{v}^{\theta} \) are uniformly bounded, we can choose a constant \( R \) large enough so that \( \mathcal{P}_f(\tilde{v}^{\theta} \ast \tilde{u}, \bar{v}^{\theta} \ast \bar{u}) = u \) outside the interval \([-R, R] \). Choose a curve \( \theta \mapsto z^{\theta} \) that has a jump at \( \hat{t} \) converging to \( A(\tilde{v}, \bar{v}) \) outside \([-R, R] \). Theorem 2.1 will thus be proved if we show that for any curve \( \theta \mapsto z^{\theta} \) that has a jump at \( \hat{t} \) converging to \( A(\tilde{v}, \bar{v}) \),
\[
\limsup_{\theta \to 0} \frac{1}{\theta} \int_{-R}^{R} \left| \mathcal{P}_f^{\theta}(\tilde{v}^{\theta} \ast \tilde{u}, \bar{v}^{\theta} \ast \bar{u})(x) - z^{\theta} \ast u(\hat{t}, x) \right| dx = 0.
\]

Call \( L \) the left-hand side above. Let \((s_i)_{i \geq 1} \) be the sequence of those points in \([-R, R] \), where \( u(\hat{t}, \cdot) \) has a jump. Fix \( \varepsilon > 0 \) and consider an integer \( N \) such that
\[
\sum_{i = N+1}^{\infty} |\Delta u(\hat{t}, s_i)| < \varepsilon,
\]
where \( \Delta u(\hat{t}, s_i) \) is the jump of \( u \) in \( s_i \). For all \( i \leq N \), we can find a positive \( \delta_i \) such that
\[
\text{TV}(u(\hat{t}, \cdot), |s_i - \delta_i, s_i| \cup |s_i + \delta_i|) < \varepsilon / N \text{ and Lemma 3.7 applies with } \hat{x} = s_i.
\]

For all \( x \) in the compact set \( K = [-R, R] \setminus \bigcup_{i=1}^{N} (|s_i - \delta_i, s_i| \cup |s_i + \delta_i|) \), let \( \delta_i \) be defined as in Lemma 3.5, 3.6 or 3.7 and such that for all \( x \in K \), the interval \([x - \delta_i, x + \delta_i] \) is contained in \( K \). Extract the finite subcovering \([x_i - \delta_{x_i}, x_i + \delta_{x_i}] \) with \( x_1, \ldots, x_M \) in \( B_1, x_{M_1} + 1, \ldots, x_{M_2} \) in \( B_2(\hat{v}, \bar{v}) \) and \( x_{M_2 + 1}, \ldots, x_{M_3} \) in \( B_3(\bar{v}, \hat{v}) \). Clearly, \( M_3 - M_2 \) is bounded by the cardinality of \( B_3 \), which is given by the number \( n \) of jumps in \( \bar{v} \) and \( \hat{v} \).

We have
\[
L \leq \limsup_{\theta \to 0} \frac{1}{\theta} \sum_{i=1}^{M_1} \int_{x_i - \delta_{x_i}}^{x_i + \delta_{x_i}} \left| P_j(\bar{v}^\theta \ast \bar{u}, \bar{v}^\theta \ast \bar{u})(x) \right| dx
\]
\[
\leq C \cdot \frac{1}{\theta} \sum_{i=1}^{M_1} \int_{x_i - \delta_{x_i}}^{x_i + \delta_{x_i}} \left| \Delta u(\hat{t}, x) \right| \leq C \cdot \sum_{i=1}^{M_1} \sum_{x \in [x_i - \delta_{x_i}, x_i + \delta_{x_i}]} \left| \Delta u(\hat{t}, x) \right|
\]
\[
+ C \cdot \sum_{i=M_1+1}^{M_2} \sum_{x \in [x_i - \delta_{x_i}, x_i + \delta_{x_i}]} \left| \Delta u(\hat{t}, x) \right| + C \cdot \sum_{i=M_1+1}^{M_2} \varepsilon
\]
\[
\leq (3C + n) \cdot \varepsilon.
\]

Since \( n \) and \( C \) are independent of \( \varepsilon \) we get the conclusion.

We conclude the proof of Theorem 2.1 with the estimate of the norm of the shift differential. Let \( (\hat{v}, \bar{v}) \) and \( (\bar{w}, \bar{w}) \) be in \( T_{(\hat{v}, \bar{v})}(X \times X) \). Introduce two families of curves \( \vartheta \mapsto (A(\hat{v}, \bar{v}))^\vartheta \), \( \vartheta \mapsto (A(\bar{v}, \bar{v}))^\vartheta \), such that \( (A(\hat{v}, \bar{v}))^\vartheta \Rightarrow A(\hat{v}, \bar{v}) \) and \( (A(\bar{v}, \bar{v}))^\vartheta \Rightarrow A(\bar{v}, \bar{v}) \). Recalling that the process \( P_j \) is Lipschitzian with constant \( L \), using (2.6) and (2.7) we get
\[
\parallel (\Delta u(\hat{v}, \bar{v}) - A(\bar{w}, \bar{w})) \parallel_{T_{(\hat{v}, \bar{v})}(X \times X)}
\]
\[
= \lim_{\theta \to 0} \frac{1}{\theta} \left\parallel (A(\hat{v}, \bar{v}))^\theta \ast u - (A(\bar{w}, \bar{w}))^\theta \ast u \right\parallel_{L^1([0, +\infty])}
\]
\[
\leq \lim_{\theta \to 0} \frac{1}{\theta} \left\parallel (A(\hat{v}, \bar{v}))^\theta \ast u - P_j(\hat{v}^\theta \ast \bar{u}, \bar{v}^\theta \ast \bar{u}) \right\parallel_{L^1([0, +\infty])}
\]
\[
+ \lim_{\theta \to 0} \frac{1}{\theta} \left\parallel P_j(\hat{v}^\theta \ast \bar{u}, \bar{v}^\theta \ast \bar{u}) - \bar{v}^\theta \ast \bar{u} \right\parallel_{L^1([0, +\infty])}
\]
\[
\leq \lim_{\theta \to 0} \frac{L}{\theta} \left( \parallel \hat{v}^\theta \ast \bar{u} - \bar{v}^\theta \ast \bar{u} \parallel_{L^1([0, +\infty])} + \parallel \bar{v}^\theta \ast \bar{u} - \bar{v}^\theta \ast \bar{u} \parallel_{L^1([0, +\infty])} \right)
\]
\[
= L \cdot \parallel (\hat{v}, \bar{v}) - (\bar{w}, \bar{w}) \parallel_{T_{(\hat{v}, \bar{v})}(X \times X)}.
\]
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