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Topology Vol. 11, pp. 223–235. Pergamon Press, 1972. Printed in Great Britain

UNIVERSAL CRUMPLED CUBES†

ROBERT J. DAVERMAN and WILLIAM T. EATON

(*Received* 23 *September* 1969; *revised* 2 *July* 1971)

§1. INTRODUCTION

IT IS known [16, 18] that the 3-cell is universal in the sense that any sewing of itself to a crumpled cube yields the 3-sphere S^3. This suggests the following definition: A crumpled cube C is *universal* if, for each crumpled cube D and sewing h of C and D, the resulting space $C \bigcup_h D$ is topologically equivalent to S^3. The main result of this paper, Theorem 5.1, implies that there exists an uncountable family of topologically distinct universal crumpled cubes.

Martin [22] has shown that a necessary condition for $C \bigcup_h D = S^3$ is that for each $p \in \text{Bd } C$ either p is a piercing point of C or $h(p)$ is a piercing point of D. On the other hand, examples of Ball [2] and Cannon [10] indicate that this condition is not sufficient. Techniques developed to prove Theorem 5.1 lead to a modification of Martin's condition, suggested by McMillan's Cellularity Criterion [19] and his characterization of piercing points [20], which is sufficient for $C \bigcup_h D = S^3$.

We say a subset K of the boundary of a crumpled cube C is *semicellular* in C if for each open subset U of C containing K there is an open set V such that $K \subset V \subset U$ and loops in $V - K$ are null-homotopic in $U - K$. The Sewing Theorem (Theorem 4.1) establishes that a sufficient condition for $C \bigcup_h D = S^3$ is that for each $p \in \text{Bd } C$ either p lies interior to a semi-cellular disk of C or $h(p)$ lies interior to a semi-cellular disk of D.

Martin's condition that the sewing h mismatch non-piercing points does imply $C \bigcup_h D = S^3$ in certain cases; for example, if C is a countably knotted crumpled cube [12, Th. 3]. In addition, Theorem 5.7 states that this condition is sufficient if Int C is homeomorphic to Euclidean 3-space E^3.

Arguments in Section 3 follow an outline similar to Section 2 of [12]; nevertheless, with the exception of Lemma 3.1, this paper can be read without reference to [12]. The crucial new idea, using the semi-cellularity condition to realize a prescribed sewing, is found in the proof of Lemma 3.3. Examples and properties of semi-cellular sets, including a characterization in terms of piercing points, are described in Section 2.

† Research supported by NSF Grant GP-8888.

A *crumpled cube* C is a topological space homeomorphic to the closure of the bounded complementary domain of a 2-sphere embedded in Euclidean 3-space E^3. The *boundary of* C, denoted Bd C, consists of the points of C where it fails to be a 3-manifold, and the *interior* of C, denoted Int C, is defined by Int $C = C -$ Bd C. A *sewing* h of two crumpled cubes C and D is a homeomorphism of Bd C onto Bd D, and the *space* $C \bigcup_h D$ *given by the sewing* is the identification space obtained from the (disjoint) union of C and D by identifying each point x in Bd C with the point $h(x)$ in Bd D.

Let C be a crumpled cube and p a point in Bd C. Then p is a *piercing point of* C if there exists an embedding f of C into S^3 such that f(Bd C) can be pierced with a tame arc at $f(p)$. McMillan has characterized the piercing points as those points p in Bd C such that C-$\{p\}$ is 1-*ULC* [20].

If X is a metric space and C is a compact subset of X, then X/C denotes the decomposition space associated with the upper semi-continuous decomposition of X whose only non-degenerate element is C.

§2. SEMI-CELLULARITY IN CRUMPLED CUBES

LEMMA 2.1. *Let S be a 2-sphere in S^3 and X a non-separating subcontinuum of S which is semi-cellular in both crumpled cubes bounded by S. Then X is a cellular subset of S^3.*

Proof. Let C_1 and C_2 denote the crumpled cubes bounded by S. If U is an open set in S^3 containing X, we find another open set V containing X such that each loop in $(C_i \cap V) - X$ is null homotopic in $(C_i \cap U) - X$. Any loop L in $V - X$ can be adjusted by a homotopy in $V - X$ so that it intersects S in a finite point set, and the adjusted L can be represented as the sum of loops $L_1, \ldots, L_n$ such that L_i is contained in either $(C_1 \cap V) - X$ or $(C_2 \cap V) - X$ $(i = 1, \ldots, n)$. Since each L_i is contractible in $U - X$, L is contractible in $U - X$; thus, the Cellularity Criterion of McMillan [19, Th. 1′] implies that X is cellular.

If S is a 2-sphere in S^3 and X is a subcontinuum of S which is cellular in S^3, X is not necessarily semi-cellular in *each* of the crumpled cubes bounded by S. However, the following theorem, combined with [22], implies that X is semi-cellular in at least one of these crumpled cubes.

THEOREM 2.2. *Let C be a crumpled cube, X a non-separating subcontinuum of* Bd C, *and π the projection map of C onto the decomposition space C/X. Then X is semi-cellular in C if and only if the following conditions hold: C/X is a crumpled cube and $\pi(X)$ is a piercing point of C/X.*

Proof. If C/X is a crumpled cube such that $\pi(X)$ is a piercing point, McMillan's characterization of piercing points [20, Th. 1] can be applied to show that X is semi-cellular in C.

In case X is semi-cellular in C, we consider C to be embedded in S^3 so that $K = Cl(S^3 - C)$ is a 3-cell [16, 18]. Then X is cellular in S^3 by Lemma 2.1, and there exists a map π of S^3 onto itself whose only (non-degenerate) inverse set is X. Since X does not separate Bd C, π(Bd C) is a 2-sphere; hence π maps C onto a crumpled cube equivalent to

C/X. Once again, Theorem 1 of [20] can be used to show that $\pi(X)$ is a piercing point of $\pi(C)$.

The following lemma is applied repeatedly in Section 3.

LEMMA 2.3. *If C is a crumpled cube and the subdisk D of* Bd C *is semi-cellular in C, then each subdisk D' of D is semi-cellular in C.*

Proof. Let U be an open set containing D'. It is sufficient to consider the case that D is the union of D' and a disk D'' such that $D' \cap D''$ is an arc (in the boundary of each). Let A be an arc in D'' separating D' from $D - U$. We find an open subset W of C containing D such that W is the union of three open sets W_1, W_2, and W_3, satisfying

(1) $D' \subset W_i \subset U$,

(2) $W_1 \cap W_3 = \emptyset$,

(3) $A \subset W_2 \subset U - D'$, and

(4) W_2 is contractible in $U - D'$.

Condition 4 holds because A is contractible and C is an absolute retract [6, Th. 4]. Consequently, any contraction f_t of A in itself extends to a contraction g_t of C in itself, and it is easy to find a neighbourhood W_2 of A such that $g_t(W_2) \subset U - D'$ $(0 \leq t \leq 1)$.

By hypothesis, there exists an open set V^* containing D such that each loop in $V^* - D$ is null-homotopic in $W - D$. We define $V = V^* \cap W_1$.

If L is a loop in $V - D'$, we shall show that L is null-homotopic in $U - D'$. By pushing L slightly, we can suppose that L is contained in $V - D$. Thus, L is contractible in $W - D$. If the image of the contraction extends into W_3, we cut it off in W_2, and we replace a portion of the original contraction with a map into $U - D'$, by using Condition 4. This establishes that L is contractible in $(W_1 - D') \cup (U - D') \subset U - D'$.

LEMMA 2.4. *If C is a crumpled cube and D is a subdisk of* Bd C *which is semi-cellular in C, then each point of D is a piercing point of C.*

Proof. This is an easy consequence of Lemma 2.3 and McMillan's characterization of piercing points.

The following theorem is an existence result for semi-cellular sets. The original argument stemmed from the geometric techniques of [8, Th. 1], but we give an alternate proof.

THEOREM 2.5. *If the interior of the crumpled cube C is an open* 3*-cell and X_1 and X_2 are disjoint, non-separating subcontinua of* Bd C, *then either X_1 or X_2 is semi-cellular in C.*

Proof. We consider C to be embedded in S^3 so that $K = Cl\,(S^3 - C)$ is a 3-cell. Both X_1 and X_2 are cellular subsets of S^3, since for each neighbourhood U_i of X_i there exists a disk D_i such that

$$X_i \subset D_i \subset \text{Bd}\, K \cap U_i (i = 1, 2).$$

But K is cellular by hypothesis and K collapses to D_i. Therefore, using [21, Th. 1] we find that there exists a 3-cell B_i such that

$$X_i \subset D_i \subset \text{Int}\, B_i \subset B_i \subset U_i (i = 1, 2).$$

As a result, the decomposition spaces C/X_1 and C/X_2 are crumpled cubes. If π denotes a map of S^3 onto itself whose only inverse sets are X_1 and X_2, then $\pi(C)$ is also a crumpled cube. Furthermore, a neighbourhood of $\pi(X_i)$ in $\pi(C)$ is homeomorphic to a neighbourhood of the image of X_i in C/X_i; thus, $\pi(X_i)$ is a piercing point of $\pi(C)$ if and only if the image of X_i is a piercing point of $C/X_i (i = 1, 2)$. Note that Int $\pi(C)$ is an open 3-cell, since it is a homeomorphic (via π) to Int C. Consequently, Theorem 2 of [21] implies that either $\pi(X_1)$ or $\pi(X_2)$ is a piercing point of $\pi(C)$. We appeal to Theorem 2.2 to complete the proof.

COROLLARY 2.6. *If the interior of a crumpled cube C is an open* 3*-cell, then there exists a point q in* Bd C *such that each point of* Bd $C - \{q\}$ *lies interior to a subdisk of* Bd C *semi-cellular in C.*

LEMMA 2.7. *If the interior of the crumpled cube C is an open* 3*-cell, and q is a non-piercing point of C, then each non-separating subcontinuum X of* Bd C *contained in* Bd $C - \{q\}$ *is semi-cellular in C.*

Proof. Let E be a subdisk of Bd $C - X$ containing q. By Theorem 2.5 either X or E is semi-cellular in C, and E is excluded by Lemma 2.4.

Remark. Although we can avoid the question in the later applications of this paper, it would be interesting to know whether each disk in the boundary of a crumpled cube C is semi-cellular in C if Int C is an open 3-cell and each point of Bd C is a piercing point of C.

§ 3. A METHOD FOR SQUEEZING CERTAIN CELLS

Consider a 3-cell C in S^3 such that Bd C contains a disk D which is semi-cellular in S^3 – Int C, and let h be a homeomorphism of D onto Bd C – Int D which leaves Bd D fixed. In this section we describe how to squeeze C to a disk by a map f of S^3 to itself such that $f|D$ is a homeomorphism and $f|D = fh$. There are two basic steps: in the first step, as suggested by techniques of [12], we squeeze C to a collection of thin tubes, which may be very long; in the second we attach portions of each tube to itself, preserving the homeomorphism h, in such a way that the resulting image of C consists of small cells. We obtain the map f by iterating these steps.

LEMMA 3.1. *Suppose C_1 and C_2 are* 3*-cells in S^3 such that $C_1 \cap C_2 =$* Bd $C_1 \cap$ Bd $C_2 = D$ *is a tame disk, B is an arc in* Bd D, *h is a homeomorphism of B onto* Cl((Bd D) $- B$) *such that $h|$*Bd $D = 1$, *U is an open set containing $D - B$, and $\varepsilon > 0$. Then there exist a map of S^3 onto S^3 and homeomorphisms g_i of C_i onto $f(C_i)$ $(i = 1, 2)$ such that*

(1) $f|S^3 - U = 1$,

(2) $f(D) = B$ *and* $f|S^3 - D$ *is a homeomorphism onto* $S^3 - B$,

(3) $fh = 1$, *and*

(4) $g_i|S^3 - U = 1$ *and* $\rho(g_i, 1) < \varepsilon$ $(i = 1, 2)$.

Proof. Without loss of generality we make the following assumptions:

(5) $D = \{(x,y,z) | x^2 + y^2 \leq 1, z = \varepsilon/12\}$,

(6) $B = \{(x,y,z) | x = -\sqrt{1 - y^2}, z = \varepsilon/12\}$,

(7) $h(-\sqrt{1-y^2}, y, \varepsilon/12) = (\sqrt{1-y^2}, y, \varepsilon/12)$,

(8) the solid geometric cylinder $A = \{(x,y,z) \mid x^2 + y^2 \le 1, 0 \le z \le \varepsilon/6\}$ lies in the 3-cell $C_1 \cup C_2$,

(9) the disk $F = \{(x,y,z) \mid x = \sqrt{1-y^2}, 0 \le z \le \varepsilon/6\}$ intersects $\mathrm{Bd}(C_1 \cup C_2)$ in a tame Sierpinski curve X,

(10) $h(B) \cup \mathrm{Bd}\, F$ lies in the inaccessible part of X, and

(11) $A - B \subset U$.

Note that the closures of the components of $(C_1 \cup C_2) - A$ which intersect Int F form a null sequence $\{K_i\}$ of disjoint 3-cells. The map f is now constructed using the technique of [12, Lemma 4] on the solid cylinder A.

We require the following definitions throughout the rest of this section.

Let K be a 3-cell in S^3 and D a disk in Bd K. The *cross sectional diameter of K with respect to D* is said to be less than the number d if there exists a homeomorphism g of $D \times I$ onto K such that $g(x, 0) = x$ for all $x \in D$ and diam $g(D \times t) < d$ for all $t \in I$.

A collection of 2-cells $D_1, \ldots, D_n$ in a disk D is a *cellular subdivision of D* if and only if Int $D_i \cap$ Int $D_j = \emptyset$ $(i \neq j)$ and $D = \cup D_i$.

LEMMA 3.2. *Suppose C is a 3-cell in S^3, D is a disk in* Bd C, *h is a homeomorphism of D onto $Cl(\mathrm{Bd}\, C) - D)$ such that $h \mid \mathrm{Bd}\, D = 1$, U is an open set containing $C - D$, and $\varepsilon > 0$. Then there exist a cellular subdivision $D_1, D_2, \ldots, D_n$ of D and a map f of S^3 onto S^3 such that*

(1) $f \mid S^3 - U = 1$,

(2) *$f \mid S^3 - C$ is a homeomorphism onto $S^3 - f(C)$,*

(3) *fh is a homeomorphism,*

(4) $fh \mid \cup \mathrm{Bd}\, D_i = 1$,

(5) *$fh(D) \cap D = \cup$ Bd D_i, and*

(6) *$D_i \cup (fh(D_i))$ bounds a 3-cell $K_i \subset U$ such that the cross-sectional diameter of K_i is less than ε.*

Proof. Let A be the standard 3-cell $\{(x,y,z) \mid x^2 + y^2 + z^2 \le 1\}$. By [5, Th. 4.2] there exists a 0-dimensional F_σ set F in Bd C such that $F \cup$ Ext C is $1 - ULC$. It is straight forward to show that there exists a homeomorphism H of $\{(x,y,z) \mid z = -\sqrt{1-x^2-y^2}\}$ onto D such that if r is a rational number in $[-1,1]$ then $H(\{(x,y,z) \mid z = -\sqrt{1-r^2-y^2}, x = r\}) \subset D - (F \cup h^{-1}(F))$ and $H(\{(x,y,z) \mid z = -\sqrt{1-x^2-r^2}, y = r\}) \subset D - (F \cup h^{-1}(F))$. Then H can be extended to the top of A by the formula $H(x,y, \sqrt{1-x^2-y^2}) = hH(x,y, -\sqrt{1-x^2-y^2})$, and this homeomorphism of Bd A onto Bd C can be extended to a homeomorphism H of A onto C. There exist rational numbers $-1 = r_0 < r_1 \ldots < r_m = 1$ such that the cross-sectional diameter of the 3-cell $C_{ij} = H(\{(x,y,z) \mid r_i \le x \le r_{i+1}, r_j \le y \le r_{j+1}, |z| \le \sqrt{1-x^2-y^2}\})$ with respect to the disk $D_{ij} = H(\{(x,y,z)) \mid r_i \le x \le r_{i+1}, r_j \le y \le r_{j+1},$

$z = -\sqrt{1 - x^2 - y^2}\})$ is less than $\varepsilon/3$. The disks $\{D_{ij}\}$ form the required cellular subdivision of D. The required f is obtained by applying Lemma 3.1 a finite number of times as follows

Let $U_1, \ldots, U_{m-1}$ be a collection of disjoint open sets in U such that the disk $E_i = H(\{(x,y,z) | x = r_i, \quad y^2 + z^2 \leq 1 - r_i^2\}$ less the arc $B_i = \{(x,y,z) | x = r_i, z = -\sqrt{1 - r_i^2 - y^2}\})$ lies in U_i. Apply Lemma 3.1 to the tame disk E_i, arc B_i, homeomorphism $h | B_i$ and open set U_i $(i = 1, \ldots, n - 1)$. The resulting maps can be pieced together to form a map f' of S^3 onto S^3. Let $\{V_{ij}\}$ be a collection of disjoint open sets in U such that the disk $E_{ij} = f'H(\{(x,y,z) | r_i \leq x \leq r_{i+1}, y = r_j, |z| \leq \sqrt{1 - r_j^2 - x^2}\}$ less the arc $B_{ij} = f'H(\{(x,y,z) | r_i \leq x \leq r_{i+1}, y = r_j, z = -\sqrt{1 - r_j^2 - x^2}\})$ lies in V_{ij}. Apply Lemma 3.1 to the tame disk E_{ij}, arc B_{ij}, homeomorphism $h | B_{ij}$ and open set V_{ij}, and collect the resulting maps to form a map f'' of S^3 onto S^3. The required map $f = f''f'$. We may assume that Lemma 3.1 was applied with small enough epsilons to insure that there exists an $\varepsilon/3$-homeomorphism g_{ij} of C_{ij} onto $f(C_{ij})$. It follows that the cross-sectional diameter of $f(C_{ij})$ with respect to D_{ij} is less than ε.

LEMMA 3.3. *Suppose* $\varepsilon > 0$, C *is a* 3-*cell in* S^3, *and* D *is a disk in* Bd C *such that* C *has cross-sectional diameter less than* ε *with respect to* D *and each subdisk of* Int D *is semi-cellular in* $Cl(S^3 - C)$, h *is a homeomorphism of* D *onto* $Cl(\text{Bd } C - D)$ *such that* $h | \text{Bd } D = 1$, *and* U *is an open set containing* $C -$ Bd D. *Then there exist a cellular subdivision* $D_0, \ldots, D_n$ *of* D *and a map* f *of* S^3 *onto* S^3 *such that*

(1) $f | S^3 - U = 1$,

(2) f *is a homeomorphism of* $S^3 - C$ *onto* $S^3 - f(C)$,

(3) *both* $f | D$ *and* $f | h(D)$ *are homeomorphisms,*

(4) $f(D) \cap fh(D) = f(\cup \text{Bd } D_i)$,

(5) $f | \cup \text{Bd } D_i = fh | \cup \text{Bd } D_i$, *and*

(6) $f(D_i) \cup fh(D_i)$ *bounds a* 3-*cell in* $f(C)$ *of diameter less than* 2ε.

Proof. Let g be a homeomorphism of $D \times I$ onto C such that $g(x,0) = x$ for all $x \in D$ and Diam $g(D \times t) < \varepsilon$ for all $t \in I$. Then there exists a sequence of positive numbers

$$0 = t_0 < t_1 < \ldots < t_n = 1$$

such that Diam $g(D \times [t_1, t_{i+1}]) < \varepsilon$. Since g can be adjusted slightly, if necessary, we assume without loss of generality that for $i = 1, \ldots, n$ the simple closed curve $J_i = h^{-1}g(\text{Bd } D \times t_i)$ is tame [4, Th. 1]. Let J_0 be a tame simple closed curve in D which separates Bd D and J_1. By pushing the interiors of each annulus on D bounded by successive J_i's toward Int C, we form a collection of tame annuli $A_0, A_1, \ldots, A_{n-1}$ in $g(D \times [0, t_1))$ whose interiors are pairwise disjoint subsets of Int C and whose boundaries satisfy Bd $A_i = J_i \cup J_{i+1}$. Similarly, there exists a tame disk A_n in $g(D \times [0, t_1))$ such that Bd $A_n = J_n$, Int $A_n \subset$ Int C, and Int $A_n \cap$ Int $A_i = \varnothing$ $(i \neq n)$. Let F_i denote the subdisk of D bounded by J_i.

In the next three paragraphs we describe the construction of a collection of disks "below" D that will control the squeezing map f. Let U_0 be an open subset of $U -$ Int A_0

containing F_1 such that $U_0 \cap C \subset F_0$. From the hypothesis we find that F_1 is semi-cellular in $Cl(S^3 - C)$. Consequently, J_0 bounds a singular disk, with no singularities near J_0, in $(F_0 - F_1) \cup (U_0 \cap (S^3 - C))$. Thus, after a careful simplicial approximation to Δ_0, making use of the Side Approximation Theorem [7, Th. 1] to avoid the introduction of singularities near J_0, we apply Dehn's Lemma [23] to obtain a tame disk G_0 with properties similar to those of Δ_0. We require, in particular, that $G_0 \cap C$ be contained in the solid torus bounded by $(F_0 - F_1) \cup A_0$.

Working our way inside, we find that J_1 bounds a singular disk, with no singularities near J_1, in $(F_1 - F_2) \cup (U_1 \cap (S^3 - C))$, where U_1 is a neighbourhood of F_2 in U so close to F_2 that $\Delta_1 \cap G_0 = \varnothing$. Once again, after adjustments to satisfy the hypotheses of Dehn's Lemma, we obtain a tame disk G_1 with similar properties.

Continuing in this fashion, we find a collection of mutually exclusive disks G_0, $G_1, \ldots, G_{n-1}$ such that

$\text{Bd } G_i = J_i$,

$G_i \cap (\cup A_i) = \text{Bd } G_i$,

$G_i \subset U$,

$G_i \cap C \subset$ the solid torus bounded by $(F_i - F_{i+1}) \cup A_i$.

Let B_0 denote an annulus in $C \cap g(D \times [0, t_1])$ bounded by J_0 and $g(\text{Bd } D \times t_1)$ such that $\text{Int } B_0 \subset \text{Int } g(D \times [0, t_1])$ and $B_0 \cap (\cup A_i) = J_0$. Let $B_i = g(\text{Bd } D \times [t_i, t_{i+1}])$ $(i = 1, \ldots, n-1)$ and $P_i = g(D \times t_i)$ $(i = 1, \ldots, n-1)$.

We define a homeomorphism f on the domain $(\cup G_i) \cup (\cup A_i)$ as follows:

$f|G_0 = 1$,

$f|A_0\colon A_0 \to B_0$ such that $f|J_1 = h|J_1$,

$f|A_i\colon A_i \to B_i$ such that $f|\text{Bd } A_i = h|\text{Bd } A_i$ $(i = 1, \ldots, n-1)$,

$f|G_i\colon G_i \to P_i$ $(i = 1, \ldots, n-1)$, and

$f|A_n\colon A_n \to g(D \times 1)$.

Extend f so as to take the 3-cell in U bounded by $G_0 \cup A_0 \cup G_1$ onto the 3-cell on U bounded by $G_0 \cup B_0 \cup P_1$, the cell bounded by $G_i \cup A_i \cup G_{i+1}$ onto the cell bounded by $P_i \cup B_i \cup P_{i+1}$ $(i = 1, \ldots, n-2)$, and the cell bounded by $G_{n-1} \cup A_{n-1} \cup A_n$ onto the cell bounded by $P_{n-1} \cup B_{n-1} \cup g(D \times 1)$.

Extend f via the identity to the remainder of $S^3 - \text{Int } C$ and to the toroidal region in C bounded by $(D - \text{Int } F_1) \cup B_0 \cup g(\text{Bd } D \times [0, t_1])$. Use Tietze's Extension Theorem to complete the definition of f from the rest of C to the disk $g(D \times 1) \cup (\cup B_i)$. A schematic view of the action of this map is given in Fig. 1.

This construction forces each component of $S^3 - f(S^3 - C)$ into some 3-cell of the form $g(D \times [t_{j-1}, t_{j+1}])$; thus, the closure of each such component has diameter less than 2ε. Let M_1 denote the closure of the component of $S^3 - f(S^3 - C)$ bounded by $f(D - \text{Int } F_1) \cup h(D - \text{Int } F_1)$; for $i = 2, \ldots, m$ let M_i denote the closure of the component of

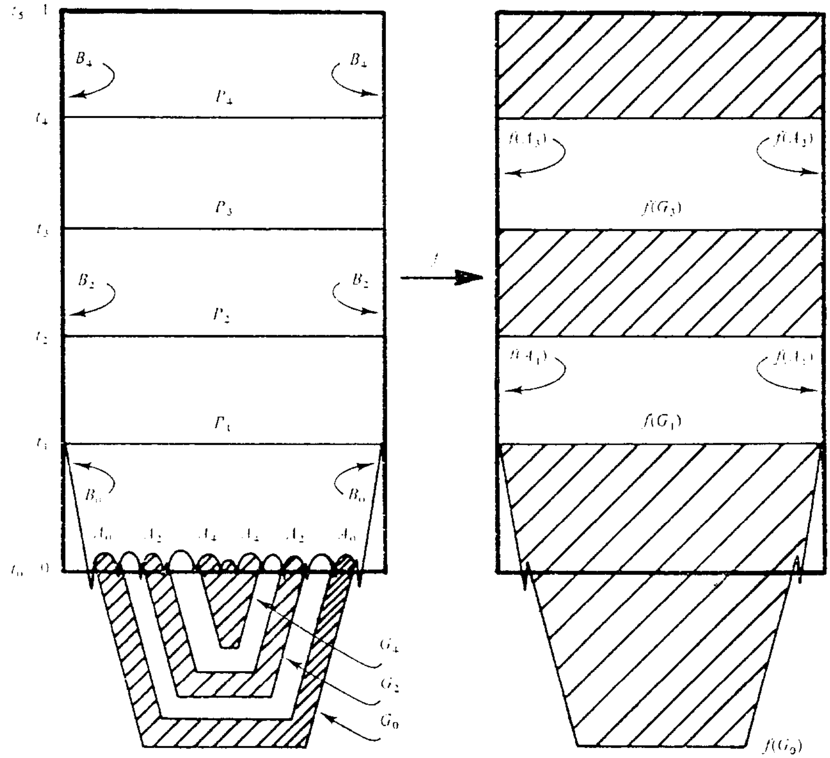

FIG. 11.

$S^3 - f(S^3 - C)$ bounded by $f(F_{i-1} - \text{Int } F_i) \cup h(F_{i-1} - \text{Int } F_i)$. One easily can show each M_i to be a solid torus. One should observe also that the only component of $S^3 - f(S^3 - C)$ not contained in $\cup M_i$ is the interior of a 3-cell bounded by $f(F_n) \cup h(F_n)$.

To finish the proof we must squeeze further so that the closure of each component of $S^3 - f(S^3 - C)$ is a small 3-cell. To do this, we choose three pairwise disjoint, tame disks $\Delta_{ij} (j = 1, 2, 3)$ in each solid torus M_i with $\text{Int } \Delta_{ij} \subset \text{Int } M_i$ and such that there exist arcs R_{ij} in D whose endpoints are points of J_{i-1} and J_i (in case $i = 1$, then the endpoints of R_{ij} are points of Bd D and J_1) satisfying Bd $\Delta_{ij} \cap f(D) = f(R_{ij})$ and Bd $\Delta_{ij} \cap h(D) = h(R_{ij})$. The only problem in locating these tame disks Δ_{ij} is to find arcs R_{ij} satisfying the preceding conditions and such that $f(R_{ij} \cup h(R_{ij}))$ is tame. From [5, Th. 6.2] it follows that there exists a 0-dimensional F_σ-subset Z_i of Bd M_i such that $Z_i \cup (S^3 - M_i)$ is 1-ULC, and it follows from [9, Th. 2.4] and [17, Th. 6] that for any such arc R_{ij} contained in

$$(F_{i-1} - \text{Int } F_i) - [(f \mid D)^{-1}(Z_i \cap f(D)) \cup h^{-1}(Z_i \cap h(D))]$$

the simple closed curve $f(R_{ij} \cup h(R_{ij}))$ is tame.

Thus, $M_i - \bigcup_{j=i}^{3} \Delta_{ij}$ is the union of three components whose closures C_{ij} $(j = 1, 2, 3)$ are each 3-cells. Applying Lemma 3.1 to each of the Δ_{ij}'s, we obtain a map f^* of S^3 onto itself such that $f^*(\Delta_{ij}) = h(R_{ij})$, f^* takes $S^3 - \cup \Delta_{ij}$ homeomorphically onto $S^3 - \cup h(R_{ij})$, $f^* | S^3 - U = 1$, for each point x in $\cup R_{ij}$,

$$f^*f(x) = f^*fh(x) = h(x),$$

and diam $f^*(C_{ij}) < 2\varepsilon$.

The required subdivision of D has its 1-skeleton the graph Bd $D \cup (\bigcup_{i=1}^{n} J_i) \cup (\bigcup_{j=1}^{3} \bigcup_{i=1}^{n} R_{ij})$; the required map is f^*f.

The following lemma is proved by first using Lemma 3.2 and then applying Lemma 3.3 to each of the resulting cross-sectionally small 3-cells.

LEMMA 3.4. *Suppose C is a 3-cell in S^3, D is a disk in* Bd C *such that each subdisk of* Int D *is semi-cellular in* S^3 − Int C, *h is a homeomorphism of D onto Cl((*Bd *C) − D) such that $h|$*Bd $D = 1$, *U is an open set containing C −* Bd D, *and $\varepsilon > 0$. Then there exist a cellular subdivision $D_1, D_2, \ldots, D_n$ of D and a map f of S^3 onto S^3 such that*

(1) $f|S^3 - U = 1$,

(2) *$f|S^3 - C$ is a homeomorphism onto $S^3 - f(C)$,*

(3) *both $f|D$ and $f|h(D)$ are homeomorphisms,*

(4) $f(D) \cap fh(D) = f(\cup \text{ Bd } D_i)$,

(5) $f|\cup \text{ Bd } D_i = fh|\cup \text{ Bd } D_i$,

(6) *$f(D_i) \cup fh(D_i)$ bounds a 3-cell in $f(C)$ of diameter less than ε, and*

(7) Diam $D_i < \varepsilon$.

THEOREM 3.5. *Suppose C is a 3-cell in S^3 and, D is a disk in* Bd C *such that each subdisk of D in* Int D *is semi-cellular in the crumpled cube $Cl(S^3 - C)$, h is a homeomorphism of D onto Cl(*Bd *$C - D$) such that $h|$*Bd $D = 1$, *and U is an open set containing C −* Bd D. *Then there exists a map f of S^3 onto S^3 such that*

(1) $f|S^3 - U = 1$,

(2) *$f|S^3 - C$ is a homeomorphism onto $S^3 - f(C)$, and*

(3) *$f|D = fh$ and fh is a homeomorphism onto $f(C)$.*

Proof. The map f is the limit of a sequence of maps $\{f_i\}$ obtained by repeated use of Lemma 3.4. Specifically, let $U_0, U_1, \ldots$ be a sequence of open sets in S^3 such that $U \supset U_0 \supset U_1 \supset \ldots$ and $\cap U_i = C -$ Bd D, and let $\varepsilon_0, \varepsilon_1, \ldots$ be a sequence of positive numbers such that $\sum_0^\infty \varepsilon_i$ is bounded. The map f_0 is the identity on S^3. It follows that there exist a cellular subdivision $\{D_i^1\}$ of D and a map f_1 of S^3 onto S^3 that satisfy the conclusions of Lemma 3.4. Let K_i^1 be the 3-cell that $f_1(D_i^1) \cup f_1h(D_i^1)$ bounds in $f_1(C)$ and let $\{V_i^1\}$ be a finite collection of disjoint open sets in $f_1(U_1)$ such that $K_i^1 - f_1(\text{Bd } D_i^1) \subset V_i^1$ and Diam $V_i^1 < \varepsilon_1$.

Inductively we assume that $\{D_i^n\}$ is a cellular subdivision of D, f_n is a map of S^3 onto S^3, K_i^n is the 3-cell bounded by $f_n(D_i^n) \cup f_n h(D_i^n)$, and $\{V_i^n\}$ is a finite collection of disjoint open sets in $f_n(U_n)$ such that $K_i^n - f_n(\text{Bd } D_i^n) \subset V_i^n$ and Diam $V_i^n < \varepsilon_n$. Lemma 3.4 is applied the 3-cell K_i^n, disk $f^n(D_i^n)$, homeomorphism $f_n h f_n^{-1}|f_n(D_i^n)$, open set V_i^n, and positive number ε_{n+1} to obtain a map f_{n+1}^i of S^3 onto S^3 and a cellular subdivision $\{E_j^i\}$ of $f_n(D_i^n)$. The map f_{n+1} is obtained by piecing together $\{f_{n+1}^i\}$; the collection $\{D_i^{n+1}\}$ is given by $\{f_n^{-1}(E_j^i)\}_{ij}$. Each 2-sphere $f_{n+1}(D_i^{n+1}) \cup f_{n+1}h(D_i^{n+1})$ bounds a 3-cell K_i^{n+1} in $f_{n+1}(C)$. A finite collection $\{V_i^{n+1}\}$ of disjoint open sets in $f_{n+1}^i(U_{n+1})$ are chosen such that $K_i^{n+1} - f_{n+1}(\text{Bd } D_i^{n+1}) \subset V_i^{n+1}$, Diam $V_i^{n+1} < \varepsilon_{n+1}$, and $\bigcup_i v_i^{n+1} \subset \bigcup_i v_i^n$.

It is straightforward to check that the limit of the maps $\{f_n\}$ satisfies the conclusions of this theorem.

§4. A GENERAL SEWING THEOREM

THEOREM 4.1. *Suppose C_1 and C_2 are crumpled cubes and h a homeomorphism of* Bd C_1 *onto* Bd C_2. *Then* $C_1 \bigcup_h C_2 = S^3$ *if there exists a set* $F \subset$ Bd C_1 *such that*

(1) $F \cup \text{Int } C_1$ *is 1-ULC, and*

(2) *for each $p \in F$ either p lies in the interior of a semi-cellular disk of C_1 or $h(p)$ lies in the interior of semi-cellular disk of C_2.*

Proof. Without loss of generality we assume that C_1 and C_2 are embedded in S^3 such that $S^3 - \text{Int } C_i$ is a 3-cell [16, 18]. Let U be an open subset of Bd C_1 containing F such that for each point $p \in U$ there exists a disk $D \subset U$ such that either D is semi-cellular in C_1 or $h(D)$ is semi-cellular in C_2. It follows that there exists a locally finite graph $G \subset U$ such that

(3) G is locally tame,

(4) the closures of the components of $U - G$ form a null sequence $\{D_i\}$ of disks, and

(5) either D_i is semi-cellular in C_1 or $h(D_i)$ is semi-cellular in C_2 ($i = 1, 2, \ldots$).

The interior of each disk D_i is pushed slightly into $S^3 - C_1$ to form a tame disk E_i such that Bd E_i = Bd D_i and Int $E_i \cap$ Int $E_j = \varnothing$ if $i \neq j$. It follows from [9] and [17] that the 2-sphere $S = (\text{Bd } C_1 - \cup D_i) \cup (\cup E_i)$ is tame and thus bounds a cell C containing C_1. Furthermore, for each i, $E_i \cup D_i$ bounds a cell K_i in C.

There is a homeomorphism g of C onto $S^3 - \text{Int } C_2$ such that $g | \text{Bd } C_1 - \cup \text{ Int } D_i = h | \text{Bd } C_1 - \cup \text{ Int } D_i$. Let $U_1, U_2, \ldots$ be a null sequence of disjoint open sets in S^3 such that $g(K_i) - g(\text{Bd } D_i) \subset U_i$. The proof is completed by applying Theorem 3.5 to each 3-cell $g(K_i)$ in one of two ways: in case D_i is semi-cellular in C_1, Theorem 3.5 is applied to the 3-cell $g(K_i)$, the disk $g(D_i)$, the homeomorphism $hg^{-1}|g(D_i)$, and the open set U_i; in case D_i fails to be semi-cellular in C_1 (hence, $h(D_i)$ is semi-cellular in C_2 by Condition 5), it is applied to the 3-cell $g(K_i)$, the disk $h(D_i)$, the homeomorphism $gh^{-1}|h(D_i)$, and the open set U_i.

§5. UNIVERSAL CRUMPLED CUBES

As a corollary to Theorem 4.1 we obtain the following sufficient condition that a crumpled cube be universal.

THEOREM 5.1. *Let C be a crumpled cube and F a subset of* Bd C *such that* Int $C \cup F$ *is* 1-*ULC and each point of F lies interior to a subdisk in* Bd C *semi-cellular in C. Then C is a universal crumpled cube.*

THEOREM 5.2. *If C is a crumpled cube such that* Int C *is an open* 3-*cell and each point of* Bd C *is a piercing point of C, then C is a universal crumpled cube.*

Proof. By Corollary 2.6, there exists a point q of Bd C such that each point of Bd $C - \{q\}$ lies interior to a subdisk of Bd C semi-cellular in C. Let $F = \text{Bd } C - \{q\}$; then McMillan's characterization of piercing points [20] implies that $F \cup \text{Int } C$ is 1-*ULC* Theorem 5.1 can be applied to show that C is universal.

COROLLARY 5.3. *Gillman's crumpled cube* [15, Section 2] *is universal.*

COROLLARY 5.4. *Alford's crumpled cube* [1, Section 2] *is universal.*

COROLLARY 5.5. *There exists an uncountable family of universal crumpled cubes, no two of which are homeomorphic.*

Proof. The family of spheres of [1, Section 3] can be constructed so as to bound crumpled cubes with the desired properties.

The boundary of a crumpled cube C is said to be *free relative to* Int C if for each positive number ε there exists a map f of Bd C into Int C such that $\rho(f, I) < \varepsilon$.

COROLLARY 5.6. *If the boundary of a crumpled cube C is free relative to* Int C, *then C is universal.*

Under this hypothesis each subdisk of Bd C is semi-cellular in C. It is not known whether such a crumpled cube must be a 3-cell.

Another Example. The crumpled cubes found in the literature which turn out to be universal all share the property that their interiors are open 3-cells. This property is not essential for universality. By modifying the hooking described by Gillman in [15, Section 2], one can construct a crumpled cube C such that (1) each Sierpinski curve in Bd C is tame, (2) each subdisk of Bd C is semi-cellular in C, and (3) Int C is not simply connected. The only difference between this construction and Gillman's is the last eyebolt in each stage of the defining sequence for C is hooked around the first eyebolt in that stage, forming a circular chain of eyebolts rather than an arc-like chain. It follows from Condition (2) and Theorem 5.1 (with $F = \text{Bd } C$) that C is universal.

Using the preceding results we obtain a necessary and sufficient condition that a sewing of two crumpled cubes gives S^3, provided the interior of one of the cubes is an open 3-cell.

THEOREM 5.7. *Suppose C_1 and C_2 are crumpled cubes, h a homeomorphism of* Bd C_1 *to* Bd C_2, *and* Int C_2 *is an open* 3-*cell. Then the sewing $C_1 \bigcup_h C_2 = S^3$ if and only if the following condition holds: each non-piercing point of C_1 is identified by h with a piercing point of C_2.*

Proof. The necessity of this condition follows from [22]. On the other hand, if h is a sewing satisfying the condition, then by [21, Th. 2] C_2 contains at most one non-piercing point. In case C_2 contains no non-piercing point, then C_2 is universal by Theorem 5.2. If C_2 contains a non-piercing point q, then by hypothesis $h^{-1}(q)$ is a piercing point of C_1.

In this case, let $F = \text{Bd } C_1 - h^{-1}(q)$. Thus, $F \cup \text{Int } C_1$ is 1-*ULC* [20], and, for each point p of F, $h(p)$ lies interior to a subdisk of Bd C_2 semi-cellular in C_2 (see Lemma 2.7). Theorem 4.1 implies that this sewing gives S^3.

The above discussion leads naturally to the following list of questions.

Question 1. Is the sufficient condition of Theorem 5.1 necessary for a crumpled cube to be universal?

Question 2. Is the crumpled cube described by Bing [3] universal?

Question 3. Is a crumpled cube C universal if each arc in Bd C is tame?

Question 4. Is there a universal crumpled cube C such that Bd C is locally tame modulo a Cantor set?

J. W. Cannon has asked whether there exists a crumpled cube D that would serve as a "test" cube for universality.

Question 5. Is there a crumpled cube D such that, if $C \bigcup_h D = S^3$ for all sewings h, then C is a universal crumpled cube?

Addendum—After submitting this paper, the authors improved its results, answering the preceding questions. Eaton [14] established that a crumpled cube C is universal if, for each Cantor set X in Bd C, $C - X$ is 1-*ULC*, and Daverman [11] discovered that this condition characterizes universal crumpled cubes. As a result, the answers to Questions 2, 3, and 5 are affirmative; to Questions 1 and 4, negative. Also, Eaton [14] characterized the sewings of two crumpled cubes C_1 and C_2 that yield S^3 as those homeomorphisms h of Bd C_1 to Bd C_2 for which there exist subsets F_i of Bd C_i $(i = 1, 2)$ such that $F_i \cup \text{Int } C_i$ is 1-*ULC* and $h(F_1) \cap F_2 = \varnothing$.

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The University of Tennessee