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Fractional calculus in the Mellin setting and Hadamard-type fractional integrals

Paul L. Butzer,[a,*] Anatoly A. Kilbas,[b] and Juan J. Trujillo[c]

[a] *Lehrstuhl A für Mathematik, Rheinisch Westfälische Technische Hochschule Aachen, Templergraben 55, D-52062 Aachen, Germany*
[b] *Department of Mathematics and Mechanics, Belarusian State University, 220050 Minsk, Belarus*
[c] *Departamento de Analisis Matematico, Universidad de La Laguna, 38271 La Laguna-Tenerife, Spain*

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Abstract

The purpose of this paper and some to follow is to present a new approach to fractional integration and differentiation on the half-axis $\mathbf{R}_+ = (0, \infty)$ in terms of Mellin analysis. The natural operator of fractional integration in this setting is not the classical Liouville fractional integral $I^\alpha_{0+} f$ but

$$\left(\mathcal{J}^\alpha_{0+,c} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_0^x \left(\frac{u}{x}\right)^c \left(\log \frac{x}{u}\right)^{\alpha-1} \frac{f(u)\,du}{u} \quad (x > 0)$$

for $\alpha > 0$, $c \in \mathbf{R}$. The Mellin transform of this operator is simply $(c-s)^{-\alpha}\mathcal{M}[f](s)$, for $s = c + it$, $c, t \in \mathbf{R}$. The Mellin transform of the associated fractional differentiation operator $\mathcal{D}^\alpha_{0+,c} f$ is similar: $(c-s)^\alpha \mathcal{M}[f](s)$. The operator $\mathcal{D}^\alpha_{0+,c} f$ may even be represented as a series in terms of $x^k f^{(k)}(x)$, $k \in \mathbf{N}_0$, the coefficients being certain generalized Stirling functions $S_c(\alpha, k)$ of second kind. It turns out that the new fractional integral $\mathcal{J}^\alpha_{0+,c} f$ and three further related ones are not the classical fractional integrals of Hadamard (J. Mat. Pure Appl. Ser. 4, 8 (1892) 101–186) but far reaching generalizations and modifications of these. These four new integral operators are first studied in detail in this paper. More specifically,

* Corresponding author.
E-mail address: butzer@rwth-aachen.de (P.L. Butzer).

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conditions will be given for these four operators to be bounded in the space X_c^p of Lebesgue measurable functions f on $(0,\infty)$, for $c \in (-\infty,\infty)$, such that $\int_0^\infty |u^c f(u)|^p \, du/u < \infty$ for $1 \leqslant p < \infty$ and $\operatorname{ess\,sup}_{u>0}[u^c|f(u)|] < \infty$ for $p = \infty$, in particular in the space $L^p(0,\infty)$ for $1 \leqslant p \leqslant \infty$. Connections of these operators with the Liouville fractional integration operators are discussed. The Mellin convolution product in the above spaces plays an important role.

Keywords: Fractional integration and differentiation; Hadamard-type fractional integrals; Mellin transform; Mellin convolution; Weighted spaces of p-summable functions; Stirling functions of second kind

1. Introduction

The purpose of this paper and some to follow is to present a new approach to fractional differentiation and integration in the Mellin setting. In the classical sense, the Mellin transform $\mathcal{M}$ of $f : \mathbf{R}_+ \to \mathbf{C}$ is defined by

$$\mathcal{M}[f](s) := \int_0^\infty u^{s-1} f(u)\, du \quad (s = c + it,\ c, t \in \mathbf{R}). \tag{1.1}$$

It is directly verified that such a transform of the classical r-derivative $D^r f$ $(D = d/dx)$, $r \in \mathbf{N} = \{1, 2, \ldots\}$, for "sufficiently good" functions f is given by

$$\mathcal{M}\big[D^r f\big](s) = \frac{(-1)^r \Gamma(s)}{\Gamma(s-r)} \mathcal{M}[f](s-r). \tag{1.2}$$

As to the rth integral

$$\begin{aligned}\big(I^r f\big)(x) &:= \int_0^x du_1 \int_0^{u_1} du_2 \ldots \int_0^{u_{r-1}} f(u_r)\, du_r \\ &= \frac{1}{(r-1)!} \int_0^x (x-u)^{r-1} f(u)\, du,\end{aligned} \tag{1.3}$$

its Mellin transform turns out to be

$$\mathcal{M}\big[I^r f\big](s) = \frac{\Gamma(1-r-s)}{\Gamma(1-s)} \mathcal{M}[f](s+r). \tag{1.4}$$

As the reader observes immediately, on the right sides of (1.2) and (1.4) there occur quotients of gamma functions, and the transforms $\mathcal{M}[f](s \mp r)$ involve $r \in \mathbf{N}$.

Now D is not the natural operator of differentiation for the Mellin setting, nor is it the customary $\delta := xD$; it is actually

$$(\Theta_c f)(x) := \big((\delta + c)f\big)(x) = xf'(x) + cf(x) \\ (c = \mathrm{Re}(s) \in \mathbf{R},\ x > 0), \tag{1.5}$$

that of order $r \in \mathbf{N}$ being defined iteratively by

$$\Theta_c^1 f := \Theta_c f, \qquad \Theta_c^r f := \Theta_c\big(\Theta_c^{r-1} f\big) \quad (r = 2, 3, \ldots). \tag{1.6}$$

Its Mellin transform is given by

$$\mathcal{M}\big[\Theta_c^r f\big](s) = (-it)^r \mathcal{M}[f](s), \quad t = \mathrm{Im}(s) \in \mathbf{R}, \tag{1.7}$$

or, more generally, for arbitrary $\mu \in \mathbf{R}$ there holds

$$\mathcal{M}\big[\Theta_\mu^r f\big](s) = (\mu - s)^r \mathcal{M}[f](s) \quad (s = c + it,\ t \in \mathbf{R}). \tag{1.8}$$

If $\mu = c$ the latter formula turns into (1.7). Observe that the right side of (1.8) does not involve the gamma function.

Now the integration operator J_c^r associated with Θ_c^r—better still the anti-differentiation operator—is not the classical integration operator I^r of (1.3) but

$$\big(J_c^r f\big)(x) = x^{-c} \int_0^x \frac{du_1}{u_1} \int_0^{u_1} \frac{du_2}{u_2} \cdots \int_0^{u_{r-1}} u_r^c f(u_r) \frac{du_r}{u_r} \\ = \frac{1}{(r-1)!} \int_0^x \left(\frac{u}{x}\right)^c \left(\log \frac{x}{u}\right)^{r-1} f(u) \frac{du}{u} \\ (c \in \mathbf{R},\ x > 0). \tag{1.9}$$

Its Mellin transform, indeed, turns out to be

$$\mathcal{M}\big[J_c^r f\big](s) = (-it)^{-r} \mathcal{M}[f](s) \tag{1.10}$$

or, more generally, for $\mu \in \mathbf{R}$

$$\mathcal{M}\big[J_\mu^r f\big](s) = (\mu - s)^{-r} \mathcal{M}[f](s) \quad (s = c + it,\ t \in \mathbf{R}). \tag{1.11}$$

Here no gamma functions occur, and the transform $\mathcal{M}[f](s)$ is independent of r. The operations Θ_c^r and J_c^r are natural in the sense that they are inverse to each other; thus there hold the relations

$$\Theta_c^r J_c^r f = f, \qquad \Theta_c^r J_c^r f = f \tag{1.12}$$

for $x \in \mathbf{R}_+$ under suitable conditions upon f. See [1].

The aim of this paper and its follow-ups is to study the foregoing matter in all details in the fractional instance when the natural r is replaced by the positive real $\alpha \in \mathbf{R}_+$. Similar investigations are well developed for the classical Riemann–Liouville and Liouville fractional integro-differentiation—see, for example, [2].

Firstly, the fractional counterpart of the integral (1.9) is

$$\left(\mathcal{J}^{\alpha}_{0+,\mu} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_0^x \left(\frac{u}{x}\right)^{\mu} \left(\log \frac{x}{u}\right)^{\alpha-1} \frac{f(u)\,du}{u}$$
$$(\alpha > 0,\ \mu \in \mathbf{R},\ x > 0), \tag{1.13}$$

which equals (1.9) for $\alpha = r \in \mathbf{N}$ and $\mu = c$. It will turn out that its Mellin transform has the form (1.11) with $r \in \mathbf{N}$ being replaced by $\alpha > 0$:

$$\mathcal{M}\left[\mathcal{J}^{\alpha}_{0+,\mu} f\right](s) = (\mu - s)^{-\alpha} \mathcal{M}[f](s) \quad (s = c + it,\ t \in \mathbf{R}). \tag{1.14}$$

It is known [2, Section 5.1] that the classical left-sided Liouville fractional integral of order $\alpha > 0$ on the half-axis $\mathbf{R}_+$ has the form (1.3)

$$\left(I^{\alpha}_{0+} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_0^x (x-u)^{\alpha-1} f(u)\,du \quad (\alpha > 0,\ x > 0), \tag{1.15}$$

while the corresponding left-sided Liouville fractional derivative associated with (1.15) is given by

$$\left(D^{\alpha}_{0+} f\right)(x) := \left(\frac{d}{dx}\right)^{m} \left(I^{m-\alpha}_{0+} f\right)(x)$$
$$(\alpha > 0,\ m = [\alpha] + 1,\ x > 0), \tag{1.16}$$

where $[\alpha]$ is the greatest integer in α. This suggests defining the fractional order derivative for $\mu = c \in \mathbf{R}$ in the framework of the Mellin transform for $x > 0$ by

$$\left(\mathcal{D}^{\alpha}_{0+,c} f\right)(x) := x^{-c} \delta^m x^c \left(\mathcal{J}^{m-\alpha}_{0+,c} f\right)(x)$$
$$(\alpha > 0,\ m = [\alpha] + 1), \quad \delta = x \frac{d}{dx}, \tag{1.17}$$

or $(\Theta^{\alpha}_c f)(x) := (\mathcal{D}^{\alpha}_{0+,c} f)(x)$ in the other notation. In particular, if $\alpha = 1$, it is easy to check that $\mathcal{D}^{1}_{0+,c} f$ coincides with $\Theta_c f$ in (1.5), while for $\alpha = 2$

$$\left(\mathcal{D}^{2}_{0+,c} f\right)(x) = x^2 f''(x) + (2c+1) x f'(x) + c^2 f(x).$$

This is indeed $(\Theta^2_c f)(x)$ as defined via (1.6) with $r = 2$; see [1]. In fact, this derivative of order $r \in \mathbf{N}$ can be written as (see [1, (8.2)])

$$\left(\mathcal{D}^{r}_{0+,c} f\right)(x) \equiv \left(\Theta^{r}_c f\right)(x) = \sum_{k=0}^{r} S_c(r,k) x^k f^{(k)}(x), \tag{1.18}$$

where $S_c(r,k)$ $(0 \leqslant k \leqslant r)$ denote the generalized Stirling numbers of second kind, defined recursively by

$$S_c(r,0)=c^r, \qquad S_c(r,r)=1,$$
$$S_c(r+1,k)=S_c(r,k-1)+(c+k)S_c(r,k);$$

an alternative definition of these numbers is

$$S_c(r,k)=\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(c+j)^r. \tag{1.19}$$

Let us return to the fractional instance. In accordance with extensions of definitions of various types of numbers in combinatorial analysis from the classical discrete to the fractional case, as developed, e.g., in [3] and [4], in the fractional version of (1.18) the finite sum would be replaced by an infinite one, resulting in

$$\left(\mathcal{D}^{\alpha}_{0+,c}f\right)(x)=\sum_{k=0}^{\infty}S_c(\alpha,k)x^k f^{(k)}(x), \tag{1.20}$$

where

$$S_c(\alpha,k)=\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(c+j)^{\alpha}. \tag{1.21}$$

When $c=0$, then $S(\alpha,k)=S_0(\alpha,k)$ are the Stirling functions of $\alpha>0$ of second kind introduced in [5] and developed in [6].

Definition (1.20) would be an alternative to (1.17). To make the matter intuitively clear, let us proceed formally as follows. There hold the relations

$$\begin{aligned} f(u)&=\sum_{k=0}^{\infty}\frac{f^{(k)}(x)}{k!}(u-x)^k \\ &=\sum_{k=0}^{\infty}\frac{f^{(k)}(x)}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}x^{k-j}u^j, \end{aligned} \tag{1.22}$$

noting the binomial expansion, and the directly checked one

$$\left(\Theta^{\alpha}_{c}t^{\mu}\right)(x)=(c+\mu)^{\alpha}x^{\mu}, \quad \operatorname{Re}(c+\mu)>0. \tag{1.23}$$

Substituting this result with $\mu=j$ in (1.22), after differentiating this Taylor series term by term, we obtain (1.20). Similar arguments yield an alternative version of (1.13) in the form (1.20)

$$\left(\mathcal{J}^{\alpha}_{0+,c}f\right)(x)=\sum_{k=0}^{\infty}S_{-c}(\alpha,k)x^k f^{(k)}(x). \tag{1.24}$$

The relations (1.24) and (1.20) present a unified representation for the fractional integrals (1.13) and derivatives (1.17), being obtained from each other by replacing c by $-c$.

Similarly to the definition of the right-sided Liouville fractional integral $I_{-}^{\alpha} f$ of the form (1.15), replacing the integration over $(0, x)$ by that over (x, ∞) [2, (5.3)], we can define the right-sided fractional integration of the form (1.13)

$$\big(\mathcal{J}_{-,\mu}^{\alpha} f\big)(x) := \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} \left(\frac{x}{u}\right)^{\mu} \left(\log \frac{u}{x}\right)^{\alpha-1} \frac{f(u)\,du}{u}$$
$$(\alpha > 0,\ \mu \in \mathbf{R},\ x > 0), \tag{1.25}$$

and the corresponding fractional differentiation

$$\big(\mathcal{D}_{-,c}^{\alpha} f\big)(x) := x^{c}(-\delta)^{m} x^{c} \big(J_{-,c}^{m-\alpha} f\big)(x)$$
$$(m = [\alpha] + 1,\ \alpha > 0), \quad \delta = x\frac{d}{dx}. \tag{1.26}$$

These may also be written in the forms

$$\big(\mathcal{J}_{-,c}^{\alpha} f\big)(x) = \sum_{k=0}^{r} S_{-c}^{*}(\alpha, k) x^{k} f^{(k)}(x), \tag{1.27}$$

$$\big(\mathcal{D}_{-,c}^{\alpha} f\big)(x) = \sum_{k=0}^{r} S_{c}^{*}(\alpha, k) x^{k} f^{(k)}(x), \tag{1.28}$$

where

$$S_{c}^{*}(\alpha, k) := \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} (c - j)^{\alpha}. \tag{1.29}$$

We hope that the above approach will be useful in generalizing several known results and to obtain new trends of research not only in the theory of fractional calculus, but also in combinatorial analysis, approximation theory, and other fields.

Note that when $\mu = 0$, (1.13) and (1.25) take on the forms

$$\big(\mathcal{J}_{0+}^{\alpha} f\big)(x) = \frac{1}{\Gamma(\alpha)} \int_{0}^{x} \left(\log \frac{x}{u}\right)^{\alpha-1} \frac{f(u)\,du}{u} \quad (\alpha > 0,\ x > 0) \tag{1.30}$$

and

$$\big(\mathcal{J}_{-}^{\alpha} f\big)(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} \left(\log \frac{u}{x}\right)^{\alpha-1} \frac{f(u)\,du}{u} \quad (\alpha > 0,\ x > 0), \tag{1.31}$$

respectively. The particular integral (1.30) was introduced by Hadamard [7] and therefore the integrals (1.30) and (1.31) are often referred to as Hadamard fractional integrals of order $\alpha > 0$; see [2, Sections 18.3 and 23.1, notes to

Section 18.3]. Therefore we may call the more general integrals in (1.13) and (1.25) Hadamard-type fractional integrals.

Our paper deals with these integral operators and two of their modifications, namely

$$\left(\mathcal{I}_{0+,\mu}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \left(\frac{u}{x}\right)^{\mu} \left(\log \frac{x}{u}\right)^{\alpha-1} \frac{f(u)\,du}{x} \quad (x>0), \tag{1.32}$$

$$\left(\mathcal{I}_{-,\mu}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty \left(\frac{x}{u}\right)^{\mu} \left(\log \frac{u}{x}\right)^{\alpha-1} \frac{f(u)\,du}{x} \quad (x>0) \tag{1.33}$$

with $\alpha > 0$ and complex $\mu \in \mathbf{C}$. We note that the operators in (1.32) and (1.33) are conjugate to those in (1.25) and (1.13), respectively.

We will study the operators $\mathcal{J}_{0+,\mu}^{\alpha} f$, $\mathcal{J}_{-,\mu}^{\alpha} f$, $\mathcal{I}_{0+,\mu}^{\alpha} f$, and $\mathcal{I}_{-,\mu}^{\alpha} f$ in the space X_c^p $(c \in \mathbf{R},\ 1 \leqslant p \leqslant \infty)$ of those complex-valued Lebesgue measurable functions f on $\mathbf{R}_+ = (0,\infty)$ for which $\|f\|_{X_c^p} < \infty$, where

$$\|f\|_{X_c^p} = \left(\int_0^\infty \left|u^c f(u)\right|^p \frac{du}{u}\right)^{1/p} \quad (1 \leqslant p < \infty,\ c \in \mathbf{R}) \tag{1.34}$$

and

$$\|f\|_{X_c^\infty} = \operatorname*{ess\,sup}_{u>0}\left[u^c |f(u)|\right] \quad (c \in \mathbf{R}). \tag{1.35}$$

In particular, when $c = 1/p$ $(1 \leqslant p \leqslant \infty)$, the space X_c^p coincides with the classical $L^p(\mathbf{R}_+)$-space: $L^p(\mathbf{R}_+) \equiv X_{1/p}^p$ with

$$\|f\|_p = \left(\int_0^\infty |f(u)|^p\,du\right)^{1/p} \quad (1 \leqslant p < \infty),$$

$$\|f\|_\infty = \operatorname*{ess\,sup}_{u>0} |f(u)|. \tag{1.36}$$

In this paper we will give conditions for the operators in (1.13), (1.25), (1.32), and (1.33) to be bounded in the space X_c^p. These results are based on the corresponding assertions for the Mellin convolution product $g * f$ of two functions $f, g : \mathbf{R}_+ \to \mathbf{C}$, defined by

$$(g * f)(x) = \int_0^\infty g\left(\frac{x}{u}\right) f(u) \frac{du}{u} \quad (x \in \mathbf{R}_+), \tag{1.37}$$

in case the integral exists. We will also obtain the corresponding properties for the Hadamard fractional integrals (1.30) and (1.31). In particular, the results in the space $L_p(\mathbf{R}_+)$ will follow.

It should be emphasized that some results of this paper will be established by three different methods of proof, which are used throughout, in order to point out their relevance and importance in our new approach to fractional calculus in the Mellin setting. The first deals with the self-contained Mellin analysis approach developed in [1,8], the second method with an operator-theoretic approach. Whereas both methods cover the mapping properties of the four new operators of fractional integration from the space X_c^p into itself, the third is connected with the mapping properties from X_c^p to X_c^q for $1 \leqslant p \leqslant q \leqslant \infty$. The results established by the third method of proof include in part those of the first two; however, in contrast to the first two methods, which are rather elementary, the third method is not self-contained but relies on the deep Riesz–Thorin theorem.

Note that another fractional calculus in the Mellin setting was introduced in [9, Definition 2.4], where the linear operators $I_m^{\eta,\alpha}$ and $K_m^{\eta,\alpha}$ were defined by

$$\big(\mathcal{M} I_m^{\eta,\alpha} f\big)(s) = \frac{\Gamma(\eta + 1 - s/m)}{\Gamma(\eta + \alpha + 1 - s/m)} (\mathcal{M} f)(s) \tag{1.38}$$

and

$$\big(\mathcal{M} K_m^{\eta,\alpha} f\big)(s) = \frac{\Gamma(\eta + s/m)}{\Gamma(\eta + \alpha + s/m)} (\mathcal{M} f)(s), \tag{1.39}$$

respectively, for f in a special Fréchet space $F_{p,\mu}$. Here $I_m^{\eta,\alpha} \equiv I_{0+;m,\eta}^{\alpha}$ and $K_m^{\eta,\alpha} \equiv I_{-;m,\eta}^{\alpha}$ are the familiar Erdélyi–Kober-type operators of fractional integration and differentiation; their explicit representations can be found in [10, Chapter 3] and [2, Section 18.1]. The boundedness of such fractional integration operators in X_μ^p-spaces together with the relations (1.38) and (1.39) were established in [11, Corollaries 3.1 and 4.1]. Observe that the right-hand sides of (1.38) and (1.39) involve quotients of gamma functions.

We also mention a series of papers devoted to the investigation of Mellin multipliers in the frame of the Mellin transform $\mathcal{M}$ defined in terms of the Fourier transform by $(\mathcal{M} f)(s) = \int_{-\infty}^{\infty} e^{u(c+it)} f(e^u)\,du$. In [12] a class of multipliers h was constructed, which lead to a bounded linear mapping T from X_μ^p into X_μ^p for $1 < p < \infty$ and suitable complex number μ, and for which the relation $(\mathcal{M} T f)(s) = h(s)(\mathcal{M} f)(s)$ holds for $f \in X_c^p \cap X_2^p$; see [13] in this connection. A similar class of Mellin multipliers in the setting of the space $F_{p,\mu}$ was examined in [14], where the fractional integration operator $K_1^{\eta,\alpha}$ in (1.39) was considered as an example [14, Example 3.4]. A Mellin transform approach to develop a theory of complex powers T^α of linear operators T on the basis of the relation $(\mathcal{M} T^\alpha f)(s - \alpha\gamma) = [h(s - \alpha\gamma)/h(s)](\mathcal{M} f)(s)$ in the spaces X_μ^p, $F_{p,\mu}$ and the corresponding space of generalized functions $F'_{p,\mu}$ was carried out in [15,16].

The paper is organized as follows. Section 2 presents isomorphic properties of some elementary operators in the space X_c^p and the connections of the Hadamard-type operators (1.13) and (1.25) considered with the Liouville fractional integral operators on the real line $\mathbf{R}$. Mapping properties of the Mellin convolution operator (1.37) in the space X_c^p are given in Section 3. Section 4 is devoted to the boundedness of the operators $\mathcal{J}_{0+,\mu}^\alpha$, $\mathcal{J}_{-,\mu}^\alpha$, $\mathcal{I}_{0+,\mu}^\alpha$, and $\mathcal{I}_{-,\mu}^\alpha$ in the space X_c^p. Sections 5 and 6 deal with such properties from the one X_c^p-space to the other X_c^q for $1 \leqslant p \leqslant q \leqslant \infty$.

2. Auxiliary results

For a function $f(x)$ defined almost everywhere on $\mathbf{R}_+$ we define the elementary operators M_ζ, τ_h^r, N_a, R, and Q as follows:

$$(M_\zeta f)(x) = x^\zeta f(x) \quad (\zeta \in \mathbf{C}), \tag{2.1}$$

$$\left(\tau_h^r f\right)(x) = h^r f(hx) \quad (h \in \mathbf{R}_+,\ r \in \mathbf{R}), \tag{2.2}$$

$$(N_{a,r} f)(x) = |a|^r f\left(x^a\right) \quad (a \in \mathbf{R},\ a \neq 0,\ r \in \mathbf{R}), \tag{2.3}$$

$$(Rf)(x) = \frac{1}{x} f\left(\frac{1}{x}\right), \tag{2.4}$$

and

$$(Qf)(x) = f\left(\frac{1}{x}\right). \tag{2.5}$$

It is clear that the inverse operators M_ζ^{-1}, $(\tau_h^r)^{-1}$, $N_{a,r}^{-1}$, R^{-1}, and Q^{-1} have the forms

$$\left(M_\zeta^{-1} f\right)(x) = (M_{-\zeta} f)(x) = x^{-\zeta} f(x) \quad (\zeta \in \mathbf{C}), \tag{2.6}$$

$$\left(\left(\tau_h^r\right)^{-1} f\right)(x) = h^{-r} f\left(\frac{x}{h}\right) \quad (h \in \mathbf{R}_+,\ r \in \mathbf{R}), \tag{2.7}$$

$$\left(N_{a,r}^{-1} f\right)(x) = |a|^{-r} f\left(x^{1/a}\right) \quad (a \in \mathbf{R},\ a \neq 0,\ r \in \mathbf{R}), \tag{2.8}$$

$$\left(R^{-1} f\right)(x) = (Rf)(x), \tag{2.9}$$

and

$$\left(Q^{-1} f\right)(x) = (Qf)(x). \tag{2.10}$$

The following assertions are easily verified.

Lemma 1. *Let $c \in \mathbf{R}$, $1 \leqslant p \leqslant \infty$ and $f \in X_c^p$.*

(a) *M_ζ with $\zeta \in \mathbf{C}$ is an isometric isomorphism of X_c^p onto $X_{c-\mathrm{Re}(\zeta)}^p$:*

$$\|M_\zeta f\|_{X_{c-\mathrm{Re}(\zeta)}^p} = \|f\|_{X_c^p}. \tag{2.11}$$

(b) *The translation operator* τ_h^r *with* $h \in \mathbf{R}_+$ *and* $r \in \mathbf{R}$ *is an isomorphism of* X_c^p *onto* X_c^p:

$$\|\tau_h^r f\|_{X_c^p} = h^{r-c}\|f\|_{X_c^p}. \tag{2.12}$$

In particular, τ_h^c *is an isometric isomorphism of* X_c^p *onto* X_c^p:

$$\|\tau_h^c f\|_{X_c^p} = \|f\|_{X_c^p}. \tag{2.13}$$

(c) $N_{a,r}$ *with* $a \in \mathbf{R}$ $(a \neq 0)$ *and* $r \in \mathbf{R}$ *is an isomorphism of* X_c^p *onto* X_{ac}^p:

$$\|N_{a,r} f\|_{X_{ac}^p} = |a|^{r-1/p}\|f\|_{X_c^p}. \tag{2.14}$$

In particular, $N_{a,1/p}$ *is an isometric isomorphism of* X_c^p *onto* X_{ac}^p:

$$\|N_{a,r} f\|_{X_{ac}^p} = \|f\|_{X_c^p}. \tag{2.15}$$

(d) R *is an isometric isomorphism of* X_c^p *onto* X_{1-c}^p:

$$\|Rf\|_{X_{1-c}^p} = \|f\|_{X_c^p}. \tag{2.16}$$

(e) Q *is an isometric isomorphism of* X_c^p *onto* X_{-c}^p:

$$\|Qf\|_{X_{-c}^p} = \|f\|_{X_c^p}. \tag{2.17}$$

Remark 1. The assertions of Lemma 1 are indicated in part in [17]; for part (b) see especially [1,8].

For a function $\varphi(x)$ defined almost everywhere on $\mathbf{R}$ we define the elementary operator A by

$$(A\varphi)(x) = \varphi(e^x). \tag{2.18}$$

It is clear its inverse A^{-1} has the form

$$(A^{-1}\psi)(x) = \psi(\log(x)) \tag{2.19}$$

for a function $\psi(x)$ defined almost everywhere on $\mathbf{R}_+$.

For $c \in \mathbf{R}$ and $1 \leqslant p \leqslant \infty$ we denote by L_c^p the space of those complex-valued Lebesgue measurable functions $\varphi(x)$ on $\mathbf{R}$ such that $\|\varphi\|_{L_c^p} < \infty$, where

$$\|\varphi\|_{L_c^p} = \left(\int_{-\infty}^{\infty} \left|e^{cu}\varphi(u)\right|^p du\right)^{1/p} \quad (1 \leqslant p < \infty,\ c \in \mathbf{R}) \tag{2.20}$$

and

$$\|\varphi\|_{L_c^\infty} = \operatorname*{ess\,sup}_{u\in\mathbf{R}}\left[e^{cu}|\varphi(u)|\right] \quad (c \in \mathbf{R}). \tag{2.21}$$

The following assertion holds.

Lemma 2. *Let* $c \in \mathbf{R}$ *and* $1 \leqslant p \leqslant \infty$.

(a) *A is isometric isomorphism of X_c^p onto L_c^p:*

$$\|A\varphi\|_{L_c^p} = \|\varphi\|_{X_c^p}. \tag{2.22}$$

(b) *A^{-1} is isometric isomorphism of L_c^p onto X_c^p:*

$$\|A^{-1}\psi\|_{X_c^p} = \|\psi\|_{L_c^p}. \tag{2.23}$$

Proof. If $1 \leqslant p < \infty$, then using (2.18) and (2.20) and making the change of variable $u = \log(x)$, we have

$$\|A\varphi\|_{L_c^p} = \left(\int_{-\infty}^{\infty} \left|e^{cu}\varphi\left(e^u\right)\right|^p du\right)^{1/p}$$
$$= \left(\int_0^{\infty} \left|x^c\varphi(x)\right|^p dx\right)^{1/p} = \|\varphi\|_{X_c^p},$$

which proves (2.22) for $1 \leqslant p < \infty$. If $p = \infty$, then by (2.18) and (2.21),

$$\|A\varphi\|_{L_c^\infty} = \operatorname*{ess\,sup}_{u\in\mathbf{R}}\left[e^{cu}\left|\varphi\left(e^u\right)\right|\right] = \operatorname*{ess\,sup}_{x>0}\left[x^c|\varphi(x)|\right] = \|\varphi\|_{X_c^\infty},$$

which yields (2.22) with $p = \infty$. The relation (2.23) is proved similarly. □

In conclusion we note that the Hadamard-type integrals (1.13) and (1.25) are closely connected with the Liouville fractional integrals $I_+^\alpha f$ and $I_-^\alpha f$, defined on the whole real line $\mathbf{R}$ by

$$\left(I_+^\alpha f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} \frac{f(u)\,du}{(x-u)^{1-\alpha}} \quad (\alpha > 0,\ x \in \mathbf{R}) \tag{2.24}$$

and

$$\left(I_-^\alpha f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} \frac{f(u)\,du}{(u-x)^{1-\alpha}} \quad (\alpha > 0,\ x \in \mathbf{R}), \tag{2.25}$$

respectively [2, Section 5.1]. It is directly checked that such connections for the operators (1.13) and (1.25) are given by the relations

$$\left(\mathcal{J}_{0+,\mu}^\alpha f\right)(x) = \left(M_{-\mu}A^{-1}I_+^\alpha A M_\mu f\right)(x) \tag{2.26}$$

and

$$\left(\mathcal{J}_{-,\mu}^\alpha f\right)(x) = \left(M_\mu A^{-1}I_-^\alpha A M_{-\mu} f\right)(x), \tag{2.27}$$

where the elementary operators M_μ and A are defined by (2.1) and (2.18).

3. Mellin convolution in X_c^p

Let $1 \leqslant p \leqslant \infty$ and p' be the exponent conjugate to p, i.e.,

$$\frac{1}{p} + \frac{1}{p'} = 1, \tag{3.1}$$

where $p' = \infty$ for $p = 1$, while $p' = 1$ for $p = 1$.

The first result for the Mellin convolution product $g * f$ of (1.37) in the space X_c^p is given by the following theorem.

Theorem 1. *Let $c \in \mathbf{R}$ and $1 \leqslant p \leqslant \infty$. If $f \in X_c^p$ and $g \in X_c^1$, then $g * f$ exists (a.e.) on $\mathbf{R}_+$, belongs to X_c^p, and*

$$\|g * f\|_{X_c^p} \leqslant \|g\|_{X_c^1} \|f\|_{X_c^p}. \tag{3.2}$$

Proof. First we consider the case $1 \leqslant p < \infty$. By [7, p. 396] the function $[g(x/u) f(u)]^c u^{cp-1}$ is measurable on $\mathbf{R}_+$. If $p > 1$, then using the property

$$g * f = f * g \tag{3.3}$$

and applying the Holder inequality, we have for almost all $x \in \mathbf{R}_+$

$$\begin{aligned}
|(g * f)(x)| &= |(f * g)(x)| \\
&\leqslant \int_0^\infty \left|g(u)u^c\right|^{1/p'} \left|g(u)u^{-cp/p'}\right|^{1/p} \left|f\left(\frac{x}{u}\right)\right| \frac{du}{u} \\
&\leqslant \left(\int_0^\infty |g(u)|u^c \frac{du}{u}\right)^{1/p'} \left(\int_0^\infty |g(u)|^{-cp/p'} \left|f\left(\frac{x}{u}\right)\right|^p \frac{du}{u}\right)^{1/p} \\
&= \left(\|g\|_{X_c^1}\right)^{1/p'} \left(\int_0^\infty |g(u)|u^{-cp/p'} \left|f\left(\frac{x}{u}\right)\right|^p \frac{du}{u}\right)^{1/p},
\end{aligned}$$

if we take into account (1.34) with $p = 1$. Using Fubini's theorem and substituting $y = x/u$, we obtain

$$\begin{aligned}
\left(\|g * f\|_{X_c^p}\right)^p &\leqslant \left(\|g\|_{X_c^1}\right)^{p/p'} \int_0^\infty x^{cp} \left(\int_0^\infty |g(u)|u^{-cp/p'} \left|f\left(\frac{x}{u}\right)\right|^p \frac{du}{u}\right) \frac{dx}{x} \\
&= \left(\|g\|_{X_c^1}\right)^{p/p'} \int_0^\infty |g(u)|u^{-cp/p'} \frac{du}{u} \int_0^\infty x^{cp} \left|f\left(\frac{x}{u}\right)\right|^p \frac{dx}{x}
\end{aligned}$$

$$= \big(\|g\|_{X_c^1}\big)^{p/p'} \int_0^\infty |g(u)|u^{-cp/p'} u^{cp} \frac{du}{u} \int_0^\infty y^{cp}|f(y)|^p \frac{dy}{y}$$
$$= \big(\|g\|_{X_c^1}\big)^{1+p/p'} \big(\|f\|_{X_c^p}\big)^p = \big(\|g\|_{X_c^1}\big)^p \big(\|f\|\big)_{X_c^p}^p.$$

Now (3.2) follows for $1 < p < \infty$. If $p = 1$, then Fubini's theorem gives

$$\|g * f\|_{X_c^1} \leqslant \int_0^\infty x^{c-1}\,dx \int_0^\infty |f(u)|\left|g\left(\frac{x}{u}\right)\right|\frac{du}{u}$$
$$= \int_0^\infty |f(u)|\frac{du}{u} \int_0^\infty x^{c-1}\left|g\left(\frac{x}{u}\right)\right| dx = \|f\|_{X_c^1}\|g\|_{X_c^1}.$$

Further, in accordance with (1.35) and (1.34) for $r = 1$, we have for almost all $x \in \mathbf{R}_+$,

$$|(g * f)(x)| \leqslant \int_0^\infty \big[u^c|f(u)|\big]\left[u^{-c}\left|g\left(\frac{x}{u}\right)\right|\right]\frac{du}{u}$$
$$\leqslant \|f\|_{X_c^\infty} \int_0^\infty \left[u^{-c}\left|g\left(\frac{x}{u}\right)\right|\right]\frac{du}{u} = x^{-c}\|f\|_{X_c^\infty}\|g\|_{X_c^1},$$

which yields (3.2) for $p = \infty$. This completes the proof of the theorem. □

The next theorem presents the second result for $g * f$ with g in the space $X_c^{p'}$.

Theorem 2. *Let* $c \in \mathbf{R}$, $1 \leqslant p \leqslant \infty$ *and* p' *be given by* (3.1). *If* $f \in X_c^p$ *and* $g \in X_c^{p'}$, *then* $g * f$ *exists as a continuous function on* $\mathbf{R}_+$ *with*

$$\|g * f\|_{X_c^\infty} \leqslant \|g\|_{X_c^{p'}}\|f\|_{X_c^p}. \tag{3.4}$$

Proof. Using (2.2) and noting the Holder inequality, we have

$$\big|(g * f)(x) - \big(\tau_h^c(g * f)\big)(x)\big|$$
$$\leqslant \int_0^\infty |f(u)|u^c\left|g\left(\frac{x}{u}\right) - h^c g\left(\frac{hx}{u}\right)\right|u^{-c}\frac{du}{u}, \tag{3.5}$$

since

$$\int_0^\infty g\left(\frac{hx}{u}\right)f(u)\frac{du}{u} = \int_0^\infty \tau_u^c g\left(\frac{x}{u}\right)f(u)h^{-c}\frac{du}{u}.$$

If $1 < p \leqslant \infty$, then there follows

$$\begin{aligned}
&\big|(g * f)(x) - \big(\tau_h^c(g * f)\big)(x)\big| \\
&\quad \leqslant \|f\|_{X_c^p} \left(\int_0^\infty \left| g\left(\frac{x}{u}\right) - h^c g\left(\frac{hx}{u}\right)\right|^{p'} u^{-cp'} \frac{du}{u} \right)^{1/p'} \\
&\quad \leqslant \|f\|_{X_c^p} \left(\int_0^\infty \big|g(y) - h^c g(hy)\big|^{p'} \left(\frac{x}{y}\right)^{-cp'} \frac{dy}{y} \right)^{1/p'}, \qquad (3.6)
\end{aligned}$$

and thus

$$\big|(g * f)(x) - \big(\tau_h^c(g * f)\big)(x)\big| \leqslant x^{-c} \|f\|_{X_c^p} \big\|g - \tau_h^c g\big\|_{X_c^{p'}} \qquad (3.7)$$

for $1 < p \leqslant \infty$. This estimate also holds for $p = 1$, because from (3.5) we have

$$\begin{aligned}
&\big|(g * f)(x) - \big(\tau_h^c(g * f)\big)(x)\big| \\
&\quad \leqslant \int_0^\infty \left| f\left(\frac{x}{y}\right)\right| y^{-c} \big[y^c \big|g(y) - h^c g(hy)\big|\big] \frac{dy}{y} \\
&\quad \leqslant \big\|g - \tau_h^c g\big\|_{X_c^\infty} \int_0^\infty \left| f\left(\frac{x}{y}\right)\right| y^{-c-1}\, dy = x^{-c} \|f\|_{X_c^1} \big\|g - \tau_h^c g\big\|_{X_c^\infty}.
\end{aligned}$$

According to the relation (7) in [8], one has

$$\lim_{h\to 1} \big\|g - \tau_h^c g\big\|_{X_c^{p'}} = 0, \quad g \in X_c^{p'}, \qquad (3.8)$$

and hence (3.7) yields the continuity of the convolution result:

$$\lim_{h\to 1} \big|(g * f)(x) - \big(\tau_h^c(g * f)\big)(x)\big| = 0.$$

To prove the estimate (3.4), for $1 < p \leqslant \infty$ similarly to (3.5) and (3.6) we have

$$\begin{aligned}
\big|(g * f)(x)\big| &\leqslant \int_0^\infty |f(u)| u^c \left| g\left(\frac{x}{u}\right)\right| u^{-c} \frac{du}{u} \\
&\leqslant \|f\|_{X_c^p} \left(\int_0^\infty \left| g\left(\frac{x}{u}\right)\right|^{p'} u^{-cp'} \frac{du}{u} \right)^{1/p'} \\
&= \|f\|_{X_c^p} \left(\int_0^\infty |g(y)|^{p'} \left(\frac{x}{y}\right)^{-cp'} \frac{dy}{y} \right)^{1/p'},
\end{aligned}$$

and hence

$$|(g*f)(x)| \leqslant x^{-c}\|f\|_{X_c^p}\|g\|_{X_c^{p'}} \tag{3.9}$$

for $1 < p \leqslant \infty$. In particular,

$$|(g*f)(x)| \leqslant x^{-c}\|f\|_{X_c^\infty}\|g\|_{X_c^1}. \tag{3.10}$$

Therefore the estimate (3.9) also holds for $p = 1$, because, in accordance with (3.3) and (3.10),

$$|(g*f)(x)| = |(f*g)(x)| \leqslant \|f\|_{X_c^1} x^{-c}\|g\|_{X_c^\infty}.$$

Now (3.9) yields (3.4), and so the theorem is proved. □

Remark 2. In the case $p = 2$ Theorems 1 and 2 were proved in [8, Lemma 2.2]. For $p = 1$ see [1].

Taking $g(u) = k(u)$ in Theorems 1 and 2 we obtain the following result for the Mellin integral convolution operator:

$$(Kf)(x) = (k*f)(x) = \int_0^\infty k\left(\frac{x}{u}\right) f(u)\frac{du}{u} \quad (x > 0). \tag{3.11}$$

Theorem 3. *Let $c \in \mathbf{R}$ and $1 \leqslant p \leqslant \infty$.*

(a) *If $k \in X_c^1$, then the operator K is bounded in X_c^p and there holds the estimate*

$$\|Kf\|_{X_c^p} \leqslant C\|f\|_{X_c^p}, \tag{3.12}$$

where

$$C = \|k\|_{X_c^1} < \infty. \tag{3.13}$$

(b) *If $k \in X_c^{p'}$, then the operator K is bounded from X_c^p into X_c^∞ and there holds the estimate*

$$\|Kf\|_{X_c^\infty} \leqslant C'\|f\|_{X_c^p}, \tag{3.14}$$

where

$$C' = \|k\|_{X_c^{p'}} < \infty. \tag{3.15}$$

Remark 3. Theorem 3(a) is the X_c^p-analogue of the Young inequality in L_p-spaces; see, for example, [19, p. 199].

4. Hadamard-type fractional integration in X_c^p

In this section we apply Theorem 3(a) to prove the boundedness property for the Hadamard-type fractional integrals (1.13), (1.25), (1.32), and (1.33) in the space X_c^p. Such a property for the Hadamard-type operators $\mathcal{J}_{0+,\mu}^{\alpha} f$ and $\mathcal{J}_{-,\mu}^{\alpha} f$ in (1.13) and (1.25) are given by the following theorem.

Theorem 4. *Let $c \in \mathbf{R}$, $1 \leqslant p \leqslant \infty$, $\alpha > 0$ and $\mu \in \mathbf{C}$.*

(a) *If $\mathrm{Re}(\mu) > c$, then the operator $\mathcal{J}_{0+,\mu}^{\alpha}$ is bounded in X_c^p, and*

$$\left\|\mathcal{J}_{0+,\mu}^{\alpha} f\right\|_{X_c^p} \leqslant C_1^+ \|f\|_{X_c^p}, \quad C_1^+ = [\mathrm{Re}(\mu) - c]^{-\alpha}. \tag{4.1}$$

(b) *If $\mathrm{Re}(\mu) > -c$, then the operator $\mathcal{J}_{-,\mu}^{\alpha}$ is bounded in X_c^p, and*

$$\left\|\mathcal{J}_{-,\mu}^{\alpha} f\right\|_{X_c^p} \leqslant C_1^- \|f\|_{X_c^p}, \quad C_1^- = [\mathrm{Re}(\mu) + c]^{-\alpha}. \tag{4.2}$$

Proof. $\mathcal{J}_{0+,\mu}^{\alpha}$ and $\mathcal{J}_{-,\mu}^{\alpha}$ are Mellin convolution operators of the form (3.11), namely

$$\left(\mathcal{J}_{0+,\mu}^{\alpha} f\right)(x) = \int_0^{\infty} k_1^+\left(\frac{x}{u}\right) f(u) \frac{du}{u} \quad (x \in \mathbf{R}_+), \tag{4.3}$$

$$k_1^+(u) = 0 \quad (0 < u < 1),$$
$$k_1^+(u) = \frac{1}{\Gamma(\alpha)} u^{-\mu} \left(\log(u)\right)^{\alpha-1} \quad (u > 1), \tag{4.4}$$

and

$$\left(\mathcal{J}_{-,\mu}^{\alpha} f\right)(x) = \int_0^{\infty} k_1^-\left(\frac{x}{u}\right) f(u) \frac{du}{u} \quad (x \in \mathbf{R}_+), \tag{4.5}$$

$$k_1^-(u) = \frac{1}{\Gamma(\alpha)} u^{\mu} \left[\log\left(\frac{1}{u}\right)\right]^{\alpha-1} \quad (0 < u < 1),$$
$$k_1^-(u) = 0 \quad (u > 1), \tag{4.6}$$

respectively. It is directly verified that the constant C of (3.13) is given by

$$C = C_1^+ = [\mathrm{Re}(\mu) - c]^{-\alpha} \tag{4.7}$$

for the operator $\mathcal{J}_{0+,\mu}^{\alpha}$, while by

$$C = C_1^- = [\mathrm{Re}(\mu) + c]^{-\alpha} \tag{4.8}$$

for the operator $\mathcal{J}_{-,\mu}^{\alpha}$. Thus Theorem 4 follows from Theorem 3(a). □

We also note that the results in Theorem 4 may be proved on the basis of the relations (2.26), (2.27) between the Hadamard-type integrals (1.13), (1.25) and the Liouville fractional integrals (2.24), (2.25) by using the following result (see [2, Theorem 5.7] with $\omega = -cp$).

Lemma 3. *Let* $c \in \mathbf{R}$, $1 \leqslant p \leqslant \infty$ *and* $\alpha > 0$.

(a) *If* $c < 0$, *then the Liouville fractional integration operator* I_+^α *is bounded in* L_c^p, *and*

$$\left\| I_+^\alpha f \right\|_{L_c^p} \leqslant C^+ \| f \|_{L_c^p}, \quad C^+ = |c|^{-\alpha}. \tag{4.9}$$

(b) *If* $c > 0$, *then the Liouville fractional integration operator* I_-^α *is bounded in* L_c^p, *and*

$$\left\| I_-^\alpha f \right\|_{L_c^p} \leqslant C^- \| f \|_{L_c^p}, \quad C^- = c^{-\alpha}. \tag{4.10}$$

Indeed, using (2.26), (2.11) with $\zeta = \mu$ and f being replaced by $A^{-1} I_+^\alpha A M_\mu f$ and (2.23) with $\varphi = I_+^\alpha A M_\mu f$, we have

$$\begin{aligned} \left\| \mathcal{J}_{0+,\mu}^\alpha f \right\|_{X_c^p} &= \left\| M_{-\mu} A^{-1} I_+^\alpha A M_\mu f \right\|_{X_c^p} \\ &= \left\| A^{-1} I_+^\alpha A M_\mu f \right\|_{X_{c-\mathrm{Re}(\mu)}^p} = \left\| I_+^\alpha A M_\mu f \right\|_{L_{c-\mathrm{Re}(\mu)}^p}. \end{aligned} \tag{4.11}$$

By Lemmas 2(a) and 1(a) $A M_\mu$ is an isometric isomorphism of X_c^p onto $L_{c-\mathrm{Re}(\mu)}^p$. Since $c < \mathrm{Re}(\mu)$, we can apply Lemma 3(a) with f being replaced by $A M_\mu f$, c replaced by $c - \mathrm{Re}(\mu)$ and $C^+ = C_1^+$ to deduce

$$\left\| I_+^\alpha A M_\mu f \right\|_{L_{c-\mathrm{Re}(\mu)}^p} \leqslant C_1^+ \| A M_\mu f \|_{L_{c-\mathrm{Re}(\mu)}^p}.$$

Substituting this estimate into (4.11), using (2.22) with $\varphi = M_\mu f$ and (2.11) with $\zeta = \mu$, we find

$$\left\| \mathcal{J}_{0+,\mu}^\alpha f \right\|_{X_c^p} \leqslant C_1^+ \| A M_\mu f \|_{L_{c-\mathrm{Re}(\mu)}^p} = C_1^+ \| M_\mu f \|_{X_{c-\mathrm{Re}(\mu)}^p} = C_1^+ \| f \|_{X_c^p},$$

which proves (4.1).

The relation (4.2) is proved similarly by applying (2.27), (2.11), (2.23), Lemma 3(b), (2.22), and (2.11), thus

$$\left\| \mathcal{J}_{-,\mu}^\alpha f \right\|_{X_c^p} = \left\| A^{-1} I_+^\alpha A M_{-\mu} f \right\|_{X_{c-\mathrm{Re}(\mu)}^p} = \left\| I_+^\alpha A M_{-\mu} f \right\|_{L_{c-\mathrm{Re}(\mu)}^p}$$

and

$$\begin{aligned} \left\| \mathcal{J}_{-,\mu}^\alpha f \right\|_{X_c^p} &= \left\| I_+^\alpha A M_{-\mu} f \right\|_{L_{c-\mathrm{Re}(\mu)}^p} \leqslant C_1^- \| A M_{-\mu} f \|_{L_{c-\mathrm{Re}(\mu)}^p} \\ &= C_1^- \| M_{-\mu} f \|_{X_{c-\mathrm{Re}(\mu)}^p} = C_1^- \| f \|_{X_c^p}. \end{aligned}$$

The next corollary follows from Theorem 4 if we take $c = 1/p$ and apply definition (1.36).

Corollary 1. *Let* $1 \leqslant p \leqslant \infty$, $\alpha > 0$ *and* $\mu \in \mathbf{C}$.

(a) *If* $\mathrm{Re}(\mu) > 1/p$, *then the operator* $\mathcal{J}^{\alpha}_{0+,\mu}$ *is bounded in* $L^p(\mathbf{R}_+)$, *and*

$$\left\| \mathcal{J}^{\alpha}_{0+,\mu} f \right\|_p \leqslant l_1^+ \| f \|_p, \quad l_1^+ = \left[\mathrm{Re}(\mu) - \frac{1}{p} \right]^{-\alpha}. \tag{4.12}$$

(b) *If* $\mathrm{Re}(\mu) > -1/p$, *then the operator* $\mathcal{J}^{\alpha}_{-,\mu}$ *is bounded in* $L^p(\mathbf{R}_+)$, *and*

$$\left\| \mathcal{J}^{\alpha}_{-,\mu} f \right\|_p \leqslant l_1^- \| f \|_p, \quad l_1^- = \left[\mathrm{Re}(\mu) + \frac{1}{p} \right]^{-\alpha}. \tag{4.13}$$

Taking into account the obvious relations

$$\mathcal{I}^{\alpha}_{0+,\mu} f = \mathcal{J}^{\alpha}_{0+,\mu+1} f, \qquad \mathcal{I}^{\alpha}_{-,\mu} f = \mathcal{J}^{\alpha}_{-,\mu-1} f \tag{4.14}$$

between Hadamard-type fractional integrals (1.32), (1.13) and (1.33), (1.25), and applying Theorem 4 with μ being replaced by $\mu + 1$ and $\mu - 1$, we obtain the X_c^p-boundedness properties of the Hadamard-type fractional integration operators $\mathcal{I}^{\alpha}_{0+,\mu} f$ and $\mathcal{I}^{\alpha}_{-,\mu} f$.

Theorem 5. *Let* $c \in \mathbf{R}$, $1 \leqslant p \leqslant \infty$, $\alpha > 0$ *and* $\mu \in \mathbf{C}$.

(a) *If* $\mathrm{Re}(\mu) > c - 1$, *then the operator* $\mathcal{I}^{\alpha}_{0+,\mu}$ *is bounded in* X_c^p, *and*

$$\left\| \mathcal{I}^{\alpha}_{0+,\mu} f \right\|_{X_c^p} \leqslant C_2^+ \| f \|_{X_c^p}, \quad C_2^+ = [\mathrm{Re}(\mu) + 1 - c]^{-\alpha}. \tag{4.15}$$

(b) *If* $\mathrm{Re}(\mu) > 1 - c$, *then the operator* $\mathcal{I}^{\alpha}_{-,\mu}$ *is bounded in* X_c^p, *and*

$$\left\| \mathcal{I}^{\alpha}_{-,\mu} f \right\|_{X_c^p} \leqslant C_2^- \| f \|_{X_c^p}, \quad C_2^- = [\mathrm{Re}(\mu) + c - 1]^{-\alpha}. \tag{4.16}$$

Corollary 2. *Let* $1 \leqslant p \leqslant \infty$, $\alpha > 0$ *and* $\mu \in \mathbf{C}$, *and* p' *be given by* (3.1).

(a) *If* $\mathrm{Re}(\mu) > -1/p'$, *then the operator* $\mathcal{I}^{\alpha}_{0+,\mu}$ *is bounded in* $L^p(\mathbf{R}_+)$, *and*

$$\left\| \mathcal{I}^{\alpha}_{0+,\mu} f \right\|_p \leqslant l_2^+ \| f \|_p, \quad l_2^+ = \left[\mathrm{Re}(\mu) + \frac{1}{p'} \right]^{-\alpha}. \tag{4.17}$$

(b) *If* $\mathrm{Re}(\mu) > 1/p'$, *then the operator* $\mathcal{I}^{\alpha}_{-,\mu}$ *is bounded in* $L^p(\mathbf{R}_+)$, *and*

$$\left\| \mathcal{I}^{\alpha}_{-,\mu} f \right\|_p \leqslant l_2^- \| f \|_p, \quad l_2^- = \left[\mathrm{Re}(\mu) - \frac{1}{p'} \right]^{-\alpha}. \tag{4.18}$$

Putting $\mu = 0$ in Theorem 4, we obtain the X_c^p-boundedness of the Hadamard fractional integration operators $\mathcal{J}^{\alpha}_{0+} f$ and $\mathcal{J}^{\alpha}_{-} f$ given by (1.30) and (1.31).

Theorem 6. *Let $c \in \mathbf{R}$, $1 \leqslant p \leqslant \infty$ and $\alpha > 0$.*

(a) *If $c < 0$, then the operator $\mathcal{J}^{\alpha}_{0+}$ is bounded in X^p_c, and*

$$\left\| \mathcal{J}^{\alpha}_{0+} f \right\|_{X^p_c} \leqslant C^+_3 \| f \|_{X^p_c}, \quad C^+_3 = |c|^{-\alpha}. \tag{4.19}$$

(b) *If $c > 0$, then the operator $\mathcal{J}^{\alpha}_{-}$ is bounded in X^p_c, and*

$$\left\| \mathcal{J}^{\alpha}_{-} f \right\|_{X^p_c} \leqslant C^-_3 \| f \|_{X^p_c}, \quad C^-_3 = c^{-\alpha}. \tag{4.20}$$

Corollary 3. *If $1 \leqslant p < \infty$ and $\alpha > 0$, then the operator $\mathcal{J}^{\alpha}_{-}$ is bounded in $L^p(\mathbf{R}_+)$, and*

$$\left\| \mathcal{J}^{\alpha}_{-} f \right\|_p \leqslant l_3 \| f \|_p, \quad l_3 = p^{\alpha}. \tag{4.21}$$

We have stated Corollaries 1–3 explicitly since the operators $\mathcal{J}^{\alpha}_{0+,\mu}$, $\mathcal{J}^{\alpha}_{-,\mu}$, $\mathcal{I}^{\alpha}_{0+,\mu}$, $\mathcal{I}^{\alpha}_{0+,\mu}$, and $\mathcal{J}^{\alpha}_{-}$ map $L^p(\mathbf{R}_+)$ into itself for any $\alpha > 0$, whereas the classical Liouville operators I^{α}_{+}, I^{α}_{-} of (2.24), (2.25) only map $L^p(\mathbf{R})$ into $L^q(\mathbf{R})$ for $0 < \alpha < 1$, $1 < p < 1/(1-\alpha)$ with $q = p/(1-\alpha p)$; see Lemma 5 below. This is one of the further advantages of the four operators introduced in this paper.

Remark 4. Theorem 6 would also follow from Lemmas 2 and 3, if we take into account the relations (2.26) and (2.27) with $\mu = 0$.

Remark 5. The boundedness of the operator $\mathcal{J}^{\alpha}_{-}$ in a weighted space of p-summable functions was indicated in [13].

Remark 6. Corollaries 1–3 (but not the more general assertions of Theorems 4–6) could be also proved on the basis of the well-known theorem on the boundedness in $L^p(\mathbf{R}_+)$ of the integral operator

$$(\mathbf{K} f)(x) = \int_0^{\infty} k(x,u) f(u)\, du \quad (x > 0), \tag{4.22}$$

with a homogeneous kernel $k(x,u)$ $(x > 0,\ u > 0)$ of degree -1: $k(\lambda x, \lambda u) = \lambda^{-1} k(x,u)$ $(\lambda > 0)$; see, for example, [2, Theorem 1.5]. According to this theorem, if $1 \leqslant p < \infty$ and

$$K' = \int_0^{\infty} |k(x,1)| x^{-1/p'}\, dx = \int_0^{\infty} |k(1,u)| u^{-1/p}\, du < \infty, \tag{4.23}$$

then the operator $\mathbf{K}$ is bounded in $L^p(\mathbf{R}_+)$, and

$$\| \mathbf{K} f \|_p \leqslant K' \| f \|_p. \tag{4.24}$$

Remark 7. As it was shown in Corollary 3, the boundedness property in $L^p(\mathbf{R}_+)$ holds for the one Hadamard operator $\mathcal{J}_-^\alpha f$, but such a property is not valid for the other $\mathcal{J}_{0+}^\alpha f$. The above theorem in Remark 6 yields more clearly this fact. Indeed, for the operator $\mathcal{J}_-^\alpha f$ the kernel $k(x,u)$ in (4.22) has the form

$$k(x,u)=0 \quad (u<x),$$
$$k(x,u)=\frac{1}{\Gamma(\alpha)}\log\left(\frac{u}{x}\right)^{\alpha-1}\frac{1}{u} \quad (u>x), \tag{4.25}$$

with the finite constant

$$K'=\frac{1}{\Gamma(\alpha)}\int_0^1 |\log x|^{\alpha-1}x^{-1/p'}\,dx=\int_0^\infty e^{-u/p}u^{\alpha-1}\,du=p^\alpha. \tag{4.26}$$

However, for the operator $\mathcal{J}_{0+}^\alpha f$,

$$k(x,u)=\frac{1}{\Gamma(\alpha)}\log\left(\frac{x}{u}\right)^{\alpha-1}\frac{1}{u} \quad (u<x),$$
$$k(x,u)=0 \quad (u>x), \tag{4.27}$$

and the constant in (4.23) yields the divergent integral

$$K'=\frac{1}{\Gamma(\alpha)}\int_1^\infty (\log x)^{\alpha-1}x^{-1/p'}\,dx=\int_0^\infty e^{u/p}u^{\alpha-1}\,du. \tag{4.28}$$

This fact leads us to conjecture that the operator $\mathcal{J}_{0+}^\alpha f$ is probably bounded from $L^p(\mathbf{R}_+)$ into another space. This is considered below.

5. Hadamard-type fractional integration from X_c^p into X_c^q

In the previous Section 4 we have studied the boundedness of the Hadamard-type fractional operators (1.13), (1.25), (1.32), and (1.33) from X_c^p into X_c^p. In this section we show that these results stay true for the mappings from the one X_c^p-space into another X_c^q. For this matter we recall the following application of the Riesz–Thorin convexity theorem [19, 195ff], namely Young's inequality.

Lemma 4. *Let $c\in\mathbf{R}$ and let p, r and q be such that*

$$1\leqslant p\leqslant\infty, \quad 1\leqslant r<\infty, \quad \frac{1}{q}=\frac{1}{p}+\frac{1}{r}-1\geqslant 0. \tag{5.1}$$

If $f\in X_c^p$ and $k\in X_c^r$, then for almost all $x>0$, there exists the integral $(Kf)(x)$, given by (3.11), *and the integral operator $Kf=k*f$ is bounded from X_c^p into X_c^q: there holds the estimate*

$$\|\mathbf{K}f\|_{X_c^q} \leqslant C_0 \|f\|_{X_c^p}, \tag{5.2}$$

where

$$C_0 := \|k\|_{X_c^r} < \infty. \tag{5.3}$$

From this lemma we deduce the corresponding statements for the Hadamard-type fractional integrals (1.13) and (1.25). Namely the following result is true.

Theorem 7. *Let* $\alpha > 0$, $c \in \mathbf{R}$ *and* $1 \leqslant p \leqslant q \leqslant \infty$ *be such that* $\alpha > (1/p) - (1/q)$, *and* $\mu \in \mathbf{C}$.

(a) *If* $\mathrm{Re}(\mu) > c$, *then the operator* $\mathcal{J}^{\alpha}_{0+,\mu}$ *is bounded from* X_c^p *into* X_c^q, *and*

$$\left\|\mathcal{J}^{\alpha}_{0+,\mu} f\right\|_{X_c^q} \leqslant C_4^+ \|f\|_{X_c^p}, \tag{5.4}$$

where

$$C_4^+ := \bigl([\mathrm{Re}(\mu) - c]r\bigr)^{-\alpha+1/r} \frac{(\Gamma[(\alpha-1)r+1])^{1/r}}{\Gamma(\alpha)},$$
$$\frac{1}{r} = \frac{1}{q} - \frac{1}{p} + 1. \tag{5.5}$$

(b) *If* $\mathrm{Re}(\mu) > -c$, *then the operator* $\mathcal{J}^{\alpha}_{-,\mu}$ *is bounded from* X_c^p *into* X_c^q, *and*

$$\left\|\mathcal{J}^{\alpha}_{-,\mu} f\right\|_{X_c^q} \leqslant C_4^- \|f\|_{X_c^p}, \tag{5.6}$$

where

$$C_4^- := \bigl([\mathrm{Re}(\mu) + c]r\bigr)^{-\alpha+1/r} \frac{(\Gamma[(\alpha-1)r+1])^{1/r}}{\Gamma(\alpha)},$$
$$\frac{1}{r} = \frac{1}{q} - \frac{1}{p} + 1. \tag{5.7}$$

Proof. The Hadamard-type integral $\mathcal{J}^{\alpha}_{0+,\mu}$ is an integral (3.11) of the form (4.3), (4.4). If $1 \leqslant p \leqslant q \leqslant \infty$, we define r as in (5.1) by $1/r = (1/q) - (1/p) + 1$. It is clear that $(1/p) + (1/r) \geqslant 1$. So the conditions in (5.1) are satisfied and we can now apply Lemma 4 to the Hadamard-type integral $\mathcal{J}^{\alpha}_{0+,\mu}$. Since $1 \leqslant r < \infty$, then in accordance with (4.4) and (1.34),

$$\begin{aligned}\|k_1^+\|_{X_c^r} &= \frac{1}{\Gamma(\alpha)} \left(\int_1^{\infty} \bigl[\bigl|x^{c-\mu}\bigr| (\log x)^{\alpha-1} \bigr]^r \frac{dx}{x} \right)^{1/r} \\ &= \frac{1}{\Gamma(\alpha)} \left(\int_0^{\infty} e^{-[\mathrm{Re}(\mu)-c]ur} u^{(\alpha-1)r} \, du \right)^{1/r}.\end{aligned} \tag{5.8}$$

Since the integral is convergent if and only if $\alpha > 1 - (1/r) = (1/p) - (1/q)$ and $\mathrm{Re}(\mu) > c$, one has $k_1^+ \in X_c^r$ $(1 \leqslant r < \infty)$ if and only if $\alpha > (1/p) - (1/q)$ and $\mathrm{Re}(\mu) > c$, which coincides with the conditions of the theorem in the case (a). Substituting $y = [\mathrm{Re}(\mu) - c]ur$ in (5.8) gives the constant C_4^+ in (5.5). Applying Lemma 4 with $k(u) = k_1^+(u)$ we obtain assertion (a) of the theorem.

To prove (b), we use the obvious property

$$\big(\mathcal{J}_{-,\mu}^{\alpha} f\big)(x) = \big(Q\mathcal{J}_{0+,\mu}^{\alpha} Qf\big)(x), \tag{5.9}$$

connecting the Hadamard-type fractional integral (1.25) and (1.13) via the elementary operator (2.5). By Lemma 1(e) Q is an isometric isomorphism of X_c^r onto X_{-c}^r. Therefore since $\mathrm{Re}(\mu) > -c$, we can apply assertion (a) with c being replaced by $-c$ to obtain (b). Using (5.9), (2.16), and the estimate (5.2) with f and c being replaced by Qf and $-c$, we have

$$\big\|\mathcal{J}_{-,\mu}^{\alpha} f\big\|_{X_c^q} = \big\|\mathcal{J}_{0+,\mu}^{\alpha} Qf\big\|_{-c,q} \leqslant C_0' \|Qf\|_{X_{-c}^p} = C_0' \|f\|_{X_c^p},$$

which gives (5.6). Here the constant C_0', being obtained from the constant C_4^+ in (5.5) when replacing c by $-c$, coincides with C_4^- given in (5.7). We also note that assertion (b) of the theorem can be proved similarly to the proof of (a) by applying Lemma 4 to the Hadamard-type integral $\mathcal{J}_{-,\mu}^{\alpha}$ in the form (4.5), (4.6). □

Putting $c = 1/p$ in Theorem 7, we obtain the following result.

Corollary 4. *Let $\alpha > 0$, $c \in \mathbf{R}$ and $1 \leqslant p \leqslant q \leqslant \infty$ be such that $\alpha > (1/p) - (1/q)$, and $\mu \in \mathbf{C}$.*

(a) *If $\mathrm{Re}(\mu) > 1/p$, then the operator $\mathcal{J}_{0+,\mu}^{\alpha}$ is bounded from $L^p(\mathbf{R}_+)$ into $X_{1/p}^q$, and*

$$\big\|\mathcal{J}_{0+,\mu}^{\alpha} f\big\|_{X_{1/p}^q} \leqslant l_4^+ \|f\|_p, \tag{5.10}$$

$$l_4^+ := \big([\mathrm{Re}(\mu) - 1/p]r\big)^{-\alpha+1/r} \frac{(\Gamma[(\alpha-1)r+1])^{1/r}}{\Gamma(\alpha)}. \tag{5.11}$$

(b) *If $\mathrm{Re}(\mu) > -1/p$, then the operator $\mathcal{J}_{-,\mu}^{\alpha}$ is bounded from $L^p(\mathbf{R}_+)$ into $X_{1/p}^q$, and*

$$\big\|\mathcal{J}_{-,\mu}^{\alpha} f\big\|_{X_{1/p}^q} \leqslant l_4^- \|f\|_p, \tag{5.12}$$

$$l_4^- := \big([\mathrm{Re}(\mu) + 1/p]r\big)^{-\alpha+1/r} \frac{(\Gamma[(\alpha-1)r+1])^{1/r}}{\Gamma(\alpha)}. \tag{5.13}$$

The corresponding statements for the Hadamard-type fractional integration operators $\mathcal{I}_{0+,\mu}^{\alpha} f$ and $\mathcal{I}_{-,\mu}^{\alpha} f$ of (1.32) and (1.33) follow from (4.14) and Theorems 7(a) and (b) with μ being replaced by $\mu + 1$ and $\mu - 1$, respectively.

Theorem 8. *Let $\alpha > 0$, $c \in \mathbf{R}$ and $1 \leqslant p \leqslant q \leqslant \infty$ be such that $\alpha > (1/p) - (1/q)$, and $\mu \in \mathbf{C}$.*

(a) *If $\operatorname{Re}(\mu) > c - 1$, then the operator $\mathcal{I}^{\alpha}_{0+,\mu}$ is bounded from X^p_c into X^q_c, and*

$$\left\|\mathcal{I}^{\alpha}_{0+,\mu} f\right\|_{X^q_c} \leqslant C_5^+ \|f\|_{X^p_c}, \tag{5.14}$$

$$C_5^+ := \bigl([\operatorname{Re}(\mu) + 1 - c]r\bigr)^{-\alpha+1/r} \frac{(\Gamma[(\alpha-1)r+1])^{1/r}}{\Gamma(\alpha)}. \tag{5.15}$$

(b) *If $\operatorname{Re}(\mu) > 1 - c$, then the operator $\mathcal{I}^{\alpha}_{-,\mu}$ is bounded from X^p_c into X^q_c, and*

$$\left\|\mathcal{I}^{\alpha}_{-,\mu} f\right\|_{X^q_c} \leqslant C_5^- \|f\|_{X^p_c}, \tag{5.16}$$

$$C_5^- := \bigl([\operatorname{Re}(\mu) + c - 1]r\bigr)^{-\alpha+1/r} \frac{(\Gamma[(\alpha-1)r+1])^{1/r}}{\Gamma(\alpha)}. \tag{5.17}$$

Corollary 5. *Let $\alpha > 0$, $c \in \mathbf{R}$ and $1 \leqslant p \leqslant q \leqslant \infty$ be such that $\alpha > (1/p) - (1/q)$, p' is given by* (3.1) *and $\mu \in \mathbf{C}$.*

(a) *If $\operatorname{Re}(\mu) > -1/p'$, then the operator $\mathcal{I}^{\alpha}_{0+,\mu}$ is bounded from $L^p(\mathbf{R}_+)$ into $X^q_{1/p}$, and*

$$\left\|\mathcal{I}^{\alpha}_{0+,\mu} f\right\|_{X^q_{1/p}} \leqslant l_5^+ \|f\|_p, \tag{5.18}$$

$$l_5^+ := \bigl([\operatorname{Re}(\mu) + 1/p']r\bigr)^{-\alpha+1/r} \frac{(\Gamma[(\alpha-1)r+1])^{1/r}}{\Gamma(\alpha)}. \tag{5.19}$$

(b) *If $\operatorname{Re}(\mu) > 1/p'$, then the operator $\mathcal{I}^{\alpha}_{-,\mu}$ is bounded from $L^p(\mathbf{R}_+)$ into $X^q_{1/p}$, and*

$$\left\|\mathcal{I}^{\alpha}_{-,\mu} f\right\|_{X^q_{1/p}} \leqslant l_5^- \|f\|_p, \tag{5.20}$$

$$l_5^- := \bigl([\operatorname{Re}(\mu) - 1/p']r\bigr)^{-\alpha+1/r} \frac{(\Gamma[(\alpha-1)r+1])^{1/r}}{\Gamma(\alpha)}. \tag{5.21}$$

Putting $\mu = 0$ in Theorem 7 we arrive at the boundedness of the Hadamard fractional integration operators (1.30) and (1.31) from X^p_c into X^q_c.

Theorem 9. *Let $\alpha > 0$, $c \in \mathbf{R}$ and $1 \leqslant p \leqslant q \leqslant \infty$ be such that $\alpha > (1/p) - (1/q)$.*

(a) *If $c < 0$, then the operator $\mathcal{J}^{\alpha}_{0+}$ is bounded from X^p_c into X^q_c, and*

$$\left\|\mathcal{J}^{\alpha}_{0+} f\right\|_{X^q_c} \leqslant C_6^+ \|f\|_{X^p_c}, \tag{5.22}$$

$$C_6^+ := (|c|r)^{-\alpha+1/r} \frac{(\Gamma[(\alpha-1)r+1])^{1/r}}{\Gamma(\alpha)}. \tag{5.23}$$

(b) *If* $c > 0$, *then the operator* $\mathcal{J}_-^\alpha$ *is bounded from* X_c^p *into* X_c^q, *and*

$$\left\|\mathcal{J}_-^\alpha f\right\|_{X_c^q} \leqslant C_6^- \|f\|_{X_c^p}, \tag{5.24}$$

$$C_6^- := (cr)^{-\alpha+1/r} \frac{(\Gamma[(\alpha-1)r+1])^{1/r}}{\Gamma(\alpha)}. \tag{5.25}$$

Corollary 6. *If* $1 \leqslant p \leqslant q \leqslant \infty$ *and* $\alpha > (1/p) - (1/q)$, *then the operator* $\mathcal{J}_-^\alpha$ *is bounded from* $L^p(\mathbf{R}_+)$ *into* $X_{1/p}^q$, *and*

$$\left\|\mathcal{J}_-^\alpha f\right\|_{X_{1/p}^q} \leqslant l_6 \|f\|_p, \tag{5.26}$$

$$l_6 := \left(\frac{r}{p}\right)^{-\alpha+1/r} \frac{(\Gamma[(\alpha-1)r+1])^{1/r}}{\Gamma(\alpha)}. \tag{5.27}$$

Remark 8. Theorems 7–9 and Corollaries 4–6 in the particular case $q = p$ imply the inequalities of Theorems 1–3 and Corollaries 1–3, respectively. They were first established directly since they do not involve deep theorems. Observe that particular attention is placed upon the many constants involved in the estimates deduced; they are generally best possible.

6. Hadamard-type fractional integration from X_c^p into X_c^q in special cases

In Sections 4 and 5 we have established the boundedness of the Hadamard-type fractional integration operators (1.13), (1.25), (1.32), and (1.33) in the cases when $\mathrm{Re}(\mu) > c$, $\mathrm{Re}(\mu) > -c$, $\mathrm{Re}(\mu) > c-1$, and $\mathrm{Re}(\mu) > 1-c$, respectively. In this section we show that such statements can be obtained in the limiting cases $\mathrm{Re}(\mu) = c$, $\mathrm{Re}(\mu) = -c$, $\mathrm{Re}(\mu) = c-1$, and $\mathrm{Re}(\mu) = 1-c$ for certain special relations between $\mathrm{Re}(\mu)$ and c. Our arguments are based on the corresponding assertions for the Liouville fractional integration operators (2.24) and (2.25).

The classical result [2, Theorem 5.3] is known as a Hardy–Littlewood theorem with limiting exponent. It states that

Lemma 5. *Let* $1 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$ *and* $\alpha > 0$. *The operators* I_+^α *and* I_-^α *are bounded from* $L^p(\mathbf{R})$ *into* $L^q(\mathbf{R})$ *if and only if*

$$0 < \alpha < 1, \qquad 1 < p < \frac{1}{\alpha}, \qquad q = \frac{p}{1-\alpha p}. \tag{6.1}$$

It follows from Lemma 5 that if the conditions in (6.1) are satisfied, then there hold the estimates

$$\left\|I_+^\alpha f\right\|_q \leqslant K^+ \|f\|_p \tag{6.2}$$

and

$$\left\| I_{-}^{\alpha} f \right\|_q \leqslant K^{-} \| f \|_p, \tag{6.3}$$

where K^{+} and K^{-} are certain unspecified positive constants.

Similar statements are true for the Hadamard-type fractional integrals (1.13) and (1.25) in the cases when $\operatorname{Re}(\mu) = c$ and $\operatorname{Re}(\mu) = -c$, respectively.

Theorem 10. *Let* $0 < \alpha < 1$, $1 < p < 1/\alpha$, $q = p/(1-\alpha p)$, $c \in \mathbf{R}$ *and* $\mu \in \mathbf{C}$.

(a) *If* $\operatorname{Re}(\mu) = c$, *then the operator* $\mathcal{J}_{0+,\mu}^{\alpha}$ *is bounded from* X_c^p *into* X_c^q, *and*

$$\left\| \mathcal{J}_{0+,\mu}^{\alpha} f \right\|_{X_c^q} \leqslant C_7^{+} \| f \|_{X_c^p}. \tag{6.4}$$

(b) *If* $\operatorname{Re}(\mu) = -c$, *then the operator* $\mathcal{J}_{-,\mu}^{\alpha}$ *is bounded from* X_c^p *into* X_c^q, *and*

$$\left\| \mathcal{J}_{-,\mu}^{\alpha} f \right\|_{X_c^q} \leqslant C_7^{-} \| f \|_{X_c^p}. \tag{6.5}$$

Here C_7^{+} *and* C_7^{-} *are certain, unspecified positive constants.*

Proof. Let $\operatorname{Re}(\mu) = c$. Then in accordance with (4.11), p being replaced by q, together with (2.20), (1.36) we have

$$\left\| \mathcal{J}_{0+,\mu}^{\alpha} f \right\|_{X_c^q} = \left\| I_{+}^{\alpha} A M_{\mu} f \right\|_{L_0^q} = \left\| I_{+}^{\alpha} A M_{\mu} f \right\|_q. \tag{6.6}$$

Since $\operatorname{Re}(\mu) = c$, then by Lemmas 2(a) and 1(a) AM_{μ} is an isometric isomorphism of X_c^p onto $L_0^p \equiv L^p(\mathbf{R}_{+})$. So we can apply (6.2) with f being replaced by $AM_{\mu} f$. Using this estimate (6.2) with a constant K^{+} being replaced by another constant C_7^{+} (which depends on μ), we obtain

$$\left\| I_{+}^{\alpha} A M_{\mu} f \right\|_q \leqslant C_7^{+} \| A M_{\mu} f \|_p.$$

Substituting this estimate into (6.6), taking into account (2.20), (1.36), and applying (2.22) with $\varphi = M_{\mu} f$ and (2.11) with $\zeta = \mu$, we find that

$$\begin{aligned} \left\| \mathcal{J}_{0+,\mu}^{\alpha} f \right\|_{X_c^q} &\leqslant C_7^{+} \| A M_{\mu} f \|_p = C_7^{+} \| A M_{\mu} f \|_{L_0^p} \\ &= C_7^{+} \| M_{\mu} f \|_{X_0^p} = C_7^{+} \| f \|_{X_{\operatorname{Re}(\mu)}^p} = C_7^{+} \| f \|_{X_c^p}, \end{aligned}$$

which proves (6.4).

When $\operatorname{Re}(\mu) = -c$, relation (6.5) is proved similarly:

$$\begin{aligned} \left\| \mathcal{J}_{-,\mu}^{\alpha} f \right\|_{X_c^q} &= \left\| I_{+}^{\alpha} A M_{-\mu} f \right\|_{L_0^q} = \left\| I_{+}^{\alpha} A M_{-\mu} f \right\|_q \\ &\leqslant C_7^{-} \| A M_{-\mu} f \|_p = C_7^{-} \| A M_{-\mu} f \|_{L_0^p} = C_7^{-} \| f \|_{-\operatorname{Re}(\mu),p} \\ &= C_7^{-} \| f \|_{X_c^p}. \end{aligned}$$

Thus, Theorem 10 is complete. □

Corollary 7. *Let* $0 < \alpha < 1$, $1 < p < 1/\alpha$, $q = p/(1-\alpha p)$ *and* $\mu \in \mathbf{C}$.

(a) *If* $\mathrm{Re}(\mu) = 1/p$, *then the operator* $\mathcal{J}^{\alpha}_{0+,\mu}$ *is bounded from* $L^p(\mathbf{R}_+)$ *into* $X^q_{1/p}$, *and*

$$\left\|\mathcal{J}^{\alpha}_{0+,\mu} f\right\|_{X^q_{1/p}} \leqslant l_7^{+} \|f\|_p. \tag{6.7}$$

(b) *If* $\mathrm{Re}(\mu) = -1/p$, *then the operator* $\mathcal{J}^{\alpha}_{-,\mu}$ *is bounded from* $L^p(\mathbf{R}_+)$ *into* $X^q_{1/p}$, *and*

$$\left\|\mathcal{J}^{\alpha}_{-,\mu} f\right\|_{X^q_{1/p}} \leqslant l_7^{-} \|f\|_p. \tag{6.8}$$

Here l_7^{+} *and* l_7^{-} *are certain positive constants.*

Corollary 7, which follows from Theorem 10 if we put $c = 1/p$, solves the problem raised in Remark 7.

Taking into account the relations (4.14) and applying Theorem 10 with μ being replaced by $\mu + 1$ and $\mu - 1$, we obtain the corresponding results for the Hadamard-type fractional integration operators $\mathcal{I}^{\alpha}_{0+,\mu} f$ and $\mathcal{I}^{\alpha}_{-,\mu} f$ of (1.32) and (1.33).

Theorem 11. *Let* $0 < \alpha < 1$, $1 < p < 1/\alpha$, $q = p/(1 - \alpha p)$, $c \in \mathbf{R}$ *and* $\mu \in \mathbf{C}$.

(a) *If* $\mathrm{Re}(\mu) = c - 1$, *then the operator* $\mathcal{I}^{\alpha}_{0+,\mu}$ *is bounded from* X^p_c *into* X^q_c, *and*

$$\left\|\mathcal{I}^{\alpha}_{0+,\mu} f\right\|_{X^q_c} \leqslant C_8^{+} \|f\|_{X^p_c}. \tag{6.9}$$

(b) *If* $\mathrm{Re}(\mu) = 1 - c$, *then the operator* $\mathcal{I}^{\alpha}_{-,\mu}$ *is bounded from* X^p_c *into* X^q_c, *and*

$$\left\|\mathcal{I}^{\alpha}_{-,\mu} f\right\|_{X^q_c} \leqslant C_8^{-} \|f\|_{X^p_c}. \tag{6.10}$$

Here C_8^{+} *and* C_8^{-} *are certain positive constants.*

In particular, the analogue of Corollary 7 follows from Theorem 11, namely the case $c = 1/p$.

Finally, by putting $\mu = c = 0$ in Theorem 10, we can also obtain the boundedness from X^p_0 into X^q_0 of the Hadamard fractional integration operators $\mathcal{J}^{\alpha}_{0+} f$ and $\mathcal{J}^{\alpha}_{-} f$ given by (1.30) and (1.31), indicated in [2, p. 331].

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