GLOBAL WELLPOSEDNESS FOR 1D NON-LINEAR SCHRÖDINGER EQUATION FOR DATA WITH AN INFINITE $L^2$ NORM

Ana VARGAS$^a$, Luis VEGA$^b$

$^a$Departamento Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Spain
$^b$Departamento de Matemáticas, Universidad del País Vasco, Apartado 644, 48080 Bilbao, Spain

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ABSTRACT. – We prove global wellposedness for the one-dimensional cubic non-linear Schrödinger equation in a space of distributions which is invariant under Galilean transformations and includes $L^2$. This space arises naturally in the study of the restriction properties of the Fourier transform to curved surfaces. The $L^p$ bounds, $p \neq 2$, for the extension operator, dual to the restriction one, plays a fundamental role in our approach. © 2001 Éditions scientifiques et médicales Elsevier SAS

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RÉSUMÉ. – On montre que l’équation de Schrödinger cubique unidimensionnelle est globalement bien posée dans un espace de distributions qui est invariant par rapport aux transformations galiléennes et qui inclut $L^2$. Cet espace apparaît naturellement dans l’étude des propriétés de restriction aux surfaces courbes de la transformation de Fourier. Les estimations $L^p$, $p \neq 2$, pour l’opérateur d’extension, le dual de celui de restriction, jouent un rôle fondamental dans notre approche. © 2001 Éditions scientifiques et médicales Elsevier SAS

1. Introduction and statement of the results

In this paper we plan to revisit the local and global wellposedness theory for the one-dimensional non-linear Schrödinger equation:

$$iu_t + u_{xx} \pm |u|^2 u = 0,$$

(1.1)

$$u(x, 0) = \phi(x),$$

from the point of view of the regularity of the initial conditions. It is a well known result that if $\phi(x)$ is in $L^2(\mathbb{R})$ then the initial value problem (1.1) is globally wellposed. This was proved by Tsutsumi in [14], following previous ideas by Ginibre and Velo [9]. We refer the reader to the monographs [4,5] where the proofs, references and related topics can be found. The global result is obtained in two steps. First a local result is proved by a Picard iteration scheme on a suitable

E-mail addresses: ana.vargas@uam.es (A. Vargas), mtpvegol@lg.ehu.es (L. Vega).
function space, in such a way that the time of existence depends just on the $L^2$ norm of the initial datum. Then the result becomes global thanks that the $L^2$ norm is a conserved quantity along the evolution.

Nevertheless, it is important to remember that the $L^2$ space is subcritical from scaling considerations. In fact given $\lambda > 0$ and $u$ a solution of (1.1), then

$$u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t)$$

is also a solution with initial datum

$$\phi_\lambda(x, t) = \lambda \phi(\lambda x),$$

which is the scaling corresponding to the Dirac $\delta$-function. Hence, if we know that the time of existence depends on the size of the $L^2$ norm, by rescaling we conclude that this time goes to infinity if the size in $L^2$ goes to zero, and from this point of view the problem is subcritical in that function space. This property remains the same if working in the Sobolev class $H^s(\mathbb{R})$, as long as $s > -1/2$.

The situation changes completely when other transformations which leave invariant the set of solutions are considered, as was observed in [11]. There the authors analyze the so-called Galilean transformations given by:

$$u^N(x, t) = e^{-iN^2t + iNx} u(x - 2Nt, t),$$

where $N \in \mathbb{R}$ and $u$ is any solution of (1.1). Then it is easy to compute that $u^N$ is also a solution of (1.1) with $u^N(x, 0) = e^{iNx} \phi(x)$. Hence at time $t = 0$ the above transformation is just a translation in Fourier space. As a consequence any Banach space with a norm such that the size of $u^N(0)$ is independent of $N$ has to be translation invariant in Fourier space. In the Sobolev class this happens just when $s = 0$, that is to say in $L^2$. In fact this observation can be in a sense quantified by saying that the map $\phi \mapsto u(t)$ can not be smooth within the class $H^s, s < 0$, see [2] and [11] for more details.

However there still exists the possibility of working with an alternative class of spaces which includes $L^2$ and such that their intersections with $H^s$ are not empty. This is precisely the main issue of this paper. Although we think this question is interesting in itself, we are also motivated by another problem. This is the LIA (from localized induction approximation) model for the flow of a vortex filament. In this model the filament in time $t$ is given by a curve $X(s, t)$ in $\mathbb{R}^3$ with $s$ the arclength parameter. The flow is precisely

$$X_t = X_s \times X_{ss}.$$  

The reader can find in [1, p. 509] how (1.5) is obtained as an approximation of Euler equations, as well as the limitations of the model from a physical point of view. Our interest comes through a remarkable transformation found by Hasimoto, [10], which links (1.5) with (1.1). Define:

$$\psi(s, t) = c(s, t) e^{i \int^t \tau(r, \tau) \, dr},$$

where $c(s, t)$ and $\tau(s, t)$ denote the curvature and the torsion of $X(s, t)$ respectively. Then Hasimoto proves that if $X$ flows according to (1.5), $\psi$ satisfies:

$$i\psi_t + \psi_{ss} + \frac{1}{2}(|\psi|^2 - A(t))\psi = 0,$$
with \( A(t) \) any function which can be taken identically zero. In this setting the \( L^2 \) conservation law,
\[
\int |\psi(s, t)|^2 \, ds = \int |\psi(s, 0)|^2 \, ds,
\]
is related to the kinetic energy of the filament, see for example [12].

It has been recently proved in [8] that under (1.5) a \( C^\infty \) filament can develop a singularity in the shape of a corner in finite time. A corner for the filament means through (1.6) a Dirac \( \delta \) singularity for \( \psi \) which due to (1.8) is forbidden if the kinetic energy is finite. Therefore it seems very natural to try to develop a satisfactory local and global mathematical theory for (1.1) which includes initial data of infinite energy.

In this paper we give a first step in this direction, without any attempt to optimize our ideas. This is mainly because we think we are still far from the desired regularity result. For example, and in order to fix the ideas, assume that for \( j = 0, 1, \ldots, 2 \),
\[
\left| \frac{d^j}{dx^j} \hat{\phi}(\xi) \right| \leq \frac{c}{(1 + |\xi|)^{\alpha} + j}
\]
for some \( \alpha > 1/6 \),

with \( \hat{\phi} \) denoting the Fourier transform of \( \phi \). Then we prove below that (1.1) is locally well posed if \( \alpha > 1/6 \) and globally well posed if
\[
\alpha > 1/3,
\]
while the expected result should be \( \alpha > 0 \). The regularity assumption in (1.9) can be relaxed and just assume conditions of size on \( \hat{\phi} \) but to the expense of a worse \( \alpha \).

In order to explain in a precise way our results we have to introduce some notation. As usual we denote by \( e^{it\partial^2_x} \phi \) the free flow which gives the solution of the linear Schrödinger equation associated to (1.1). For a function \( z(x, t) \) and an interval \( I \subset \mathbb{R} \), we denote:
\[
\|z\|_{L^p(I)} = \left[ \int_I \left( \int_{\mathbb{R}} |z(x, t)|^p \, dx \right)^{\frac{1}{p}} \, dt \right].
\]
(1.11)

We also define the space \( Y_{p, q} \) as the set of tempered distributions:
\[
Y_{p, q} = \left\{ \phi \in \mathcal{S}'(\mathbb{R}) : \|\phi\|_{Y_{p, q}} = : \max_{I} \|e^{it\partial^2_x} \phi\|_{L^p(I)} < +\infty \right\},
\]
(1.12)

where the supremum is taken over all intervals of unit length in \( \mathbb{R} \).

Spaces as \( Y_{p, q} \) usually appear related to the local solvability of (1.1) and this will be also the case in this paper. To go further and to obtain a global result we need a useful description of \( Y_{p, q} \). We do that through an extension – dual to restriction theorem for the Fourier transform, which is expressed in terms of another spaces, this time of functions which are locally in \( L^2 \). We call this spaces \( X_{p, \gamma} \), \( p \geq 1 \) and \( \gamma > 0 \). In order to define them consider for \( j = 0, 1, 2, 3, \ldots \) \( \mathcal{D}_j = \{ [k2^j, (k+1)2^j) : k \in \mathbb{Z} \} \), the family of dyadic intervals of length \( 2^j \) in \( \mathbb{R} \). Then, for \( \gamma > 0 \) and \( p \geq 1 \) and \( f : \mathbb{R} \rightarrow \mathbb{R} \) we define:
\[
\|f\|_{p, \gamma} = \left( \sum_{j \geq 0} \left( \sum_{\tau \in \mathcal{D}_j} 2^{-p\gamma j} \|f \chi_{\tau}\|_{L^p}^2 \right)^{\frac{1}{p}} \right)^{1/p},
\]
(1.13)
and the space
\[(1.14) \quad X_{p,\gamma} = \{ f : \mathbb{R} \to \mathbb{C} \mid \| f\|_{p,\gamma} < \infty \}. \]

Our conditions on \( \phi \) will be given by saying that \( \hat{\phi} \in X_{p,\gamma} \) for some \( p > 2 \) and \( 1/6 > \gamma > 0 \). See Propositions 4 and 5 in Section 3 below for the detailed results.

Our first result is about local wellposedness of (1.1) and is the following:

**Theorem 1.** – Assume that \( \phi \) is a tempered distribution defined on \( \mathbb{R} \), such that there is \( \tilde{T} > 0 \) with
\[(1.15) \quad \| e^{it\partial_x^2} \phi \|_{L^3_t(L^6_x)} < +\infty, \quad \tilde{I} = [-\tilde{T}, \tilde{T}]. \]

Then, there is \( T \leq \tilde{T} \) and a unique solution \( u \) of (1.1) such that
\[(1.16) \quad \| u \|_{L^3_t(L^6_x)} < +\infty, \quad I = [-T, T]. \]

Moreover for \( |t| \leq T \):
\[(1.17) \quad u(t) - e^{it\partial_x^2} \phi \in L^2(\mathbb{R}), \]
and
\[(1.18) \quad \lim_{t \to 0} \| u(t) - e^{it\partial_x^2} \phi \|_{L^2} = 0. \]

**Remark.** – By a solution of (1.1) we mean a solution of the integral equation
\[
u(t) = e^{it\partial_x^2} \phi \pm i \int_0^t e^{i(t-\tau)\partial_x^2} |u|^2 u(\tau) \, d\tau. \]

**Remark.** – From Van der Corput’s lemma it follows that if \( \hat{\phi} \in C^1 \) and for \( j = 0, 1 \),
\[(1.19) \quad \left| \frac{d^j}{d\xi^j} \hat{\phi}(\xi) \right| \leq \frac{c}{(1+|\xi|)^{\alpha+j}} \quad \text{for some } \alpha > 1/6, \]
then \( \phi \) satisfies (1.15).

We will prove this remark in Section 3.

There is nothing new in the proof of Theorem 1, except the minor point that the Picard iteration is done in the ball given by (1.15) instead of in a ball of \( L^6_tL^6_x \). This has as immediate consequence a gain of regularity as showed in (1.19). It would be very interesting to improve this result up to \( \alpha > 0 \).

Our global wellposedness results are gathered in the following theorem:

**Theorem 2.** – Let \( \phi \) be a distribution defined on \( \mathbb{R} \), that can be decomposed as \( \phi = \phi_0 + \psi_0 \) with
\[(1.20) \quad \| \phi_0 \|_{L^2} = N^\alpha, \quad \| \psi_0 \|_{Y_{3,8}} = N^{-1}, \quad \text{for some } \alpha < 1 \text{ and some } N > 1. \]

Then, there are constants \( C, c > 0 \) independent of \( \alpha \) and \( N \), such that, the solution of the initial value problem:
\[
u_t + u_{xx} \pm |u|^2 u = 0, \quad u(x, 0) = \phi(x), \]
is well defined for $0 \leq t \leq T = cN^{1-\alpha} = c\|\psi_0\|_{Y_{3,6}}\|\phi_0\|_{L^2}^{-1}$ and it satisfies

\begin{equation}
\lim_{t \to 0} \left\| u(x, t) - e^{it\partial^2_x} \phi \right\|_{L^2} = 0,
\end{equation}

\begin{equation}
\|u\|_{L^3_t(L^6_x)} \leq CT^{\frac{1+4p}{3-6p}},
\end{equation}

for the interval $I = [0, T]$, and

\begin{equation}
\left\| u(x, t) - e^{it\partial^2_x} \phi \right\|_{L^2_x} \leq T^{\frac{1+4p}{3-6p}},
\end{equation}

for any $t \leq T$. As a consequence if for some $\alpha < 1$ and for all $N > 0$, $\phi$ can be decomposed as $\phi = \phi^N + \psi^N$ satisfying (1.20), then, there is a unique global solution of (1.1) such that (1.21)–(1.23) hold for all $T$.

In Section 3 we include several propositions which give sufficient conditions to apply the above theorems. In Proposition 4 we assume regularity and decay as in (1.10) and (1.19). Proposition 5 gives the relation between the spaces $Y_{p,q}$ and $X_{p,\gamma}$. Finally in Proposition 6 we obtain the decomposition which allows to use Theorem 2 for general data. As a consequence we will obtain the following corollary:

**Corollary 3.** – For every $2 < p < 3$ and $0 < \gamma < 1/6$ with $\gamma \leq \frac{12-5p}{6(12-2p-5p)}$ and every $\phi$ such that $\hat{\phi} \in X_{p,\gamma}$, the solution of the problem (1.1) is well defined for all $t \in \mathbb{R}$ and it satisfies the estimates given by Theorem 2.

As we see the conditions on the above corollary are not so simple to verify. For some specific examples is more useful to forget about Proposition 6 and use directly in Theorem 2 the Proposition 5 by exhibiting a good decomposition of the initial datum. This is the case if $\phi$ verifies

\begin{equation}
|\hat{\phi}(\xi)| \leq \frac{c}{(1 + |\xi|)^\alpha}, \quad \alpha > 5/12,
\end{equation}

and without any assumption on the regularity of $\hat{\phi}$. In fact a straightforward computation using Proposition 5 and Theorem 2 gives that if $\phi$ verifies (1.24) then the solution of (1.1) is global and satisfies (1.21)–(1.23).

The proofs of the two theorems are given in Section 2. The more delicate one is the proof of Theorem 2 which follows the ideas developed by Bourgain in [3]. In fact in order to prove Theorem 2 we need to solve equation (2.5) below which is more general than (1.1) – take $u_0 = 0$ in (2.5). Therefore the proof of Theorem 1 will be omitted. In Section 3 we state and prove the extension results for the Fourier transform. We go back to the classical argument of Fefferman and Stein which can be found in [7]. Finally the gain of regularity is due to the quasiorthogonality property given by Córdoba in [6] in its bilinear version as done in [15].

### 2. Proof of Theorem 2

Our proof follows the argument by Bourgain [3]. We are interested in the problem:

\begin{equation}
\begin{aligned}
iu_t + u_{xx} \pm |u|^2 u &= 0, \\
u(x, 0) &= \phi(x),
\end{aligned}
\end{equation}
where the function $\phi$ can be decomposed as $\phi = \phi_0 + \psi_0$ with

\begin{align}
\|\phi_0\|_{L^2} &= N^\alpha, \\
\|\psi_0\|_{Y_{3,6}} &= N^{-1}, \quad \alpha < 1.
\end{align}

Hereafter we just consider the positive sign in equation (2.1), following the other one with the same arguments. We consider the initial value problem:

\begin{align}
&i(u_0)_t + (u_0)_{xx} + |u_0|^2 u_0 = 0, \\
&u_0(x, 0) = \phi_0(x).
\end{align}

It is well known that $u_0$ is well defined for all $t \in (-\infty, \infty)$, and that it satisfies

\[ \|u_0(\cdot, t)\|_{L^2} = \|\phi_0\|_{L^2} = N^\alpha, \quad \text{for all } t. \]

We will also show that there are constants $C, c > 0$ independent of $\phi_0$ such that, for all $\delta > 0$, $\delta \leq c\|\phi_0\|_{L^2}^{-4/3}$, it holds

\begin{align}
\|u_0\|_{L^6([0, \delta])} &\leq C\delta^{1/2}\|\phi_0\|_{L^2}, \\
\|u_0(x, \delta) - e^{i\delta^2} \phi_0\|_{L^2} &\leq C\delta^{3/2}\|\phi_0\|_{L^2}^3.
\end{align}

We write the solution $u$ of (2.1) as $u = v + u_0$ where $v$ satisfies

\begin{align}
&iv_t + v_{xx} + 2|u_0|^2 v + |u_0|^2 \bar{v} + 2u_0|v|^2 + \bar{u}_0 v^2 + |v|^2 v = 0, \\
&v(x, 0) = \psi_0(x).
\end{align}

By Duhamel’s formula, $v = e^{it\delta^2}\psi_0 + w$, where:

\[ w(x, t) = \int_0^t e^{i(t-\tau)\delta^2} \left[ 2|u_0|^2 v + |u_0|^2 \bar{v} + 2u_0|v|^2 + \bar{u}_0 v^2 + |v|^2 v \right] \mathrm{d}\tau. \]

We will use a fixed point argument and (2.4) to show that there is a universal constant $c > 0$, such that, for any $\delta \leq cN^{-4\alpha/3}$, the solution of (2.5) is well defined for $t \in [0, \delta]$ and that $v$ and $w$ satisfy:

\begin{align}
\|v\|_{L^6([0, \delta])} &\leq 4N^{-1}, \\
\|w\|_{L^6([0, \delta])} &\leq 4N^{-1}, \\
\|w(\cdot, \delta)\|_{L^2} &\leq 64N^{2\alpha/3}N^{-1} \quad \text{and} \quad \lim_{t \to 0} \|w(\cdot, t)\|_{L^2} = 0.
\end{align}

Besides, we will deduce that

\begin{align}
\lim_{t \to 0} \|u(x, t) - e^{it\delta^2} \phi\|_{L^2} = 0 \quad \text{and} \quad \|u\|_{L^6([0, \delta])} &\leq C\delta^{3/2}\|\phi_0\|_{L^2}^3, \\
\|u(x, \delta) - e^{i\delta^2} \phi\|_{L^2} &\leq C\delta^{3/2}\|\phi_0\|_{L^2}^3 + N^{2\alpha/3-1}.
\end{align}
Let us first complete the proof assuming that (2.5), (2.7) and (2.8) hold. We take \( \delta = cN^{-4\alpha/3} \). We have shown that \( u = u_0 + e^{it\alpha/3} \psi_0 + w \) is well defined for \( t \in [0, \delta] \). We now define:

\[ \phi_1(x) = u_0(x, \delta) + w(x, \delta) \quad \text{and} \quad \psi_1(x) = e^{it\alpha/3} \psi_0. \]

We have the estimates

\[ \|\phi_1\|_{L^2} \leq N^{\alpha} + 64N^{2\alpha/3 - 1} \leq 2N^{\alpha}, \quad \|\psi_1\|_{Y_{\alpha, \delta}} = N^{-1} \]

similar to (2.1). Then we can repeat the previous argument and show that the solution \( u \) of the problem:

\[ iu_t + u_{xx} + |u|^2u = 0, \quad u(x, \delta) = \phi_1(x, \delta) + \psi_1(x, \delta), \]

is well defined for \( t \in [\delta, 2\delta] \).

We can follow the argument by induction. After \( k \) steps we will have:

\[ \|\phi_k\|_{L^2} \leq N^{\alpha} + CkN^{2\alpha/3 - 1} \leq 2N^{\alpha}, \quad \|\psi_k\|_{Y_{\alpha, \delta}} = N^{-1} \]

so that we show that \( u \) is well defined for \( t \in [k\delta, (k + 1)\delta] \). We iterate as long as \( N^{\alpha} \geq CkN^{2\alpha/3 - 1} \), i.e. \( k \leq cN^{\alpha/3 + 1} \). This shows that \( u \) is well defined for all \( t \leq \delta cN^{\alpha/3 + 1} = cN^{1-\alpha}/T \).

Moreover we have the estimates on \( u \) which follow from (2.8),

\[ \|u\|_{L^3_{[0, T]}(L^6)} \leq \left( \sum \|u\|_{L^3_{[0, (k + 1)\delta]}(L^6)} \right)^{1/3} \leq (N^{1+4\alpha/3})^{1/3} = T^{(3+4\alpha)/(9(1-\alpha))}, \]

and

\[ \|u - e^{it\alpha/3} \phi\|_{L^2_t L^2_x} \leq \int_0^T \left( \int e^{i(t-\tau)\alpha/3} (|u|^2u) \, d\tau \right) \, dt \leq \|u\|_{L^3_{[0, T]}(L^6)} \leq CT^{(3+4\alpha)/(3(1-\alpha))}, \]

for \( t \leq T \).

We now have to prove (2.4), (2.7) and (2.8), and also that \( u \), the solution of (2.1) is well defined for \( t \leq cN^{-4\alpha/3} \). We take \( \delta \leq cN^{-4\alpha/3} \), for a constant \( c > 0 \) small.

To prove (2.4), we write

\[ u_0(x, t) = e^{it\alpha/3} \phi_0 + i \int_0^t e^{i(t-\tau)\alpha/3} (|u_0|^2u_0) \, d\tau. \]

Then,

\[ \|u_0\|_{L^6_{[0, T]}(L^6)} \leq \|e^{it\alpha/3} \phi_0\|_{L^6_{[0, T]}(L^6)} + \| \int_0^t e^{i(t-\tau)\alpha/3} (|u_0|^2u_0) \, d\tau \|_{L^6_{[0, T]}(L^6)}. \]

By Cauchy–Schwarz’s inequality and Stein–Tomas restriction theorem, we have:

\[ \|e^{it\alpha/3} \phi_0\|_{L^6_{[0, T]}(L^6)} \leq \delta^{1/2} \|e^{it\alpha/3} \phi_0\|_{L^6_{[0, T]}(L^6)} \leq C \delta^{1/2} \|\phi_0\|_{L^2}. \]
On the other hand, for $0 < t < \delta$,
\[
\left\| \int_0^t e^{i(t-\tau)\partial_x^2} (|u_0|^2 u_0) \, d\tau \right\|_{L^3_{[0,\delta]}(L^6)} \leq \left\| \int_0^\delta e^{i(t-\tau)\partial_x^2} (|u_0|^2 u_0) \, d\tau \right\|_{L^3_{[0,\delta]}}
\]
\[
\leq \int_0^\delta \left\| e^{i(t-\tau)\partial_x^2} (|u_0|^2 u_0) \right\|_{L^6} \, d\tau \leq \int_0^\delta \left\| e^{i(t-\tau)\partial_x^2} (|u_0|^2 u_0) \right\|_{L^6} \, d\tau.
\]

Again by Cauchy–Schwarz’s inequality, this is bounded by
\[
\int_0^\delta \frac{1}{2} \left\| e^{i(t-\tau)\partial_x^2} (|u_0|^2 u_0) \right\|_{L^6} \, d\tau.
\]
Stein–Tomas theorem gives
\[
\leq C \int_0^\delta \delta^{1/2} \left\| u_0 \right\|_{L^2} \, d\tau = C \delta^{1/2} \left\| u_0 \right\|_{L^3_{[0,\delta]}(L^6)}.
\]

Hence,
\[
\left\| u_0 \right\|_{L^3_{[0,\delta]}(L^6)} \leq C \delta^{1/2} \left\| \phi_0 \right\|_{L^2} + C \delta^{1/2} \left\| u_0 \right\|_{L^3_{[0,\delta]}(L^6)}.
\]

When $\delta < c \left\| \phi_0 \right\|_{L^2}^{-4/3}$, the previous estimate is enough to show that
\[
\left\| u_0 \right\|_{L^3_{[0,\delta]}(L^6)} \leq 2C \delta^{1/2} \left\| \phi_0 \right\|_{L^2}.
\]

Moreover,
\[
\left\| u_0(x, \delta) - e^{i\delta \partial_x^2} \phi_0 \right\|_{L^2}^2 = \left\| \int_0^\delta e^{i(\delta-\tau)\partial_x^2} (|u_0|^2 u_0) \, d\tau \right\|_{L^2}^2
\]
\[
\leq \int_0^\delta \left\| e^{i(\delta-\tau)\partial_x^2} (|u_0|^2 u_0) \right\|_{L^2} \, d\tau
\]
\[
= \int_0^\delta \left\| (|u_0|^2 u_0) \right\|_{L^2} \, d\tau
\]
\[
= \left\| u_0 \right\|_{L^3_{[0,\delta]}(L^6)}^3 \leq C (\delta^{1/2} \left\| \phi_0 \right\|_{L^2})^3,
\]
which finishes the proof of (2.4).
Moreover, as in the proof of (2.4),

\[ T v(x, t) = e^{it\partial^2_x} \psi_0 + i \int_0^t e^{i(t-\tau)\partial^2_x} \left[ 2|\alpha|\nabla^2 v + |\alpha|^2 \nabla^2 v + \nabla \cdot \nabla v^2 + |v|^2 v(x, \tau) \right] d\tau \]

satisfies

\[
\| T v_1 - T v_2 \|_{L^3_{|\alpha|}(L^6)} \leq C \delta^{1/2} \left[ \|u_0\|_{L^2_{|\alpha|}(L^6)}^2 \|v\|_{L^3_{|\alpha|}(L^6)} + \|v\|_{L^3_{|\alpha|}(L^6)}^3 + \|\nabla v\|_{L^6}^3 \right]
\]

Moreover,

\[
\| T v \|_{L^3_{|\alpha|}(L^6)} \leq C \delta^{1/2} \left[ \|u_0\|_{L^2_{|\alpha|}(L^6)}^2 \|v\|_{L^3_{|\alpha|}(L^6)} + \|v\|_{L^3_{|\alpha|}(L^6)}^3 + \|\nabla v\|_{L^6}^3 \right]
\]

Using (2.4),

\[
\| T v \|_{L^3_{|\alpha|}(L^6)} \leq C \delta^{1/2} \left( \delta^{1/2} \|\phi_0\|_{L^2} \right)^2 \|v\|_{L^3_{|\alpha|}(L^6)} + C \delta^{1/2} \|v\|_{L^3_{|\alpha|}(L^6)}^3 + N^{-1},
\]

so that, for \( \delta \leq c N^{-4\alpha/3} \) small enough and for \( \|v\|_{L^3_{|\alpha|}(L^6)} \leq 4N^{-1} \), implies

\[
\| T v \|_{L^3_{|\alpha|}(L^6)} \leq 4N^{-1}.
\]

Therefore a fixed point argument shows that the solution of (2.5) is well defined and it satisfies

\[
\| v \|_{L^3_{|\alpha|}(L^6)} \leq 4N^{-1}.
\]

To prove (2.7) note that,

\[
\| w \|_{L^3_{|\alpha|}(L^6)} \leq C \delta^{1/2} \left[ \|u_0\|_{L^2_{|\alpha|}(L^6)}^2 \|v\|_{L^3_{|\alpha|}(L^6)} + \|v\|_{L^3_{|\alpha|}(L^6)}^3 \right].
\]

Hence,

\[
\| w \|_{L^3_{|\alpha|}(L^6)} \leq C \delta^{1/2} \left( \delta^{1/2} \|\phi_0\|_{L^2} \right)^2 \|v\|_{L^3_{|\alpha|}(L^6)} + C \delta^{1/2} \|v\|_{L^3_{|\alpha|}(L^6)}^3
\]

\[
\leq \| v \|_{L^3_{|\alpha|}(L^6)} \left[ C \delta^{3/2} \|\phi_0\|_{L^2}^2 + C \delta^{1/2} \|v\|_{L^2}^2 \right]
\]

\[
\leq 4N^{-1}.
\]

Moreover, as in the proof of (2.4),

\[
\| w(\cdot, \delta) \|_{L^2} \leq 2 \left[ \|u_0\|_{L^2_{|\alpha|}(L^6)}^2 \|v\|_{L^3_{|\alpha|}(L^6)} + \|v\|_{L^3_{|\alpha|}(L^6)}^3 \right]
\]

\[
\leq 2 \left[ (C \delta^{1/2} \|\phi_0\|_{L^2})^2 4N^{-1} + (4N^{-1})^3 \right]
\]

\[
\leq 64N^{2\alpha/3-1}.
\]
Note also that, for \( t \leq \delta \) the same argument shows that

\[
\|v\|_{L^3_{[0,\delta]}(L^6)} \leq C \delta^{1/2} \left[ \|u_0\|_{L^3_{[0,\delta]}(L^6)}^2 \|v\|_{L^3_{[0,\delta]}(L^6)} + \|v\|_{L^3_{[0,\delta]}(L^6)}^3 \right] + \|e^{it\partial_x^2} \psi_0\|_{L^3_{[0,\delta]}(L^6)}.
\]

By (2.4) again,

\[
\|v\|_{L^3_{[0,\delta]}(L^6)} \leq C \delta^{1/2} \left( \delta^{1/2} \|\phi_0\|_{L^2} \right)^2 \|v\|_{L^3_{[0,\delta]}(L^6)} + C \delta^{1/2} \|v\|_{L^3_{[0,\delta]}(L^6)}^3 + \|e^{it\partial_x^2} \psi_0\|_{L^3_{[0,\delta]}(L^6)},
\]

so that, for \( \delta \leq cN^{-4a/3} \),

\[
\|v\|_{L^3_{[0,\delta]}(L^6)} \leq C \delta^{1/2} \|v\|_{L^3_{[0,\delta]}(L^6)}^3 + \|e^{it\partial_x^2} \psi_0\|_{L^3_{[0,\delta]}(L^6)}.
\]

This, when \( \delta^{1/2} \) is smaller than \( 1/10 \), suffices to prove the a priori inequality,

\[
\|v\|_{L^3_{[0,\delta]}(L^6)} \leq 4 \|e^{it\partial_x^2} \psi_0\|_{L^3_{[0,\delta]}(L^6)}
\]

and

\[
\|w\|_{L^3_{[0,\delta]}(L^6)} \leq 4 \|e^{it\partial_x^2} \psi_0\|_{L^3_{[0,\delta]}(L^6)}.
\]

Hence,

\[
\|w(\cdot,t)\|_{L^2} \leq 2 \left[ (C\delta^{1/2} \|\phi_0\|_{L^2})^2 4 \|e^{it\partial_x^2} \psi_0\|_{L^3_{[0,\delta]}(L^6)}^3 + 4 \|e^{it\partial_x^2} \psi_0\|_{L^3_{[0,\delta]}(L^6)}^3 \right] \to 0
\]

when \( t \to 0 \).

Finally, we have to go to (2.8). We have all the ingredients that we need. From (2.4) and (2.7),

\[
\|u\|_{L^3_{[0,\delta]}(L^6)} \leq \|u_0\|_{L^3_{[0,\delta]}(L^6)} + \|v\|_{L^3_{[0,\delta]}(L^6)} \leq C \delta^{1/2} \|\phi_0\|_{L^2} + 4N^{-1} \leq 8N^{a/3}
\]

and

\[
\|u(x,\delta) - e^{it\partial_x^2} \phi\|_{L^2} \leq \|u_0(x,\delta) - e^{it\partial_x^2} \phi\|_{L^2} + \|w(\cdot,\delta)\|_{L^2} \leq 8\delta^{3/2} N^{3a} + 64N^{2a/3 - 1}.
\]

Using again (2.4) and (2.7)

\[
\lim_{t \to 0} \|u(x,t) - e^{it\partial_x^2} \phi\|_{L^2} = \lim_{t \to 0} \|u_0(x,t) - e^{it\partial_x^2} \phi\|_{L^2} + \lim_{t \to 0} \|w(\cdot,t)\|_{L^2} \leq \lim_{t \to 0} 8t^{3/2} N^{3a} + \lim_{t \to 0} \|w(\cdot,t)\|_{L^2} = 0.
\]

3. Some function spaces satisfying the hypothesis of the theorems

In this section we give sufficient conditions on the initial data \( \phi \) such that Theorems 1 and 2 apply. We consider two different situations. In the first one, quite straightforward, we assume regularity and decay assumptions as those of (1.9), (1.10), and (1.19). In the second one we consider just size conditions on \( \phi \). Let us start with the following elementary proposition.
Proposition 4. Assume that \( \hat{\phi} \in C^1 \) and that, for some \( \beta > 0 \), and for \( j = 0, 1, \)

\[
| \frac{d^j}{dx^j} \hat{\phi}(\xi) | \leq \frac{c}{(1 + |\xi|)^{\beta+j}}
\]

Then

(i) if \( \beta > 1/6 \),

\[
\| e^{it\beta \hat{\phi}} \|_{L^1_t(L^6_x)} < +\infty, \quad I = [-1, 1]
\]

and Theorem 1 applies;

(ii) if \( \beta > 1/3 \) Theorem 2 applies for every \( N > 1 \), and therefore (1.1) is globally well posed.

Proof. Assume first that \( \hat{\phi} \in C[-1, 1] \). Then from Van der Corput’s lemma, see [13], we have:

\[
|e^{it\beta \hat{\phi}}| \leq 10 \| \hat{\phi} \|_{L^\infty} (1 + |t|)^{-1/2}, \quad |x| < 10|t|,
\]

and

\[
|e^{it\beta \hat{\phi}}| \leq 10 \| \hat{\phi} \|_{L^\infty} (1 + |x|)^{-1}, \quad |x| > 10|t|.
\]

Take \( \lambda > 1 \) and \( \phi_\lambda(x) = \lambda \phi(\lambda x) \). Hence \( e^{it\beta \hat{\phi}_\lambda} = \lambda e^{it\beta \hat{\phi}}(\lambda x) \) and we easily obtain:

\[
\| e^{it\beta \hat{\phi}_\lambda} \|_{L^1_t(L^6_x)} \leq C \| \hat{\phi} \|_{L^\infty} \lambda^{1/6} \log \lambda, \quad I = [-1, 1].
\]

For general \( \phi \) satisfying (3.1) just decompose it in a dyadic series and apply (3.3) to each term of the series to obtain (3.2). The second part of the proposition follows by cutting smoothly \( \hat{\phi} \) for \( |\xi| \sim N^{\frac{1}{6\beta+1}} \). For \( |\xi| > N^{\frac{1}{6\beta+1}} \) proceed as before. The remainder is in \( L^2 \) and has the suitable size. \( \square \)

Next we are going to show some estimates for \( e^{it\beta \hat{\phi}} \) with \( \phi \notin L^2(\mathbb{R}) \), but \( \hat{\phi} \) belonging to some function spaces contained in \( L^2_{loc}(\mathbb{R}) \).

We recall the definition of those spaces given in the introduction. For \( j = 0, 1, 2, 3, \ldots \), we consider \( D_j = \{ k2^j, (k+1)2^j \}: k \in \mathbb{Z} \}, \) the family of dyadic intervals of length \( 2^j \) in \( \mathbb{R} \). Then, for \( \gamma > 0 \) and \( \rho \geq 1 \) and \( f: \mathbb{R} \rightarrow \mathbb{C} \) we define:

\[
\| f \|_{p,\gamma} = \sum_{j \geq 0} \left( \sum_{r \in D_j} 2^{-j\gamma} \| f \chi_r \|_2^p \right)^{1/p},
\]

and the space

\[
X_{p,\gamma} = \{ f: \mathbb{R} \rightarrow \mathbb{R} \mid \| f \|_{p,\gamma} < \infty \}.
\]

We are going to show that if \( \hat{\phi} \in X_{p,\gamma} \), for some \( p \) and \( \gamma \), then \( \phi \) satisfies the hypothesis of Theorem 2. Observe that if \( p > 2 \) and \( \gamma > 0 \) then \( L^2 \subset X_{p,\gamma} \).

We begin by proving the following proposition:

Proposition 5. There is a constant \( C > 0 \) such that for any \( \phi \) with \( \hat{\phi} \in X_{3.1/6} \), we have:

\[
\| \phi \|_{X_{4,6}} = \sup_{|f| < 1} \| e^{it\beta \hat{\phi}} \|_{L^1_t(L^6_x)} \leq C \| \hat{\phi} \|_{X_{3.1/6}}.
\]
Notice that, \( \|\phi\|_{Y_{3,6}} \leq \|\phi\|_{Y_{k,\delta}} \) and hence, Proposition 5 gives us a space of functions satisfying the hypothesis of Theorem 1.

A refinement of the previous proposition will give us a space of functions which satisfy the hypothesis of Theorem 2 for every \( N > 1 \).

**Proposition 6.** For every \( 2 < p < 12/5 \) and \( 0 < \gamma < 1/6 \) with

\[
\gamma < \frac{12 - 5p}{6(12 - 2p - p^2)}
\]

there is a constant \( C = C(p, \gamma) > 0 \) and \( 1 > \alpha = \alpha(p, \gamma) > 0 \), such that for any \( f \in X_{p, \gamma} \) and \( N > 1 \), we have a decomposition \( f = g + h \) where \( \|g\|_{3,1/6} \leq CN^{-1} \|f\|_{p, \gamma} \) and \( \|h\|_{L^2} \leq CN^\alpha \|f\|_{p, \gamma} \).

An immediate consequence of the above proposition is the Corollary 3 stated in the introduction.

For a function \( f : \mathbb{R} \rightarrow \mathbb{R} \) we define:

\[
\mathcal{R}^* f(x, t) = \int f(\xi) e^{it\xi + u|\xi|^\gamma} d\xi.
\]

Notice that \( \mathcal{R}^* \) is essentially the adjoint of the operator of restriction of the Fourier transform to the parabola and that \( e^{it\xi} \phi(x, t) = \mathcal{R}^* \phi \). Hence, Proposition 5 is equivalent to the following “adjoint restriction theorem”.

**Proposition 5'.** There is a constant \( C > 0 \) such that for any \( f \in X_{3,1/6} \), we have:

\[
\|\mathcal{R}^* f\|_{Y_{3,6}} \leq C \|f\|_{3,1/6}.
\]

**Proof.** We will use the following notation. For any \( \tau = \{k2^j, (k + 1)2^j\} \in \mathcal{D}_j \) we denote \( \tau(i), i = 1, 2, 3 \), the intervals “close” to \( \tau \) (i.e. those intervals with the same length which are not adjacent but have adjacent parents). Also, we define a translation of the family of dyadic intervals of length \( 2^j \), namely \( \tilde{\mathcal{D}}_j = \{2k + 1, 2(k + 1) + 1: k \in \mathbb{Z}\} \). Let \( I \subset \mathbb{R} \) be and interval of length 1. We can write

\[
\|\mathcal{R}^* f\|_{L^2(L^6)}^2 \leq \sum_{i=1,2,3} \left\| \sum_{j \geq 0} \sum_{\tau \in \mathcal{D}_i} \mathcal{R}^* (f_\tau) \mathcal{R}^* (f_\tau) \right\|_{L^2(L^3)} + \sum_{\tau \in \mathcal{D}_1} \left( \mathcal{R}^* (f_\tau) \right)^2_{L^2(L^3)}
\]

\[
+ \left\| \sum_{\tau \in \mathcal{D}_1} \sum_{j \geq 0} \mathcal{R}^* (f_\tau) \mathcal{R}^* (f_\tau) \right\|_{L^2(L^3)} + \left( \mathcal{R}^* (f_\tau) \right)^2_{L^2(L^3)}
\]

\[
\leq \sum_{i=1,2,3} \sum_{j \geq 0} \left\| \sum_{\tau \in \mathcal{D}_i} \mathcal{R}^* (f_\tau) \right\|_{L^2(L^3)} + \sum_{\tau \in \mathcal{D}_1} \left( \mathcal{R}^* (f_\tau) \right)^2_{L^2(L^3)}
\]

\[
+ \sum_{\tau \in \mathcal{D}_1} \left( \mathcal{R}^* (f_\tau) \right)^2_{L^2(L^3)}.
\]

By Strichartz estimate and Cauchy–Schwarz’s inequality,

\[
\|\mathcal{R}^* f \mathcal{R}^* g\|_{L^2(L^\infty)} \leq \|\mathcal{R}^* f\|_{L^4(L^\infty)} \|\mathcal{R}^* g\|_{L^4(L^\infty)} \leq \|f\|_{L^2} \|g\|_{L^2}.
\]
Moreover, we have the bilinear estimate
\[ \| \mathcal{R}^* f \mathcal{R}^* g \|_{L^2(L^2)} \leq \| f \|_{L^2} \| g \|_{L^2}, \]
whenever the distance between \( \text{supp} \ f \) and \( \text{supp} \ g \) is bigger than or equal to 1. This is a consequence of the classical argument of Fefferman and Stein as can be found in [7]. Therefore, by interpolation
\[ \| \mathcal{R}^* f \mathcal{R}^* g \|_{L^2(L^3)} \leq C \| f \|_{L^2} \| g \|_{L^2}, \]
when \( \text{supp} \ f \) and \( \text{supp} \ g \) are 1-separated. From this inequality and scaling, we obtain that, for \( \tau \in D_j \),
\[ \left\| \sum_{\tau \in D_j} \mathcal{R}^* (f_\tau) \mathcal{R}^* (f_{\tau(i)}) \right\|_{L^2(L^3)} \leq C 2^{-j/3} \left( \sum_{\tau \in D_j} \left\| \mathcal{R}^* (f_\tau) \mathcal{R}^* (f_{\tau(i)}) \right\|_{L^2(L^3)}^2 \right)^{1/2}. \]
(3.4)

For each fixed \( j \), triangular inequality gives,
\[ \left\| \sum_{\tau \in D_j} \mathcal{R}^* (f_\tau) \mathcal{R}^* (f_{\tau(i)}) \right\|_{L^2(L^\infty)} \leq \sum_{\tau \in D_j} \left\{ \left\| \mathcal{R}^* (f_\tau) \mathcal{R}^* (f_{\tau(i)}) \right\|_{L^2(L^\infty)}^3 / 2 \right\}^{2/3}. \]
Moreover, the sets \( \{ \tau + \tau(i) : \tau \in D_j \} \) are almost disjoint and therefore,
\[ \left\| \sum_{\tau \in D_j} \mathcal{R}^* (f_\tau) \mathcal{R}^* (f_{\tau(i)}) \right\|_{L^2(L^3)} \leq C \left( \sum_{\tau \in D_j} \left\| \mathcal{R}^* (f_\tau) \mathcal{R}^* (f_{\tau(i)}) \right\|_{L^2(L^3)}^2 \right)^{1/2}. \]
(3.5)

By interpolation (see more details at the end) we obtain:
\[ \left\| \sum_{\tau \in D_j} \mathcal{R}^* (f_\tau) \mathcal{R}^* (f_{\tau(i)}) \right\|_{L^2(L^3)} \leq C \left( \sum_{\tau \in D_j} \left\| \mathcal{R}^* (f_\tau) \mathcal{R}^* (f_{\tau(i)}) \right\|_{L^2(L^3)}^{3/2} \right)^{2/3}. \]
From (3.4) and (3.5),
\[ \sum_{j \geq 0} \left\| \sum_{\tau \in D_j} \mathcal{R}^* (f_\tau) \mathcal{R}^* (f_{\tau(i)}) \right\|_{L^2(L^3)} \leq C \sum_{j \geq 0} \left[ \sum_{\tau \in D_j} 2^{-j/2} \| f_\tau \|_{L^3}^3 \right]^{1/3}. \]

On the other hand, by a similar argument, taking \( \eta \) a bump function, \( \hat{\eta} \in C^\infty_0(\mathbb{R}) \) and \( \eta \geq 1 \) on \( I \),
\[ \left\| \sum_{\tau \in D_1} (\mathcal{R}^* (f_\tau))^2 \right\|_{L^2(L^3)} \leq \left\| \sum_{\tau \in D_1} (\mathcal{R}^* (f_\tau))^2 \eta(t) \right\|_{L^2(L^3)} \]
\[ \leq \left[ \sum_{\tau \in D_1} \left\| (\mathcal{R}^* (f_\tau))^2 \eta(t) \right\|_{L^2(L^3)}^{3/2} \right]^{2/3} \]
(3.6)
\[ = \left[ \sum_{\tau \in D_1} \left\| (\mathcal{R}^* (f_\tau))^3 \sqrt{\eta(t)} \right\|_{L^4(L^6)}^{3} \right]^{2/3}. \]
Now, since $I$ has length 1, we can use Hölder’s inequality and then Stein–Tomas theorem to bound

$$\leq \left[ \sum_{\tau \in D_1} \left\| \mathcal{R}^*(f_\tau) \right\|_{L^6}^3 \right]^{1/3} \leq \left[ \sum_{\tau \in D_1} \left\| f_\tau \right\|_{L^2}^{3/2} \right]^{2/3}. $$

The estimates for the terms corresponding to $D_0$ and $\tilde{D}_1$ can be proved similarly.

To obtain the inequality (3.5) by interpolation, we define $10^\tau$ to be the interval concentric to $\tau$ and with length 10 times longer than $\tau$, and $A_\tau = 10^\tau \times \mathbb{R}$. We consider the linear operators $T_\tau$ to be smooth multipliers adapted to $A_\tau$. We fix $j$. By orthogonality we have:

$$\left\| \sum_{\tau \in D_j} T_\tau g_\tau \right\|_{L^2(L^2)} \leq \left( \sum_{\tau \in D_j} \left\| g_\tau \right\|_{L^2(L^2)}^2 \right)^{1/2}. $$

Also, by the triangular inequality,

$$\left\| \sum_{\tau \in D_j} T_\tau g_\tau \right\|_{L^2(L^\infty)} \leq \sum_{\tau \in D_j} \left\| g_\tau \right\|_{L^2(L^\infty)}. $$

Thus, by complex interpolation,

$$\left\| \sum_{\tau \in D_j} T_\tau g_\tau \right\|_{L^2(L^3)} \leq \left( \sum_{\tau \in D_j} \left\| g_\tau \right\|_{L^2(L^3)}^{3/2} \right)^{2/3}. $$

Then (3.5) follows applying this inequality to $g_\tau = \mathcal{R}^*(f_\tau) \mathcal{R}^*(f_\tau(i))$ and (3.6) applying it to $g_\tau = (R^*(f_\tau)(x,t))^2 \eta(t)$.

**Proof of Proposition 6.** – We are going to build $h$ selecting the intervals $\tau$ where $f$ has big mass. To do that, we fix $\delta > 0$ and $j_0 \geq 0$ to be chosen appropriately. Given $j \leq j_0$, an interval $\tau \in D_j$ is selected if $\left\| f_\tau \right\|_{L^2} \geq \delta \left\| f \right\|_{p,\gamma}$. For $j > j_0$ we do not choose any interval.

We define $h = f \chi_E$, where $E$ is the union of the selected intervals and $g = f - h$.

Let us see first that $\left\| g \right\|_{3,1/6}$ is small.

$$\left\| g \right\|_{3,1/6} \leq \sum_{0 \leq j < j_0} \left[ \sum_{\tau \in D_j, \text{not selected}} 2^{-j/2} \left\| f_\tau \right\|_2^3 \right]^{1/3} + \sum_{j > j_0} \left[ \sum_{\tau \in D_j} 2^{-j/2} \left\| f_\tau \right\|_2^3 \right]^{1/3}. $$

To bound the first term, since $p < 3$ and $\gamma p < 1/2$:

$$\left[ \sum_{\tau \in D_j, \text{not selected}} 2^{-j/2} \left\| f_\tau \right\|_2^3 \right]^{1/3} \leq \left[ \sum_{\tau \in D_j, \text{not selected}} 2^{-j(1/2-\gamma p)/2} \left\| f_\tau \right\|_2^p \left\| f_\tau \right\|_{L^2}^{3-p} \right]^{1/3} \leq 2^{-j(1/6-\gamma p)/3} \left[ \sum_{\tau \in D_j, \text{not selected}} 2^{-\gamma p j} \left\| f_\tau \right\|_2^p \left( \left\| f \right\|_{p,\gamma} \delta \right)^{2-p} \right]^{1/3} \leq 2^{-j(1/6-\gamma p)/3} \left\| f \right\|_{p,\gamma}^p \left( \left\| f \right\|_{p,\gamma} \delta \right)^{(3-p)/p}.$$
Now, we can sum in $j$ for $j \leq j_0$. For the second term, we use that $\gamma < 1/6$ and $p < 3$,

$$\sum_{j > j_0} \left( \sum_{\tau \in D_j} 2^{-j/2} \| f_\tau \|_2^2 \right)^{1/3} \leq \sum_{j > j_0} \left( \sum_{\tau \in D_j} 2^{-j(1/2-3\gamma)} \| f_\tau \|_2^3 \right)^{1/3} \leq \sum_{j > j_0} 2^{-j(1/2-3\gamma)} \left( \sum_{\tau \in D_j} 2^{-3\gamma j} \| f_\tau \|_2^3 \right)^{1/3} \leq C 2^{-j_0(1/6-\gamma)} \sum_{j > j_0} \left( \sum_{\tau \in D_j} 2^{-p\gamma j} \| f_\tau \|_2^p \right)^{1/p} = C 2^{-j_0(1/6-\gamma)} \| f \|_{p,\gamma}.$$ 

Hence,

$$\| g \|_{3,1/6} \leq C \| f \|_{p,\gamma} [\delta^{(3-p)/3} + 2^{-j_0(1/6-\gamma)}].$$

We choose $\delta^{(3-p)/3} = 2^{-j_0(1/6-\gamma)}$ and $2^{-j_0(1/6-\gamma)} = N$.

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