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A class of AS-regular algebras of dimension five

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ABSTRACT

We classify 5-dimensional Artin–Schelter regular algebras generated by two generators of degree 1 with three generating relations of degree 4 under a generic condition. All the algebras obtained are proved to be strongly noetherian, Auslander regular and Cohen–Macaulay with respect to the Gelfand–Kirillov dimension.

1. Introduction

One of the most important questions in noncommutative algebraic projective geometry is to classify the quantum projective space $\mathbb{P}^n$s—noncommutative analogues of projective n-spaces. In fact, this is a challenging and hard project, even for $n = 4$. An algebraic approach to construct quantum $\mathbb{P}^n$ is to form the noncommutative projective scheme $\operatorname{Proj} A$ [AZ], where A is a noetherian connected graded Artin–Schelter regular algebras of global dimension $n + 1$. Then the question turns out to be the classification of Artin–Schelter regular algebras.

The quantum $\mathbb{P}^2$s were classified by Artin and Schelter [AS] and by Artin, Tate and Van den Bergh [ATV] using geometric method. As to the quantum $\mathbb{P}^3$s, many researchers have studied them in terms of Artin–Schelter regular algebras. The most famous 4-dimensional Artin–Schelter regular algebra is

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the Sklyanin algebra of dimension 4, introduced by Sklyanin [Sk1,Sk2]. Homological properties and the representations of the Sklyanin algebra were studied by Smith and Stafford [SS], Levasseur and Smith [LS] respectively. Normal extensions of 3-dimensional Artin–Schelter regular algebras, which are 4-dimensional Artin–Schelter regular algebras, were studied by Le Bruyn, Smith and Van den Bergh [LSV]. The quantum 2×2-matrix algebra was studied by Vancliff [Va1,Va2]. Some classes of Artin–Schelter regular algebras containing a commutative quadric were studied by Shelton, Van Rompay, Vancliff, Willaert, etc. [SV1,SV2,VV1,VV2,VVW].

Several years ago, Lu, Palmieri, Zhang and the second author [LPWZ1,LPWZ2,LPWZ3] started the project to classify quantum $\mathbb{P}^3$s, or 4-dimensional Artin–Schelter regular algebras by using A_∞-algebraic methods. In general, 4-dimensional Artin–Schelter regular algebras have three resolution types if they are domains, i.e., the so-called type (12221), (13431) and (14641). Under some generic conditions, Lu, Palmieri, Zhang and the second author classified the type (12221) [LPWZ2], i.e., the type of 4-dimensional Artin–Schelter regular algebras generated by two generators of degree 1 with two relations—one of degree 3, the other of degree 4. Type (13431) is the type of 4-dimensional Artin–Schelter regular algebras generated by three generators of degree 1 with four relations—two of degree 2, the other two of degree 3. This type has been studied by Rogalski and Zhang recently [RZ], where they gave all the families of Artin–Schelter regular algebras which are not normal extensions of 3-dimensional Artin–Schelter regular algebras. Type (14641) is the type of 4-dimensional Artin–Schelter regular algebras generated by four generators of degree 1 with six quadratic relations. Zhang and Zhang introduced a new construction, which is called double Ore extension, and they found some new families of this type (see [ZZ1,ZZ2]).

Recently, Floystad and Vatne studied 5-dimensional Artin–Schelter regular algebras [FV]. All the possible resolution types were given for the trivial modules of all 5-dimensional Artin–Schelter regular algebras generated by two elements of degree 1 which are domains.

Theorem 1.1. *(See [FV, Lemma 5.4 and Theorem 5.6].) Let A be an AS-regular algebra of global dimension 5 which is a domain of GK-dim $A \geqslant 4$. If A has two generators of degree 1, then the minimal resolution of the trivial module k_A has the form*

$$0 \to A(-l) \to A(-l+1)^{\oplus 2} \to \bigoplus_{i=1}^{n} A(a_i - l) \to \bigoplus_{i=1}^{n} A(-a_i) \to A(-1)^{\oplus 2} \to A \to k_A \to 0$$

for some integers $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$ and l, such that one of the following holds:

(1) $n = 3$ *and* (a_1, a_2, a_3) *is* $(3, 5, 5)$, $(4, 4, 4)$ *or* $(3, 4, 7)$ *with* $l = 11, 10, 12$ *respectively*;
(2) $n = 4$ *and* (a_1, a_2, a_3, a_4) *is* $(4, 4, 4, 5)$ *with* $l = 10$;
(3) $n = 5$ *and* $(a_1, a_2, a_3, a_4, a_5)$ *is* $(4, 4, 4, 5, 5)$ *with* $l = 10$.

There are 5-dimensional AS-regular algebras with the resolution types for $n = 3$, where the first two cases can be realized by the enveloping algebras of 5-dimensional graded Lie algebras, while the third one cannot be realized in such a way [FV, Proposition 3.4]. It is open that whether there is a 5-dimensional AS-regular algebra with the resolution type for $n = 4$ or $n = 5$.

In this paper, we focus on the classification of quantum $\mathbb{P}^4$s and consider the Artin–Schelter regular algebras of global dimension 5. The general ideas used here are similar as in [LPWZ2]. Under a generic condition, we classify 5-dimensional Artin–Schelter regular algebras generated by two generators of degree 1 with three generating relations of degree 4.

Theorem A. *There are* 9 *types of Artin–Schelter regular algebras of dimension* 5 *which are generated by two elements of degree* 1 *with three generating relations of degree* 4, *as listed in Section* 4 *as algebras* **A**, **B**, **C**, **D**, **E**, **F**, **G**, **H** *and* **I**. *Under the generic condition* (GM2) (*see Section* 4), *this is a complete list of* 5-*dimensional Artin–Schelter regular algebras which are domains generated by two generators of degree* 1 *with three generating relations of degree* 4.

The generic condition (GM2) mainly means that the structure matrices $\mathcal{R} = (r_{ij})_{2\times 2}$ and $\mathcal{T} = (t_{ks})_{3\times 3}$ of the corresponding Ext-algebra in (3.1) have distinct eigenvalues.

All these algebras enjoy many nice homological properties.

Theorem B. *All the algebras* **A**, **B**, **C**, **D**, **E**, **F**, **G**, **H** *and* **I** *are strongly noetherian, Auslander regular and Cohen–Macaulay* (*see Theorems* 5.4, 5.5, 5.8, 5.9).

Corollary C. *Let $\mathcal{A}$ be a* 5*-dimensional AS-regular algebra generated by two elements of degree* 1 *with three relations of degree* 4*. Suppose it is a domain and satisfies the generic condition* (GM2)*. If it is not a normal extension of some* 4*-dimensional AS-regular algebra, then it is either an iterated Ore extension of the polynomial ring in one variable or falls into one of the families* **A**, **B** *and* **F**, *up to isomorphism.*

The paper is organized as follows. In Section 2, we recall the definition of Artin–Schelter regular algebras and their properties. The canonical A_∞-structures on the Yoneda Ext-algebras and the general ideas used for the classification of AS-regular algebras by using A_∞-methods are explained also in this section. In Section 3, we analyze the Frobenius structure and A_∞-structure of the Yoneda Ext-algebras $E(A)$ for 5-dimensional AS-regular algebras A generated by two elements of degree 1 with three relations of degree 4. Several systems of equations satisfied by the structural coefficients are obtained following the Stasheff's identities. In Section 4, we introduce a generic condition (GM2) on the algebra structure on E, and give all the possible AS-regular algebras of global dimension 5 of the type considered. In Section 5, we prove all the possible algebras listed in Section 4 are strongly noetherian, Auslander regular and Cohen–Macaulay with respect to the Gelfand–Kirillov dimension, thus proving the main results.

2. Preliminaries

Throughout the paper, k is an algebraically closed field of characteristic zero and all algebras are connected graded k-algebras generated in degree 1. Now we recall the definition of Artin–Schelter regular algebras.

2.1. Artin–Schelter regular algebras

Definition 2.1. A connected graded algebra A is called **Artin–Schelter regular** (AS-regular, for short) if the following three conditions hold:

(AS1) A has finite global dimension d,

(AS2) A is Gorenstein, i.e., there exists an integer l such that

$$\operatorname{Ext}^i_A(k, A) \cong \begin{cases} k(l), & i = d, \\ 0, & i \neq d \end{cases}$$

where k is the trivial left or right A-module $A/A_{\geqslant 1}$, and the notation (l) is the degree l-shifting on graded modules,

(AS3) A has finite Gelfand–Kirillov dimension (GK dimension).

2.2. A_∞-algebras

We recall the definition of A_∞-algebras and the A_∞-structure on the Yoneda Ext-algebras in this subsection.

Definition 2.2. An A_∞**-algebra** over k is a $\mathbb{Z}$-graded vector space $A = \bigoplus_{i\in\mathbb{Z}} A^i$ endowed with a family of graded k-linear maps $m_n : A^{\otimes n} \to A$ of degree $2-n$ $(n \geqslant 1)$, such that the following Stasheff identities SI(n) hold:

$$\sum(-1)^{r+st}m_u\left(\mathrm{id}^{\otimes r}\otimes m_s\otimes \mathrm{id}^{\otimes t}\right)=0 \qquad \mathrm{SI}(n)$$

for all $n \geqslant 1$, where the sum runs over all the decompositions $n = r + s + t$ ($r, t \geqslant 0$ and $s \geqslant 1$) and $u = r + 1 + t$.

We assume that every A_∞-algebra in this paper satisfies the **strictly unital condition**: there is an element $1 \in A^0$, which is called the *strict unit* or *identity* of A, such that

- 1 is an identity with respect to the multiplication m_2, and
- if $n \neq 2$ and $a_i = 1$ for some i, then $m_n(a_1, \ldots, a_n) = 0$.

Note that when the formulas are applied to elements, additional signs appear due to the Koszul sign rule as usual in the graded setting.

A differential graded algebra (A, d) can be viewed as an A_∞-algebra by setting $m_1 = d$, m_2 be the multiplication and $m_n = 0$ for all $n \geqslant 3$. On the other hand, for any differential graded algebra A, there is a canonical A_∞-algebra structure on its cohomology algebra HA which is unique in some sense [Ka,Me].

Let A be a connected graded algebra, and k_A be the trivial A-module. The Ext-algebra $\mathrm{Ext}^*_A(k_A, k_A)$, viewed as the cohomology algebra of some differential graded algebra, is equipped with an A_∞-algebra structure. We use $\mathrm{Ext}^*_A(k_A, k_A)$ to denote both the usual associative Ext-algebra and the Ext-algebra with the canonical A_∞-structure. Occasionally we use E also to denote Ext with its A_∞-algebra structure.

We assume also that the A_∞-algebras in this paper are $\mathbb{Z}^2$-graded. In fact, the Ext-algebra $\mathrm{Ext}^*_A(k_A, k_A)$ of a connected graded algebra A is a typical example of $\mathbb{Z}^2$-graded A_∞-algebras; the first grading, written as a superscript, is the homological one, and the other grading, which is sometimes called the Adams grading, written as subscript, is induced by the grading on the original graded algebra A. The degree of the multiplication maps m_n in $\mathbb{Z}^2$-graded A_∞-algebras is $(2-n, 0)$, i.e., m_n preserves the Adams grading. For the construction of the A_∞-structure of the Ext-algebra $\mathrm{Ext}^*_A(k_A, k_A)$, see also [LPWZ3, Proposition 1.2].

2.3. A_∞-Ext-algebras of AS-regular algebras

The following theorem is one bridge for the classification of AS-regular algebras by A_∞-methods.

Theorem 2.3. *(See [LPWZ1, Theorem 12.9] or [LPWZ4, Corollary D].) Let A be a connected graded algebra and let E be the* Ext*-algebra of A. Then A satisfies the conditions* (AS1) *and* (AS2) *in Definition* 2.1 *if and only if E is a Frobenius algebra.*

This was proved by using A_∞-algebra methods. Theorem 2.3 is a generalization of a result of Smith in [Sm], where A is assumed to be noetherian Koszul.

If A is not Koszul, then the associative algebra $\mathrm{Ext}^*_A(k_A, k_A)$ does not contain enough information to recover A (see, say, [LPWZ1, Ex. 13.4]). Generally speaking, the information from the A_∞-algebra $\mathrm{Ext}^*_A(k_A, k_A)$ is sufficient to recover A. This is the main point of the following theorem, which serves as another bridge for the classification of AS-regular algebras by A_∞-methods.

Theorem 2.4. *(See [LPWZ3, Corollary B].) Let A be a connected graded algebra which is finitely generated in degree* 1, *and let E be the corresponding A_∞-algebra* $\mathrm{Ext}^*_A(k_A, k_A)$. *Let $R = \bigoplus_{n\geqslant 2} R_n$ be the minimal graded space of relations of A such that $R_n \subset A_1 \otimes A_{n-1} \subset A_1^{\otimes n}$. Let $i : R_n \to A_1^{\otimes n}$ be the inclusion map and i^* be its k-linear dual. Then the multiplication m_n of E restricted to $(E^1)^{\otimes n}$ is equal to the map*

$$i^* : \left(E^1\right)^{\otimes n} = \left(A_1^*\right)^{\otimes n} \to R_n^* \subset E^2.$$

Keller stated the result for quiver algebras kQ/I where Q is a finite quiver and I is an admissible ideal of kQ [Ke, Proposition 2]. Theorem 2.4, giving an explicit correspondence between the minimal graded space of relations of A and the A_∞-multiplications of the Ext-algebra $\operatorname{Ext}_A^*(k_A, k_A)$, works for graded algebras generated in degree 1.

Let us give an example to illustrate this.

Example 1. Let A be a 3-dimensional AS-regular algebra of Type A in Artin–Schelter's classification [AS], i.e.,

$$A = k\langle x, y\rangle \Big/ \begin{pmatrix} x^3 + axy^2 + ay^2x + byxy \\ y^3 + ayx^2 + ax^2y + bxyx \end{pmatrix}$$

with $a, b \in k \setminus \{0\}$. Then minimal projective resolution of the trivial module k_A has the following form

$$0 \to A(-4) \to A(-3)^{\oplus 2} \to A(-1)^{\oplus 2} \to A \to k_A \to 0,$$

and the Yoneda Ext-algebra $E = \operatorname{Ext}_A^*(k_A, k_A) = k \oplus E_{-1}^1 \oplus E_{-3}^2 \oplus E_{-4}^3$ as a $\mathbb{Z}^2$-graded vector space, where the lower index is the Adams grading and the upper index is the homological grading. The dimensions of the subspaces are

$$\dim E_{-1}^1 = \dim E_{-3}^2 = 2, \qquad \dim E_{-4}^3 = 1.$$

By choosing the basis suitably, let $E_{-1}^1 = k\alpha_1 \oplus k\alpha_2$, $E_{-3}^2 = k\beta_1 \oplus k\beta_2$, and $E_{-4}^3 = k\delta$. Then the A_∞-multiplication m_3 on $(E^1)^{\otimes 3}$ is

$$\begin{aligned}
&m_3(\alpha_1 \otimes \alpha_1 \otimes \alpha_1) = \beta_1, && m_3(\alpha_1 \otimes \alpha_1 \otimes \alpha_2) = a\beta_2,\\
&m_3(\alpha_1 \otimes \alpha_2 \otimes \alpha_1) = b\beta_2, && m_3(\alpha_1 \otimes \alpha_2 \otimes \alpha_2) = a\beta_1,\\
&m_3(\alpha_2 \otimes \alpha_1 \otimes \alpha_1) = a\beta_2, && m_3(\alpha_2 \otimes \alpha_1 \otimes \alpha_2) = b\beta_1,\\
&m_3(\alpha_2 \otimes \alpha_2 \otimes \alpha_1) = a\beta_1, && m_3(\alpha_2 \otimes \alpha_2 \otimes \alpha_2) = \beta_2.
\end{aligned}$$

The following is also needed later in the classification.

Theorem 2.5. *(See [Ke, Proposition 1].) As an A_∞-algebra, $E = E(A)$ can be generated by E^0 and E^1, i.e., E itself is the smallest k-subspace of E which is closed under the A_∞-multiplications m_n's containing E^0 and E^1.*

The process of recovering the algebra from its Ext-algebra is the main idea used in [LPWZ2] to classify a type of 4-dimensional AS-regular algebras. This is also the idea in this paper. We analyze the A_∞-structures of the Ext-algebras, then we recover the original algebras and check the homological properties.

3. A_∞-structural analysis on $E(A)$

In this paper we focus on the 5-dimensional AS-regular algebras which are generated by two elements with three relations of degree 4. We classify the algebras of this type under a generic condition. Following [FV] (see Theorem 1.1), the proof of the following proposition is an easy exercise.

Proposition 3.1. *Let A be a 5-dimensional AS-regular algebra which is generated by two elements of degree 1 with three generating relations of degree 4. Then the minimal resolution of the trivial module k_A is of the following form*:

$$0 \to A(-10) \to A(-9)^{\oplus 2} \to A(-6)^{\oplus 3} \to A(-4)^{\oplus 3} \to A(-1)^{\oplus 2} \to A \to k_A \to 0,$$

and the Hilbert series of A is $(1-t)^{-2}(1-t^2)^{-1}(1-t^3)^{-2}$. The Yoneda Ext*-algebra E of A is isomorphic to*

$$k \oplus E^1_{-1} \oplus E^2_{-4} \oplus E^3_{-6} \oplus E^4_{-9} \oplus E^5_{-10}$$

as a $\mathbb{Z}^2$-graded vector space, where the lower index is the Adams grading inherited from the grading of A and the upper index is the homological grading of the Ext*-group. The dimensions of the subspaces are*

$$\dim E^1_{-1} = \dim E^4_{-9} = 2, \qquad \dim E^2_{-4} = \dim E^3_{-6} = 3, \qquad \dim E^5_{-10} = 1.$$

With the canonical A_∞-algebra structure, $E = (E, m_2, m_3, m_4)$, that is, $m_n = 0$ for all $n \geqslant 5$.

3.1. Frobenius algebra structures on E

Now we start to classify all possible Frobenius algebra structures on the bigraded space

$$E = k \oplus E^1_{-1} \oplus E^2_{-4} \oplus E^3_{-6} \oplus E^4_{-9} \oplus E^5_{-10}$$

with $\dim E^1_{-1} = \dim E^4_{-9} = 2$, $\dim E^2_{-4} = \dim E^3_{-6} = 3$, $\dim E^5_{-10} = 1$. All possible non-trivial actions of the higher multiplications m_n are listed as follows.

The possible non-trivial actions of m_2 on $E^{\otimes 2}$ are

$$E^1_{-1} \otimes E^4_{-9} \to E^5_{-10}, \qquad E^4_{-9} \otimes E^1_{-1} \to E^5_{-10};$$
$$E^2_{-4} \otimes E^3_{-6} \to E^5_{-10}, \qquad E^3_{-6} \otimes E^2_{-4} \to E^5_{-10}.$$

The multiplication m_2 gives a Frobenius structure on E if and only if that there exists a basis $\{\alpha_1, \alpha_2\}$ of E^1_{-1}, a basis $\{\beta_1, \beta_2, \beta_3\}$ of E^2_{-4}, a basis $\{\eta_1, \eta_2, \eta_3\}$ of E^3_{-6}, a basis $\{\gamma_1, \gamma_2\}$ of E^4_{-9} and a basis $\{\delta\}$ of E^5_{-10} such that

$$\begin{aligned} &\alpha_i \gamma_j = \delta_{ij}\delta, \qquad \gamma_i \alpha_j = r_{ij}\delta, \quad r_{ij} \in k; \\ &\beta_k \eta_s = \delta_{ks}\delta, \qquad \eta_k \beta_s = t_{ks}\delta, \quad t_{ks} \in k, \end{aligned} \tag{3.1}$$

with the matrices $\mathcal{R} = (r_{ij})_{2\times 2}$ and $\mathcal{T} = (t_{ks})_{3\times 3}$ non-singular.

3.2. Non-trivial actions of m_3 and m_4 on E

Possible non-trivial actions of m_3 on $E^{\otimes 3}$ are

$$E^1_{-1} \otimes E^1_{-1} \otimes E^2_{-4} \to E^3_{-6}, \qquad E^1_{-1} \otimes E^2_{-4} \otimes E^1_{-1} \to E^3_{-6}, \qquad E^2_{-4} \otimes E^1_{-1} \otimes E^1_{-1} \to E^3_{-6};$$
$$E^1_{-1} \otimes E^2_{-4} \otimes E^2_{-4} \to E^4_{-9}, \qquad E^2_{-4} \otimes E^1_{-1} \otimes E^2_{-4} \to E^4_{-9}, \qquad E^2_{-4} \otimes E^2_{-4} \otimes E^1_{-1} \to E^4_{-9}.$$

Now, for $1 \leqslant i, j \leqslant 2$ and $1 \leqslant k, s \leqslant 3$, we assume that

$$m_3(\alpha_i, \alpha_j, \beta_k) = a_{13ijk}\eta_1 + a_{23ijk}\eta_2 + a_{33ijk}\eta_3,$$
$$m_3(\alpha_i, \beta_k, \alpha_j) = a_{12ijk}\eta_1 + a_{22ijk}\eta_2 + a_{32ijk}\eta_3,$$

$$m_3(\beta_k, \alpha_i, \alpha_j) = a_{11ijk}\eta_1 + a_{21ijk}\eta_2 + a_{31ijk}\eta_3,$$
$$m_3(\alpha_i, \beta_k, \beta_s) = b_{11iks}\gamma_1 + b_{21iks}\gamma_2,$$
$$m_3(\beta_k, \alpha_i, \beta_s) = b_{12iks}\gamma_1 + b_{22iks}\gamma_2,$$
$$m_3(\beta_k, \beta_s, \alpha_i) = b_{13iks}\gamma_1 + b_{23iks}\gamma_2,$$

where the coefficients are scalars in k.

Possible non-trivial actions of m_4 on $E^{\otimes 4}$ are

$$\left(E^1_{-1}\right)^{\otimes 4} \to E^2_{-4},$$
$$\left(E^1_{-1}\right)^{\otimes 3} \otimes E^3_{-6} \to E^4_{-9}, \qquad \left(E^1_{-1}\right)^{\otimes 2} \otimes E^3_{-6} \otimes E^1_{-1} \to E^4_{-9},$$
$$E^3_{-6} \otimes \left(E^1_{-1}\right)^{\otimes 3} \to E^4_{-9}, \qquad E^1_{-1} \otimes E^3_{-6} \otimes \left(E^1_{-1}\right)^{\otimes 2} \to E^4_{-9}.$$

Then, for $1 \leqslant i, j, h, m \leqslant 2$ and $1 \leqslant s \leqslant 3$, we assume that

$$m_4(\alpha_i, \alpha_j, \alpha_h, \alpha_m) = x_{1ijhm}\beta_1 + x_{2ijhm}\beta_2 + x_{3ijhm}\beta_3,$$
$$m_4(\alpha_i, \alpha_j, \alpha_h, \eta_s) = y_{14ijhs}\gamma_1 + y_{24ijhs}\gamma_2,$$
$$m_4(\alpha_i, \alpha_j, \eta_s, \alpha_h) = y_{13ijhs}\gamma_1 + y_{23ijhs}\gamma_2,$$
$$m_4(\eta_s, \alpha_i, \alpha_j, \alpha_h) = y_{11ijhs}\gamma_1 + y_{21ijhs}\gamma_2,$$
$$m_4(\alpha_i, \eta_s, \alpha_j, \alpha_h) = y_{12ijhs}\gamma_1 + y_{22ijhs}\gamma_2,$$

where all the coefficients are scalars in k.

3.3. Stasheff identities for the A_∞-algebra E

We assume first that the structure matrices $\mathcal{R}$ and $\mathcal{T}$ given in (3.1) are diagonal for simplicity, and let

$$\mathcal{R} = \begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix} \quad \text{and} \quad \mathcal{T} = \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix}.$$

It is easy to see that SI(n) holds trivially for $n = 1, 2, 3$ and $n \geqslant 7$. Now we look at SI(n) for $n = 4, 5$ and 6.

SI(4) is equivalent to

$$m_3\left(m_2 \otimes \mathrm{id}^{\otimes 2} - \mathrm{id} \otimes m_2 \otimes \mathrm{id} + \mathrm{id}^{\otimes 2} \otimes m_2\right) = m_2(m_3 \otimes \mathrm{id} + \mathrm{id} \otimes m_3).$$

By applying to elements, it is easy to see that if one of the components is in $E^0 = k$ then the formula holds trivially. If no component is in $E^0 = k$, then the action of the left-hand side of the above formula is always zero. The possible non-trivial actions of the right-hand side of the above formula are on

$$E^1_{-1} \otimes E^1_{-1} \otimes E^2_{-4} \otimes E^2_{-4}, \quad E^1_{-1} \otimes E^2_{-4} \otimes E^1_{-1} \otimes E^2_{-4}, \quad E^1_{-1} \otimes E^2_{-4} \otimes E^2_{-4} \otimes E^1_{-1},$$
$$E^2_{-4} \otimes E^1_{-1} \otimes E^1_{-1} \otimes E^2_{-4}, \quad E^2_{-4} \otimes E^1_{-1} \otimes E^2_{-4} \otimes E^1_{-1}, \quad E^2_{-4} \otimes E^2_{-4} \otimes E^1_{-1} \otimes E^1_{-1}.$$

By applying SI(4) to $(\alpha_i, \alpha_j, \beta_k, \beta_c)$, $(\alpha_i, \beta_k, \alpha_j, \beta_c)$, $(\alpha_i, \beta_k, \beta_c, \alpha_j)$, $(\beta_k, \alpha_i, \alpha_j, \beta_c)$, $(\beta_k, \alpha_i, \beta_c, \alpha_j)$ and $(\beta_k, \beta_c, \alpha_i, \alpha_j)$, it follows that SI(4) holds if and only if

$$\begin{cases} b_{i1jkc} = a_{c3ijk}t_c, & b_{i2jkc} = a_{c2ijk}t_c, \\ b_{i3jkc} = g_j b_{j1ikc}, & a_{k3ijc} = -a_{c1ijk}t_c, \\ a_{k2ijc} = -g_j b_{j2ikc}, & a_{k1ijc} = -g_j b_{j3ikc}. \end{cases} \tag{3.2}$$

It follows from (3.2) that for any $i, j \in \{1,2\}$ and $k, c \in \{1,2,3\}$ by eliminating the b's,

$$\begin{cases} a_{k1ijc} + g_i g_j t_c a_{c3ijk} = 0, \\ a_{k1ijc}t_k + a_{c3ijk} = 0, \\ g_j t_c a_{c2jik} + a_{k2ijc} = 0. \end{cases} \tag{3.3}$$

SI(5) is equivalent to

$$\begin{aligned} & m_3\big(m_3 \otimes \mathrm{id}^{\otimes 2} + \mathrm{id} \otimes m_3 \otimes \mathrm{id} + \mathrm{id}^{\otimes 2} \otimes m_3\big) \\ & \quad = m_2(\mathrm{id} \otimes m_4 - m_4 \otimes \mathrm{id}) + m_4\big(\mathrm{id}^{\otimes 3} \otimes m_2 - \mathrm{id}^{\otimes 2} \otimes m_2 \otimes \mathrm{id} + \mathrm{id} \otimes m_2 \otimes \mathrm{id}^{\otimes 2} - m_2 \otimes \mathrm{id}^{\otimes 3}\big). \end{aligned}$$

The left-hand side of the above formula is always zero. If one of the components is in $E^0 = k$, the formula holds trivially by applying it to elements. The possible non-trivial actions of the right-hand side of the above formula are $m_2(\mathrm{id} \otimes m_4 - m_4 \otimes \mathrm{id})$ acting on

$$\begin{gathered} \big(E^1_{-1}\big)^{\otimes 4} \otimes E^3_{-6}, \quad \big(E^1_{-1}\big)^{\otimes 3} \otimes E^3_{-6} \otimes E^1_{-1}, \quad \big(E^1_{-1}\big)^{\otimes 2} \otimes E^3_{-6} \otimes \big(E^1_{-1}\big)^{\otimes 2}, \\ E^1_{-1} \otimes E^3_{-6} \otimes \big(E^1_{-1}\big)^{\otimes 3}, \quad E^3_{-6} \otimes \big(E^1_{-1}\big)^{\otimes 4}. \end{gathered}$$

By applying $m_2(\mathrm{id} \otimes m_4 - m_4 \otimes \mathrm{id})$ to $(\alpha_i, \alpha_j, \alpha_h, \alpha_m, \eta_s)$, $(\alpha_i, \alpha_j, \alpha_h, \eta_s, \alpha_m)$, $(\alpha_i, \alpha_j, \eta_s, \alpha_h, \alpha_m)$, $(\alpha_i, \eta_s, \alpha_j, \alpha_h, \alpha_m)$ and $(\eta_s, \alpha_i, \alpha_j, \alpha_h, \alpha_m)$, it follows that SI(5) holds if and only if, for any $i, j, h, m \in \{1,2\}$ and $s \in \{1,2,3\}$,

$$\begin{gathered} x_{sijhm} = y_{i4jhms}, \qquad g_m y_{m4ijhs} = y_{i3jhms}, \qquad g_m y_{m3ijhs} = y_{i2jhms}, \\ g_m y_{m2ijhs} = y_{i1jhms}, \qquad g_m y_{m1ijhs} = t_s x_{sijhm}. \end{gathered} \tag{3.4}$$

It follows that for any $i, j, h, m \in \{1,2\}$ and $s \in \{1,2,3\}$

$$x_{sijhm}(t_s - g_i g_j g_h g_m) = 0. \tag{3.5}$$

By Theorem 2.4, for any fixed $s \in \{1,2,3\}$, there exist some i, j, h and m such that

$$t_s = g_i g_j g_h g_m. \tag{3.6}$$

SI(6) is equivalent to

$$\begin{aligned} & m_4\big(m_3 \otimes \mathrm{id}^{\otimes 3} + \mathrm{id} \otimes m_3 \otimes \mathrm{id}^{\otimes 2} + \mathrm{id}^{\otimes 2} \otimes m_3 \otimes \mathrm{id} + \mathrm{id}^{\otimes 3} \otimes m_3\big) \\ & \quad = m_3\big(m_4 \otimes \mathrm{id}^{\otimes 2} - \mathrm{id} \otimes m_4 \otimes \mathrm{id} + \mathrm{id}^{\otimes 2} \otimes m_4\big). \end{aligned}$$

The possible non-trivial actions of the above formula are on

$$\left(E^1_{-1}\right)^{\otimes 6},\quad \left(E^1_{-1}\right)^{\otimes 5}\otimes E^2_{-4},\quad \left(E^1_{-1}\right)^{\otimes 4}\otimes E^2_{-4}\otimes E^1_{-1},\quad \left(E^1_{-1}\right)^{\otimes 3}\otimes E^2_{-4}\otimes \left(E^1_{-1}\right)^{\otimes 2},$$
$$\left(E^1_{-1}\right)^{\otimes 2}\otimes E^2_{-4}\otimes \left(E^1_{-1}\right)^{\otimes 3},\quad E^1_{-1}\otimes E^2_{-4}\otimes \left(E^1_{-1}\right)^{\otimes 4},\quad E^2_{-4}\otimes \left(E^1_{-1}\right)^{\otimes 5}.$$

By applying SI(6) to $(\alpha_i,\alpha_j,\alpha_h,\alpha_m,\alpha_n,\alpha_l)$, $(\alpha_i,\alpha_j,\alpha_h,\alpha_m,\alpha_n,\beta_k)$, $(\alpha_i,\alpha_j,\alpha_h,\alpha_m,\beta_k,\alpha_n)$, $(\alpha_i,\alpha_j,\alpha_h,\beta_k,\alpha_m,\alpha_n)$, $(\alpha_i,\alpha_j,\beta_k,\alpha_h,\alpha_m,\alpha_n)$, $(\alpha_i,\beta_k,\alpha_j,\alpha_h,\alpha_m,\alpha_n)$, $(\beta_k,\alpha_i,\alpha_j,\alpha_h,\alpha_m,\alpha_n)$, it follows that SI(6) holds if and only if

$$\begin{aligned}
&\sum_{s=1}^3 x_{sijhm}a_{c1nls}-\sum_{s=1}^3 x_{sjhmn}a_{c2ils}+\sum_{s=1}^3 x_{shmnl}a_{c3ijs}=0,\\
&\sum_{s=1}^3 a_{s3mnk}y_{l4ijhs}+\sum_{s=1}^3 x_{sijhm}b_{l2nsk}-\sum_{s=1}^3 x_{sjhmn}b_{l1isk}=0,\\
&\sum_{s=1}^3 a_{s3hmk}y_{l3ijns}-\sum_{s=1}^3 a_{s2mnk}y_{l4ijhs}-\sum_{s=1}^3 x_{sijhm}b_{l3nsk}=0,\\
&\sum_{s=1}^3 a_{s3jhk}y_{l2imns}-\sum_{s=1}^3 a_{s2hmk}y_{l3ijns}+\sum_{s=1}^3 a_{s1mnk}y_{l4ijhs}=0,\\
&\sum_{s=1}^3 a_{s3ijk}y_{l1hmns}-\sum_{s=1}^3 a_{s2jhk}y_{l2imns}+\sum_{s=1}^3 a_{s1hmk}y_{l3ijns}=0,\\
&\sum_{s=1}^3 a_{s2ijk}y_{l1hmns}-\sum_{s=1}^3 a_{s1jhk}y_{l2imns}-\sum_{s=1}^3 x_{sjhmn}b_{l1iks}=0,\\
&\sum_{s=1}^3 a_{s1ijk}y_{l1hmns}+\sum_{s=1}^3 x_{sijhm}b_{l3nks}-\sum_{s=1}^3 x_{sjhmn}b_{l2iks}=0,
\end{aligned}\tag{3.7}$$

where $i,j,h,m,n,l\in\{1,2\}$ and $k,c\in\{1,2,3\}$.

Using (3.2), (3.3) and (3.4), plugging b_{lcnsk}, a_{c1nls} and y_{lcijns} in (3.7), we obtain a system of equations with respect to a_{c2nls}, a_{c3nls} and x_{sjhmn} as in the following:

$$\begin{aligned}
&\sum_{s=1}^3 a_{s3nlc}g_ng_lt_sx_{sijhm}+\sum_{s=1}^3 a_{c2ils}x_{sjhmn}-\sum_{s=1}^3 a_{c3ijs}x_{shmnl}=0,\\
&\sum_{s=1}^3 a_{s3mnk}x_{slijh}+\sum_{s=1}^3 a_{k2lns}t_kx_{sijhm}-\sum_{s=1}^3 a_{k3lis}t_kx_{sjhmn}=0,\\
&\sum_{s=1}^3 a_{s3hmk}g_nx_{snlij}-\sum_{s=1}^3 a_{s2mnk}x_{slijh}-\sum_{s=1}^3 a_{k3nls}g_nt_kx_{sijhm}=0,\\
&\sum_{s=1}^3 a_{s3jhk}g_mx_{smnli}-\sum_{s=1}^3 a_{s2hmk}x_{snlij}-\sum_{s=1}^3 a_{k3mns}g_mt_kx_{slijh}=0,
\end{aligned}$$

$$\sum_{s=1}^{3} a_{s3ijk} g_h x_{shmnl} - \sum_{s=1}^{3} a_{s2jhk} x_{smnli} - \sum_{s=1}^{3} a_{k3hms} g_h t_k x_{snlij} = 0,$$

$$\sum_{s=1}^{3} a_{s2ijk} g_m g_h g_n x_{shmnl} + \sum_{s=1}^{3} a_{k3jhs} g_h g_j g_m g_n t_k x_{smnli} - \sum_{s=1}^{3} a_{s3lik} t_s x_{sjhmn} = 0,$$

$$\sum_{s=1}^{3} a_{k3ijs} g_h g_i g_j g_m g_n t_k x_{shmnl} - \sum_{s=1}^{3} a_{s3nlk} g_n t_s x_{sijhm} + \sum_{s=1}^{3} a_{s2lik} t_s x_{sjhmn} = 0 \tag{3.8}$$

where $i, j, h, m, n, l \in \{1, 2\}$ and $k, c \in \{1, 2, 3\}$.

In fact, the seven families of equations in (3.8) is just equivalent to one family by (3.3) and (3.5), say the first one:

$$\sum_{s=1}^{3} a_{s3nlc} g_n g_l t_s x_{sijhm} + \sum_{s=1}^{3} a_{c2ils} x_{sjhmn} - \sum_{s=1}^{3} a_{c3ijs} x_{shmnl} = 0 \tag{3.9}$$

where $i, j, h, m, n, l \in \{1, 2\}$ and $c \in \{1, 2, 3\}$.

4. Classifications

4.1. A generic condition on the algebra structure of E

Let

$$\mathcal{R} = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \quad \text{and} \quad \mathcal{T} = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix}$$

be as given in (3.1) and let g_1, g_2 and t_1, t_2, t_3 be the eigenvalues of $\mathcal{R}$ and $\mathcal{T}$, respectively.

Lemma 4.1. *Let E be the Yoneda* Ext-*algebra of A as considered, $\mathcal{R}$ and $\mathcal{T}$ be the structure matrices as given in* (3.1)*. If $\mathcal{R}$ is diagonal, then so is $\mathcal{T}$.*

Proof. Let $\{f_1, f_2, f_3\}$ be a minimal generating relation of A. We may assume that, for any $1 \leqslant l \leqslant 3$, there exists a monomial in x and y appearing only in f_l (that is, its coefficient is non-zero). Let

$$\mathcal{R} = \begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix}.$$

The first four identities in (3.4) still hold. By applying SI(5) to $E^3_{-6} \otimes (E^1_{-1})^{\otimes 4}$ (see (3.4)), we get $\sum_{c=1}^{3} t_{sc} x_{cijhm} = g_m y_{m1ijhs}$. It follows that $\sum_{c=1}^{3} t_{sc} x_{cijhm} = g_m g_h g_i g_j x_{sijhm}$, that is,

$$(t_{ss} - g_m g_h g_i g_j) x_{sijhm} = \sum_{c \neq s} t_{sc} x_{cijhm}. \tag{4.1}$$

Now for any $1 \leqslant l \leqslant 3$, there exist some i, j, h, m such that $x_{cijhm} = 0$ if and only if $c \neq l$, by Theorem 2.4 and the discussion in the first paragraph.

Taking some i, j, h, m so that $x_{1ijhm} \neq 0$ and $x_{2ijhm} = x_{3ijhm} = 0$, it follows from (4.1) for $s = 2$ (respectively, $s = 3$) that $t_{21} x_{1ijhm} = 0$ (respectively, $t_{31} x_{1ijhm} = 0$). Hence $t_{21} = 0$ (respectively, $t_{31} = 0$). Similarly, we have $t_{12} = t_{32} = t_{13} = t_{23} = 0$. So $\mathcal{T}$ is diagonal. □

We introduce a generic condition (GM2) for m_2, which is suggested by (3.6).

$$\left(g_1 g_2^{-1}\right)^i \neq 1 \quad \text{for } 1 \leqslant i \leqslant 4 \quad \text{and} \quad t_s \neq t_j \quad \text{for } 1 \leqslant s \neq j \leqslant 3. \tag{GM2}$$

From now on, we assume that the algebra structure on E satisfies the condition (GM2). Then, without loss of generality, we may assume

$$\mathcal{R} = \begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix} \quad \text{and} \quad \mathcal{T} = \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix}.$$

If E is the Yoneda Ext-algebra of some domain A, then (GM2) implies that $t_s \neq g_i^4$ for any i and s. Again, by (GM2), without loss of generality, we may assume that

$$t_1 = g_1^3 g_2, \qquad t_2 = g_1^2 g_2^2, \qquad t_3 = g_1 g_2^3. \tag{4.2}$$

By (3.5), (4.2) and (GM2), all other x_{sijhm}'s are zero except

$$x_{11112}, \quad x_{11121}, \quad x_{11211}, \quad x_{12111};$$
$$x_{21122}, \quad x_{21212}, \quad x_{22121}, \quad x_{22211}, \quad x_{21221}, \quad x_{22112};$$
$$x_{31222}, \quad x_{32122}, \quad x_{32212}, \quad x_{32221}.$$

For convenience, let

$$x_{11112} = a, \qquad x_{11121} = p, \qquad x_{11211} = q, \qquad x_{12111} = r;$$
$$x_{21122} = l_1, \qquad x_{21212} = l_2, \qquad x_{22121} = l_3, \qquad x_{22211} = l_4, \qquad x_{21221} = l_5, \qquad x_{22112} = l_6;$$
$$x_{31222} = b, \qquad x_{32122} = d, \qquad x_{32212} = u, \qquad x_{32221} = v \tag{4.3}$$

with $a, b, p, q, r, d, u, v, l_1, l_2, l_3, l_4, l_5, l_6 \in k$.

So, by Theorem 2.4, the possible AS-regular algebras are of the form $A = k\langle x, y\rangle/(f_1, f_2, f_3)$, with the generating relations f_1, f_2 and f_3 as in the following:

$$\begin{aligned} f_1 &= ax^3y + px^2yx + qxyx^2 + ryx^3, \\ f_2 &= l_1x^2y^2 + l_2xyxy + l_3yxyx + l_4y^2x^2 + l_5xy^2x + l_6yx^2y, \\ f_3 &= bxy^3 + dyxy^2 + uy^2xy + vy^3x. \end{aligned} \tag{4.4}$$

If A is a domain, then $ab \neq 0$, $vr \neq 0$ and none of (l_1, l_2, l_5), (l_1, l_2, l_6), (l_3, l_4, l_5) and (l_3, l_4, l_6) equals $(0, 0, 0)$. We may assume that $a = b = 1$.

4.2. Classification under the generic condition (GM2)

Proposition 4.2. *Suppose that E is the Yoneda* Ext*-algebra of some AS-regular algebra considered, satisfying the generic condition* (GM2)*. Then, for some* $2 \leqslant n \leqslant 8$,

$$g_1^n g_2^{10-n} = 1.$$

Proof. By Theorem 2.5, the Yoneda Ext-algebra E should be A_∞-generated by E^0 and E^1. It follows that m_3 is non-trivial. So, not all the parameters a_{scijk} and b_{lciks} are zero. By (3.2), not all the parameters a_{s2ijk} and a_{s3ijk} are zero for $i, j \in \{1,2\}$ and $s, k \in \{1,2,3\}$.

If all the a_{s3ijk}'s are zero for $i, j \in \{1,2\}$ and $s, k \in \{1,2,3\}$, then there exists $a_{c2mnh} \neq 0$ for some $m, n \in \{1,2\}$ and $c, h \in \{1,2,3\}$. It follows from (3.3) that

$$0 = a_{c2mnh} + g_n t_h a_{h2nmc} = a_{c2mnh} + g_n t_h(-g_m t_c a_{c2mnh}) = (1 - g_n g_m t_h t_c) a_{c2mnh}.$$

So $1 - g_n g_m t_h t_c = 0$, which implies that $g_n g_m g_1^{8-h-c} g_2^{h+c} = 1$ by (4.2).

If there exists $a_{s3ijk} \neq 0$ for some $i, j \in \{1,2\}$ and $k, s \in \{1,2,3\}$, then the first two equations in (3.3) have non-zero solutions. So

$$1 - g_i g_j t_s t_k = 0.$$

It follows that $g_i g_j g_1^{8-k-s} g_2^{k+s} = 1$ by (4.2).

Both of the two cases imply that there exists an integer n with $2 \leqslant n \leqslant 8$ such that $g_1^n g_2^{10-n} = 1$. □

By Proposition 4.2, there are only four cases need to be considered, i.e.,

$$\text{(i)} \quad g_1^2 g_2^8 = 1, \qquad \text{(ii)} \quad g_1^3 g_2^7 = 1, \qquad \text{(iii)} \quad g_1^4 g_2^6 = 1, \qquad \text{(iv)} \quad g_1^5 g_2^5 = 1.$$

Proposition 4.3. *Except the case* (iv)*, any other case gives no AS-regular algebras.*

Proof. Case (i): By (3.3), $(1 - g_n g_m t_h t_c) a_{c2mnh} = (1 - g_n g_m g_1^{8-h-c} g_2^{h+c}) a_{c2mnh} = 0$. It follows from (GM2) that $g_n g_m g_1^{8-h-c} g_2^{h+c} = 1$ if and only if $h = c = 3$ and $n = m = 2$. So, except a_{32223}, all other a_{c2mnh}'s are zero.

By the first two equations in (3.3), $(1 - g_i g_j t_c t_k) a_{c3ijk} = 0$. So, except a_{33223}, all other a_{c3ijk}'s are zero. Since $a_{c3ijk} = -t_k a_{k1ijc}$, all other a_{k1ijc}'s are zero except a_{31223}.

In summary, except a_{31223}, a_{32223}, a_{33223}, all other a_{csijk}'s are zero.

It follows that $\eta_1, \eta_2 \in E^3_{-6}$ are not contained in $\operatorname{Im} m_3$. So, E can not be A_∞-generated by E^0 and E^1, and E is not an Ext-algebra of some AS-regular algebra.

Case (ii): In this case, $g_n g_m g_1^{8-h-c} g_2^{h+c} = 1$ if and only if that $h = c = n + m = 3$ or $h + c = 5$, $n = m = 2$ by (GM2). It follows that except

$$a_{32213}, \quad a_{32123}, \quad a_{32222}, \quad a_{22223},$$

all other a_{c2mnh}'s are zero. In particular, a_{12mnh}'s are zero.

Similarly, by (GM2) and $(1 - g_i g_j t_c t_k) a_{c3ijk} = 0$, except

$$a_{33213}, \quad a_{33123}, \quad a_{33222}, \quad a_{23223},$$

all other a_{c3ijk}'s are zero. In particular, all a_{13ijk}'s are zero. Since $a_{c3ijk} = -t_k a_{k1ijc}$, all a_{11ijc}'s are zero.

So, $\eta_1 \in E^3_{-6}$ is not contained in $\operatorname{Im} m_3$ and E is not A_∞-generated by E^0 and E^1. In this case, E is not an Ext-algebra of some AS-regular algebra either.

Case (iii): By (GM2) and $(1 - g_n g_m g_1^{8-h-c} g_2^{h+c}) a_{c2mnh} = 0$, except

$$a_{32113}, \quad a_{22123}, \quad a_{22213}, \quad a_{32122}, \quad a_{32212}, \quad a_{12223}, \quad a_{32221}, \quad a_{22222},$$

all other a_{c2mnh}'s are zero. In particular, except a_{12223} all a_{12mnh}'s are zero.

Similarly, by (GM2) and $(1-g_ig_jt_ct_k)a_{c3ijk}=0$, except

$$a_{33113},\quad a_{23123},\quad a_{23213},\quad a_{33122},\quad a_{33212},\quad a_{13223},\quad a_{33221},\quad a_{23222},$$

all other a_{c3ijk}'s are zero. In particular, except a_{13223} all a_{13ijk}'s are zero.

Since $a_{c3ijk}=-t_ka_{k1ijc}$, except a_{11223} all other a_{11ijc}'s are zero.

Let $l=k=1$ and $i=j=h=m=n=2$ in the second, third and fourth equations of (3.8), we get the following equations:

$$\begin{cases} a_{33221}x_{31222}=0, \\ a_{33221}x_{32122}g_2=a_{32221}x_{31222}, \\ a_{33221}x_{32212}g_2=a_{32221}x_{32122}+a_{13223}x_{31222}g_2t_1. \end{cases}$$

It follows from $x_{31222}=1$ that

$$a_{33221}=a_{32221}=a_{13223}=0.$$

Since $a_{33221}=-t_1a_{11223}$ and $a_{12223}=-g_2t_3a_{32221}$, $a_{11223}=a_{12223}=a_{13223}=0$. Hence all a_{1sijk}'s are zero.

So, in this case, $\eta_1\in E^3_{-6}$ is also not contained in $\operatorname{Im} m_3$ and E is not A_∞-generated by E^0 and E^1. Hence E is not an Ext-algebra of some AS-regular algebra. □

The only interesting case left is the case (iv) $g_1^5g_2^5=1$, which will be discussed in the next subsection.

4.3. Case $g_1^5g_2^5=1$

Using the third equation in (3.3) for a_{k2ijc}, we have $(1-g_ig_jt_ct_k)a_{k2ijc}=0$ with $i,j\in\{1,2\}$ and $k,c\in\{1,2,3\}$. Then we get the following equations:

$$a_{22113}=-g_1^2g_2^3a_{32112},\qquad a_{32121}=-g_1^3g_2^2a_{12213},\qquad a_{22122}=-g_1^2g_2^3a_{22212},$$
$$a_{12123}=-g_1g_2^4a_{32211},\qquad a_{12222}=-g_1^2g_2^3a_{22221},$$

and all other a_{k2ijc}'s are zero.

Solving the first and second equations in (3.3) for a_{k3ijc} with $i,j\in\{1,2\}$ and $k,c\in\{1,2,3\}$, we get all a_{k3ijc}'s are zero except

$$a_{23113},\quad a_{33112},\quad a_{33121},\quad a_{13213},\quad a_{23122},\quad a_{23212},\quad a_{13123},\quad a_{33211},\quad a_{13222},\quad a_{23221}.$$

Plugging the x_{sijhm}'s with the parameters as listed in (4.3) in the family of Eqs. (3.9), we get the following 50 equations:

$$g_1^3g_2^2a_{12213}+l_1a_{33112}=0,\qquad pg_1^3g_2^2a_{12213}-g_1^4g_2^2a_{13123}+l_2a_{33112}=0,$$
$$-g_1^4g_2^2a_{13213}-l_1a_{32112}+l_5a_{33112}=0,\qquad -g_1^3g_2^3a_{13222}+g_1^2g_2^3l_1a_{22212}+a_{23113}=0,$$
$$qg_1^3g_2^2a_{12213}-pg_1^4g_2^2a_{13123}+l_6a_{33112}=0,\qquad -pg_1^4g_2^2a_{13213}-l_2a_{32112}+l_3a_{33112}=0,$$
$$-pg_1^3g_2^3a_{13222}+g_1^2g_2^3l_2a_{22212}+da_{23113}=0,\qquad -g_1^4g_2^2l_1a_{23113}-l_5a_{32112}+l_4a_{33112}=0,$$
$$g_1^2g_2^3l_5a_{22212}+ua_{23113}-g_1^3g_2^3l_1a_{23122}=0,\qquad va_{23113}-g_1^3g_2^3l_1a_{23212}+g_1^2g_2^3a_{32112}=0,$$
$$-g_1^2g_2^4l_1a_{23221}+g_1g_2^4a_{32211}=0,\qquad rg_1^3g_2^2a_{12213}-qg_1^4g_2^2a_{13123}+a_{33121}=0,$$

$$\begin{aligned}
&-qg_1^4g_2^2a_{13213}-l_6a_{32112}+pa_{33121}=0, && -qg_1^3g_2^3a_{13222}+g_1^2g_2^3l_6a_{22212}+l_1a_{23122}=0,\\
&-g_1^4g_2^2l_2a_{23113}-l_3a_{32112}+qa_{33121}=0, && g_1^2g_2^3l_3a_{22212}+l_2a_{23122}-g_1^3g_2^3l_2a_{23122}=0,\\
&l_5a_{23122}-g_1^3g_2^3l_2a_{23212}+dg_1^2g_2^3a_{32112}=0, && a_{13123}-g_1^2g_2^4l_2a_{23221}+dg_1g_2^4a_{32211}=0,\\
&-g_1^4g_2^2l_5a_{23113}-l_4a_{32112}+ra_{33121}=0, && g_1^2g_2^3l_4a_{22212}-g_1^3g_2^3l_5a_{23122}+l_6a_{23122}=0,\\
&l_3a_{23122}-g_1^3g_2^3l_5a_{23212}+ug_1^2g_2^3a_{32112}=0, && da_{13123}-g_1^2g_2^4l_5a_{23221}+ug_1g_2^4a_{32211}=0,\\
&l_4a_{23122}+vg_1^2g_2^3a_{32112}-g_1^3g_2^3a_{33112}=0, && ua_{13123}+vg_1g_2^4a_{32211}-g_1^2g_2^4a_{33121}=0,\\
&va_{13123}-g_1^2g_2^4a_{33211}=0, && -rg_1^4g_2^2a_{13123}+a_{33211}=0,\\
&-rg_1^4g_2^2a_{13213}-a_{32211}+pa_{33211}=0, && -rg_1^3g_2^3a_{13222}-a_{22221}+l_1a_{23212}=0,\\
&-g_1^4g_2^2l_6a_{23113}-pa_{32211}+qa_{33211}=0, && -pa_{22221}-g_1^3g_2^3l_6a_{23122}+l_2a_{23212}=0,\\
&-l_1a_{22212}+l_5a_{23212}-g_1^3g_2^3l_6a_{23212}=0, && a_{13213}+g_1^2g_2^3l_1a_{22221}-g_1^2g_2^4l_6a_{23221}=0,\\
&-g_1^4g_2^2l_3a_{23113}-qa_{32211}+ra_{33211}=0, && -qa_{22221}-g_1^3g_2^3l_3a_{23122}+l_6a_{23212}=0,\\
&-l_2a_{22212}+l_3a_{23212}-g_1^3g_2^3l_3a_{23212}=0, && da_{13213}+g_1^2g_2^3l_2a_{22221}-g_1^2g_2^4l_3a_{23221}=0,\\
&-l_5a_{22212}+l_4a_{23212}-dg_1^3g_2^3a_{33112}=0, && ua_{13213}+g_1^2g_2^3l_5a_{22221}-dg_1^2g_2^4a_{33121}=0,\\
&-a_{12213}+va_{13213}-dg_1^2g_2^4a_{33211}=0, && -g_1^4g_2^2l_4a_{23113}-ra_{32211}=0,\\
&-ra_{22221}-g_1^3g_2^3l_4a_{23122}+a_{23221}=0, && -l_6a_{22212}-g_1^3g_2^3l_4a_{23212}+pa_{23221}=0,\\
&l_1a_{13222}+g_1^2g_2^3l_6a_{22221}-g_1^2g_2^4l_4a_{23221}=0, && -l_3a_{22212}+qa_{23221}-ug_1^3g_2^3a_{33112}=0,\\
&l_2a_{13222}+g_1^2g_2^3l_3a_{22221}-ug_1^2g_2^4a_{33121}=0, && -da_{12213}+l_5a_{13222}-ug_1^2g_2^4a_{33211}=0,\\
&-l_4a_{22212}+ra_{23221}-vg_1^3g_2^3a_{33112}=0, && l_6a_{13222}+g_1^2g_2^3l_4a_{22221}-vg_1^2g_2^4a_{33121}=0,\\
&-ua_{12213}+l_3a_{13222}-vg_1^2g_2^4a_{33211}=0, && -va_{12213}+l_4a_{13222}=0.
\end{aligned} \tag{4.5}$$

To find all the possible generating relations, it suffices to find all the solutions of the system of equations (4.5) for $p, q, r, d, u, v, l_1, l_2, l_3, l_4, l_5, l_6$ as defined in (4.3). It follows from the middle two equations in (4.5) that if $a_{13123}\neq 0$ then $v=g_1g_2r$.

Now we start to solve (4.5) in the following four subcases:

- Subcase $a_{13123}\neq 0$, $l_1=0$.
- Subcase $a_{13123}=0$, $l_1=0$.
- Subcase $a_{13123}=0$, $l_1\neq 0$.
- Subcase $a_{13123}\neq 0$, $l_1\neq 0$.

To save the tedious work, we will just list the relations f_1, f_2 and f_3 in the form as in (4.4).

4.4. Subcase $a_{13123}\neq 0, l_1=0$

In this case, the system of equations (4.5) gives only one solution:

$$p=0,\quad q=0,\quad r\neq 0,\quad d=0,\quad u=0,\quad v=r;$$
$$l_1=0,\quad l_2\neq 0,\quad l_3\neq 0,\quad l_4=0,\quad l_5=0,\quad l_6=0$$

with $v^2 + (l_3/l_2)^3 = 0$, which gives the relations

$$f_1 = x^3y - c^3yx^3,$$
$$f_2 = xyxy - c^2yxyx,$$
$$f_3 = xy^3 - c^3y^3x, \quad c \in k \setminus \{0\}.$$

There are four overlap ambiguities $xyxy^3$, x^3yxy, x^3y^3 and $xyxyxy$ if one uses the diamond lemma [Be]. The first three are resolvable. Resolving $xyxyxy$ gives a relation $yxyx^2y = xy^2xyx$. It follows that

$$\big(yxyx^2y\big)y = \big(xy^2xyx\big)y = xy^2(xyxy) = c^2xy^3xyx = c^5y^3x^2yx.$$

Then $y(xyx^2y^2 - c^5y^2x^2yx) = 0$ while $xyx^2y^2 - c^5y^2x^2yx \neq 0$. So the given algebra is not a domain.

4.5. Subcase $a_{13123} = 0$, $l_1 = 0$

In this case, except a_{23122} and a_{23212}, all other a_{c3ijk}'s and all a_{c2ijk}'s are zero by solving (4.5). In particular, all a_{1sijk}'s and a_{3sijk}'s are zero. So, in this case, neither η_1 nor η_3 is contained in $\operatorname{Im} m_3$ and E can not be A_∞-generated by E^0 and E^1. So this case gives no AS-regular algebras.

In fact, if neither a_{23122} nor a_{23212} is zero, then we have $l_4 = 0$, $l_5 = l_6$ and $l_2l_3 = l_5^2$. In this case, $f_2 = l_2xyxy + l_3yxyx + l_5xy^2x + l_5yx^2y = l_2^{-1}(l_2xy + l_5yx)^2$ and this case gives no AS-regular algebras which are domains.

4.6. Subcase $a_{13123} = 0$, $l_1 \neq 0$

Then $a_{33211} = 0$ and we may assume $l_1 = 1$.

If $a_{32211} = 0$, then all the a_{csijk}'s are zero by (4.5) and no desired algebra arises in this sub-subcase.

If $a_{32211} \neq 0$ and $l_2 = 0$, there is one solution

$$p = 0, \quad q = 0, \quad r \neq 0, \quad d = 0, \quad u = 0, \quad v = r;$$
$$l_1 = 1, \quad l_2 = 0, \quad l_3 = 0, \quad l_4 \neq 0, \quad l_5 = 0, \quad l_6 = 0$$

with $r^4 + l_4^3 = 0$, which gives the relations

$$f_1 = xy^3 + ry^3x,$$
$$f_2 = x^2y^2 + ly^2x^2,$$
$$f_3 = x^3y + ryx^3$$

where $r, l \in k \setminus \{0\}$ such that $r^4 + l^3 = 0$. The given algebra is not a domain because

$$y^2\big(r^2yx^2 + lx^2y\big) = x^2y^3 + \big(-x^2y^3\big) = 0.$$

If $a_{32211} \neq 0$ and $l_2 \neq 0$, there is one solution

$$p \neq 0, \quad q = p^2, \quad r = p^3, \quad d = p, \quad u = q, \quad v = r;$$
$$l_1 = 1, \quad l_2 = p, \quad l_3 = p^3, \quad l_4 = p^4, \quad l_5 = p^2, \quad l_6 = p^2$$

which gives the relations:

$$f_1 = x^3y + px^2yx + p^2xyx^2 + p^3yx^3,$$
$$f_2 = x^2y^2 + pxyxy + p^3yxyx + p^4y^2x^2 + p^2xy^2x + p^2yx^2y,$$
$$f_3 = xy^3 + pyxy^2 + p^2y^2xy + p^3y^3x, \quad p \in k \setminus \{0\}.$$

By the diamond lemma [Be], a monomial is irreducible if and only if it does not contain x^3y, x^2y^2 or xy^3 as a sub-word. Such monomials are of the form

$$y^i\left(xy^2\right)^{j_1}(xy)^{k_1}\left(x^2y\right)^{l_1}\cdots\left(xy^2\right)^{j_n}(xy)^{k_n}\left(x^2y\right)^{l_n}x^m,$$

where all the power indices are non-negative integers. It follows that the subalgebra generated by xy^2, xy and x^2y is a free algebra in three variables. So this solution gives an algebra with infinite GK dimension.

In fact, the Hilbert series of the given algebra is $1 + 2t + 4t^2 + 8t^3 + 13t^4 + 22t^5 + 36t^6 + \cdots$, which is different from the standard Hilbert series $1 + 2t + 4t^2 + 8t^3 + 13t^4 + 20t^5 + 31t^6 + \cdots$ of the 5-dimensional AS-regular algebras considered. So, we can also get that the given algebra is not AS-regular.

4.7. Subcase $a_{13123} \neq 0$, $l_1 \neq 0$

Without loss of generality we assume that $l_1 = 1$. As we noted before that if $a_{13123} \neq 0$ then $v = g_1g_2r$. By using the first two equations and the last one in (4.5), we know $l_4 \neq 0$. It follows also from the first two equations in (4.5) that $p \neq l_2$. If further $d = p$, then $a_{22212} = 0$ by using the fourth and seventh equations in (4.5). The discussion in this subcase is divided into the following five sub-subcases:

- Sub-subcase $a_{23212} = 0$.
- Sub-subcase $a_{23212} \neq 0$, $d = p$, $q = 0$.
- Sub-subcase $a_{23212} \neq 0$, $d = p$, $q \neq 0$, $l_2 = 0$.
- Sub-subcase $a_{23212} \neq 0$, $d = p$, $ql_2 \neq 0$.
- Sub-subcase $a_{23212} \neq 0$, $d \neq p$.

4.7.1. Sub-subcase $a_{23212} = 0$

There is one solution

$$p = 0, \quad q = 0, \quad r \neq 0, \quad d = 0, \quad u = 0, \quad v = r;$$
$$l_1 = 1, \quad l_2 \neq 0, \quad l_3 \neq 0, \quad l_4 \neq 0, \quad l_5 = 0, \quad l_6 = 0$$

with $l_3 = -r^4g_2^2l_2$, $l_4g_1 = r^3$ and $l_4g_2^2r^2 = -1$ where $g_1g_2 = 1$. Then $r^5 = -g_1^3$. Let $r = t^3$ for some $t \in k \setminus \{0\}$. Then $r = v = t^3$, $l_3 = -t^2l_2$ and $l_4 = -t^4$. This gives an algebra:

Algebra **A**:

$$f_1 = x^3y + t^3yx^3,$$
$$f_2 = x^2y^2 + l_2xyxy - t^2l_2yxyx - t^4y^2x^2,$$
$$f_3 = xy^3 + t^3y^3x, \quad t, l_2 \in k \setminus \{0\}.$$

By the diamond lemma [Be], we have that $\{y^i(xy^2)^j(xy)^k(x^2y)^lx^m \mid i, j, k, l, m \in \mathbb{N}\}$ is a k-linear basis. Algebra **A** is indeed an AS-regular algebra and enjoys many good homological properties as proved in Theorems 5.2 and 5.4.

4.7.2. Sub-subcase $a_{23212} \neq 0, d = p, q = 0$

There is no solution.

4.7.3. Sub-subcase $a_{23212} \neq 0, d = p, q \neq 0, l_2 = 0$

If $l_3 = 0$, then (4.5) has one solution

$$p \neq 0, \quad q = p^2, \quad r = p^2, \quad d = p, \quad u = q, \quad v = r;$$
$$l_1 = 1, \quad l_2 = 0, \quad l_3 = 0, \quad l_4 \neq 0, \quad l_5 = 0, \quad l_6 = 0$$

with $l_4^2 = p^8$ and $p^5 = -g_1$, which gives two algebras:

Algebra **B**:

$$f_1 = x^3y + px^2yx + p^2xyx^2 + p^3yx^3,$$
$$f_2 = x^2y^2 + p^4y^2x^2,$$
$$f_3 = xy^3 + pyxy^2 + p^2y^2xy + p^3y^3x, \quad p \in k \setminus \{0\};$$

Algebra **C**:

$$f_1 = x^3y + px^2yx + p^2xyx^2 + p^3yx^3,$$
$$f_2 = x^2y^2 - p^4y^2x^2,$$
$$f_3 = xy^3 + pyxy^2 + p^2y^2xy + p^3y^3x, \quad p \in k \setminus \{0\}.$$

By the diamond lemma [Be], $\{y^i(xy^2)^j(xy)^k(x^2y)^lx^m \mid i, j, k, l, m \in \mathbb{N}\}$ is a k-linear basis of algebra **B** and algebra **C**. The algebra **C** has a normal regular element of degree 3, but the algebra **B** does not have any normal element of degree 3. Both algebra **B** and algebra **C** are strongly noetherian, AS-regular, Auslander regular and Cohen–Macaulay (see Theorems 5.2, 5.5 and 5.9).

If $l_3 \neq 0$, then (4.5) has one solution

$$p \neq 0, \quad q \neq 0, \quad r = -p\big(2p^2 + q\big), \quad d = p, \quad u = q, \quad v = r;$$
$$l_1 = 1, \quad l_2 = 0, \quad l_3 = -p\big(p^2 + q\big), \quad l_4 = -q^2, \quad l_5 = q - p^2, \quad l_6 = q - p^2$$

where $p, q \in k \setminus \{0\}$ satisfy $2p^4 - p^2q + q^2 = 0$, which gives an algebra:

Algebra **D**:

$$f_1 = x^3y + px^2yx + qxyx^2 - p\big(2p^2 + q\big)yx^3,$$
$$f_2 = x^2y^2 - p\big(p^2 + q\big)yxyx - q^2y^2x^2 + \big(q - p^2\big)xy^2x + \big(q - p^2\big)yx^2y,$$
$$f_3 = xy^3 + pyxy^2 + qy^2xy - p\big(2p^2 + q\big)y^3x,$$

where $p, q \in k \setminus \{0\}$ satisfy $2p^4 - p^2q + q^2 = 0$.

By the diamond lemma [Be], $\{y^i(xy^2)^j(xy)^k(x^2y)^lx^m \mid i, j, k, l, m \in \mathbb{N}\}$ is a k-linear basis of **D**, and **D** does not have any normal element of degree 3. Algebra **D** is an iterated Ore extension of a polynomial ring, so it is strongly noetherian, AS-regular, Auslander regular and Cohen–Macaulay (see Theorem 5.8).

4.7.4. Sub-subcase $a_{23212} \neq 0$, $d = p$, $ql_2 \neq 0$

By the first, second and fifth equations in (4.5), $q - l_6 = p(p - l_2)$. By the seventh, eighteenth and twenty-second equations in (4.5), $u - l_5 = d(d - l_2)$. It follows from the seventh and eighteenth equations in (4.5) that

$$a_{13123} + g_1^2 g_2^4 (d - l_2) a_{23221} = 0.$$

Since $a_{13123} \neq 0$, $d \neq l_2$. By the fourth and seventh equations in (4.5) and $d = p$, $(p - l_2)a_{22212} = 0$. So $a_{22212} = 0$. Then it is easy to see that $a_{23122} \neq 0$, $g_1 g_2 = 1$ and $l_5 = l_6$. Hence $v = r$ and $u = q$.

If $l_5 = l_6 = 0$, then (4.5) has one solution:

$$p \neq 0, \quad q = p(p - l_2), \quad r = (p - l_2)^3, \quad d = p, \quad u = q, \quad v = r;$$
$$l_1 = 1, \quad l_2 \neq 0, \quad l_3 = -l_2(p - l_2)^2, \quad l_4 = -(p - l_2)^4, \quad l_5 = 0, \quad l_6 = 0$$

with $(p - l_2)^5 = -g_1$, which gives an algebra:

Algebra **E**:

$$f_1 = x^3 y + px^2 yx + ptxyx^2 + t^3 yx^3,$$
$$f_2 = x^2 y^2 + (p - t)xyxy + t^2(t - p)yxyx - t^4 y^2 x^2,$$
$$f_3 = xy^3 + pyxy^2 + pty^2 xy + t^3 y^3 x, \quad p, t \in k \setminus \{0\}, \; p \neq t.$$

Algebra **E** is a normal extension of a 4-dimensional AS-regular algebra (see Theorem 5.9), so **E** is AS-regular of dimension 5, strongly noetherian, Auslander regular and Cohen–Macaulay.

If $l_5 = l_6 \neq 0$, and $q = p^2$ (which is equivalent to that $l_5 = pl_2$ or $a_{22221} = 0$), then (4.5) has one solution:

$$p \neq 0, \quad q = p^2, \quad r = p^3, \quad d = p, \quad u = q, \quad v = r;$$
$$l_1 = 1, \quad l_2 \neq 0, \quad l_3 = p^2 l_2, \quad l_4 = p^4, \quad l_5 = pl_2, \quad l_6 = pl_2,$$

which gives an algebra:

Algebra **F**:

$$f_1 = x^3 y + px^2 yx + p^2 xyx^2 + p^3 yx^3,$$
$$f_2 = x^2 y^2 + l_2 xyxy + l_2 p^2 yxyx + p^4 y^2 x^2 + l_2 pxy^2 x + l_2 pyx^2 y,$$
$$f_3 = xy^3 + pyxy^2 + p^2 y^2 xy + p^3 y^3 x, \quad p, l_2 \in k \setminus \{0\}, \; p \neq l_2.$$

By the diamond lemma [Be], $\{y^i (xy^2)^j (xy)^k (x^2 y)^l x^m \mid i, j, k, l, m \in \mathbb{N}\}$ is a k-linear basis of **F**. Algebra **F** is strongly noetherian, AS-regular, Auslander regular and Cohen–Macaulay (see Theorem 5.5).

If $l_5 = l_6 \neq 0$, $q \neq p^2$, then the solution gives the following:

Algebra **G**:

$$f_1 = x^3 y + px^2 yx + qxyx^2 + ryx^3,$$
$$f_2 = x^2 y^2 + l_2 xyxy + l_3 yxyx + l_4 y^2 x^2 + l_5 xy^2 x + l_5 yx^2 y,$$
$$f_3 = xy^3 + pyxy^2 + qy^2 xy + ry^3 x,$$

where

$$p=-\frac{r^5+qrg^2+g^3}{r^3g},\qquad l_2=\frac{r^2(g-qr)}{g(g+qr)},\qquad l_3=r-\frac{pg(pr-q^2)}{q(qr+g)},$$
$$l_4=-\frac{g^2}{r^2},\qquad l_5=\frac{pr^2+qg}{qr+g},$$

$g\neq 0$, q satisfies the equation $q^3r^8g^3+(r^5+qrg^2+g^3)^3=0$, $q^2r^2\neq g^2$, $r^5+qrg^2+g^3+q^2r^2g\neq 0$ and $r^5+g^3\neq 0$.

Algebra **G** is an iterated Ore extension of a polynomial ring, so it is strongly noetherian, AS-regular, Auslander regular and Cohen–Macaulay (see Theorem 5.8).

4.7.5. Sub-subcase $a_{23212}\neq 0$, $d\neq p$

If $l_2=0$, then it follows from $d\neq p$ that $a_{22212}\neq 0$ and $l_3=0$. Since $a_{23212}\neq 0$, both l_5 and l_6 are not zero.

Suppose $l_2=0$ and $l_5\neq 0$, then $l_6=0$ and (4.5) has one solution

$$p\neq 0,\qquad q=p^2,\qquad r=p^3,\qquad d=ip,\qquad u=-iq,\qquad v=r;$$
$$l_1=1,\qquad l_2=0,\qquad l_3=0,\qquad l_4=-ip^4,\qquad l_5=p^2(1-i),\qquad l_6=0$$

where $p\in k\setminus\{0\}$ and $i\in k$ satisfies $i^2+1=0$ which gives an algebra:

Algebra **H**:

$$f_1=x^3y+px^2yx+p^2xyx^2+p^3yx^3,$$
$$f_2=x^2y^2-ip^4y^2x^2+p^2(1-i)xy^2x,$$
$$f_3=xy^3+ipyxy^2-ip^2y^2xy+p^3y^3x,\quad p\in k\setminus\{0\},\ i^2+1=0.$$

Algebra **H** is a normal extension of a 4-dimensional AS-regular algebra (see Theorem 5.9), so **H** is AS-regular of dimension 5, strongly noetherian, Auslander regular and Cohen–Macaulay.

Suppose $l_2=0$ and $l_5=0$, then $l_6\neq 0$ and (4.5) has one solution

$$p=di,\qquad q=-iu,\qquad r=v,\qquad d\neq 0,\qquad u=d^2,\qquad v=d^3;$$
$$l_1=1,\qquad l_2=0,\qquad l_3=0,\qquad l_4=-d^4i,\qquad l_5=0,\qquad l_6=d^2(1-i),$$

where $d\in k\setminus\{0\}$ and $i\in k$ satisfies $i^2+1=0$ which gives an algebra:

Algebra **H**′:

$$f_1=x^3y+dix^2yx-d^2ixyx^2+d^3yx^3,$$
$$f_2=x^2y^2-d^4iy^2x^2+d^2(1-i)yx^2y,$$
$$f_3=xy^3+dyxy^2+d^2y^2xy+d^3y^3x,\quad d\in k\setminus\{0\},\ i^2+1=0.$$

After changing x and y, algebra **H**′ is in fact isomorphic to algebra **H** with $p=d^{-1}$.

Suppose $l_2\neq 0$. If neither l_5 nor l_6 is zero, then (4.5) has no solution.

If $l_2\neq 0$ and $l_6=0$, then $l_5\neq 0$ and (4.5) has one solution:

$$p=-c\bigl(1+g^3\bigr),\qquad q=-c^2g^2\bigl(1+g^2\bigr),\qquad r=c^3g^2,$$
$$d=-cg^3(1+g),\qquad u=-c^2g^4(1+g),\qquad v=c^3g^3;$$

$$l_1 = 1, \qquad l_2 = cg(1+g), \qquad l_3 = -c^3 g(1+g),$$
$$l_4 = -c^4 g^3, \qquad l_5 = -c^2\bigl(1-g^3\bigr), \qquad l_6 = 0,$$

where $c \in k \setminus \{0\}$ and $g = g_1 g_2$, which gives an algebra:

Algebra **I**:

$$f_1 = x^3y - c\bigl(1+g^3\bigr)x^2yx - c^2g^2\bigl(1+g^2\bigr)xyx^2 + c^3g^2yx^3,$$
$$f_2 = x^2y^2 + cg(1+g)xyxy - c^3g(1+g)yxyx - c^4g^3y^2x^2 - c^2\bigl(1-g^3\bigr)xy^2x,$$
$$f_3 = xy^3 - cg^3(1+g)yxy^2 - c^2g^4(1+g)y^2xy + c^3g^3y^3x,$$

where $c \in k \setminus \{0\}$ and $g \in k$ satisfies the equation $1 + g + g^2 + g^3 + g^4 = 0$.

Algebra **I** is a normal extension of a 4-dimensional AS-regular algebra (see Theorem 5.9), so **I** is AS-regular of dimension 5, strongly noetherian, Auslander regular and Cohen–Macaulay.

If $l_2 \neq 0$ and $l_5 = 0$, then $l_6 \neq 0$ and (4.5) has one solution:

$$p = -c(1+g), \qquad q = -c^2g^3(1+g), \qquad r = c^3g^4,$$
$$d = -cg\bigl(1+g^3\bigr), \qquad u = -c^2g\bigl(1+g^3\bigr), \qquad v = c^3;$$
$$l_1 = 1, \qquad l_2 = cg^2(1+g), \qquad l_3 = -c^3(1+g),$$
$$l_4 = -c^4g^3, \qquad l_5 = 0, \qquad l_6 = c^2\bigl(1-g^3\bigr)$$

where $c \in k \setminus \{0\}$ and $g = g_1 g_2$, which gives an algebra:

Algebra **I**′:

$$f_1 = x^3y - c(1+g)x^2yx - c^2g^3(1+g)xyx^2 + c^3g^4yx^3,$$
$$f_2 = x^2y^2 + cg^2(1+g)xyxy - c^3(1+g)yxyx - c^4g^3y^2x^2 + c^2\bigl(1-g^3\bigr)yx^2y,$$
$$f_3 = xy^3 - cg\bigl(1+g^3\bigr)yxy^2 - c^2g\bigl(1+g^3\bigr)y^2xy + c^3y^3x,$$

where $c \in k \setminus \{0\}$ and $g \in k$ satisfies the equation $1 + g + g^2 + g^3 + g^4 = 0$.

Algebra **I**′ is isomorphic to algebra **I** by exchanging x and y.

5. Proof of the AS-regularity and other properties

In this section, we study homological properties of the algebras given in the previous section.

5.1. Algebras A, B and F

Let $\mathcal{A}$ be the quotient algebra $k\langle x, y\rangle/(f_1, f_2, f_3)$, where the generating relations f_1, f_2 and f_3 are

$$f_1 = x^3y + px^2yx + qxyx^2 + ryx^3,$$
$$f_2 = x^2y^2 + l_2xyxy + l_3yxyx + l_4y^2x^2 + l_5xy^2x + l_5yx^2y,$$
$$f_3 = xy^3 + pyxy^2 + qy^2xy + ry^3x,$$

with the parameters $p, q, r, l_2, l_3, l_4, l_5 \in k$, $p \neq l_2$ and $r \neq 0$.

The algebras **A**, **B** and **F** are of this type, and $\{y^i(xy^2)^j(xy)^k(x^2y)^lx^m \mid i, j, k, l, m \in \mathbb{N}\}$ is a k-linear basis for each of them, as we have already seen by using the diamond lemma.

Lemma 5.1. *Suppose that* $\{y^i(xy^2)^j(xy)^k(x^2y)^lx^m \mid i, j, k, l, m \in \mathbb{N}\}$ *is a k-linear basis of* $\mathcal{A}$*, and that there is a complex of right* $\mathcal{A}$*-modules of the form*:

$$0 \to \mathcal{A}(-10) \xrightarrow{d_5} \mathcal{A}(-9)^{\oplus 2} \xrightarrow{d_4} \mathcal{A}(-6)^{\oplus 3} \xrightarrow{d_3} \mathcal{A}(-4)^{\oplus 3} \xrightarrow{d_2} \mathcal{A}(-1)^{\oplus 2} \xrightarrow{d_1} \mathcal{A} \xrightarrow{\epsilon} k_{\mathcal{A}} \to 0, \quad (5.1)$$

where ϵ *is the augmented map and each* d_i *is the left multiplication of a matrix given by*

$$\begin{aligned}
d_1 &= \begin{pmatrix} x & y \end{pmatrix},\\
d_2 &= \begin{pmatrix} x^2y + pxyx + qyx^2 & xy^2 + l_2yxy + l_5y^2x & y^3 \\ rx^3 & l_3xyx + l_4yx^2 + l_5x^2y & pxy^2 + qyxy + ry^2x \end{pmatrix},\\
d_3 &= \begin{pmatrix} 0 & Dy^2 & Hxy + Kyx \\ Ay^2 & Exy + Fyx & Lx^2 \\ Bxy + Cyx & Gx^2 & 0 \end{pmatrix},\\
d_4 &= \begin{pmatrix} px^2y + qxyx + ryx^2 & x^3 \\ l_3yxy + l_4y^2x + l_5xy^2 & x^2y + l_2xyx + l_5yx^2 \\ ry^3 & xy^2 + pyxy + qy^2x \end{pmatrix},\\
d_5 &= \begin{pmatrix} x \\ y \end{pmatrix},
\end{aligned}$$

for some $A, B, C, D, E, F, G, H, K, L \in k$ *such that* $ADGL \neq 0$ *and* $K \neq pH$*. Then the complex* (5.1) *is exact and* $\mathcal{A}$ *is an AS-regular algebra of dimension* 5*.*

Proof. Since $\{y^i(xy^2)^j(xy)^k(x^2y)^lx^m \mid i, j, k, l, m \in \mathbb{N}\}$ is a k-linear basis of $\mathcal{A}$, the Hilbert series of $\mathcal{A}$ is $(1-t)^{-2}(1-t^2)^{-1}(1-t^3)^{-2}$ and $\text{GK-dim}\,\mathcal{A} = 5$. Since y is not a left zero-divisor, the complex (5.1) is exact at $\mathcal{A}(-10)$.

The composition $d_1 \circ d_2$ is exactly the generating relations of $\mathcal{A}$. The complex (5.1) is exact at $\mathcal{A}(-1)$, $\mathcal{A}$ and k by [AS, (1.4)].

To show (5.1) is exact, it suffices to check the exactness at $\mathcal{A}(-9)^{\oplus 2}$ and $\mathcal{A}(-6)^{\oplus 3}$ by using the Hilbert series.

Suppose that $(f, g)^T \in \operatorname{Ker} d_4$. Writing g in the standard form, by modulo $\operatorname{Im} d_5$ we may assume that no monomial appearing in g starts with y. Since $ry^3f + xy^2g + pyxyg + qy^2xg = 0$, then $(xy^2)g = -y(ry^2f + pxyg + qyxg)$. It follows that $g = 0$. Hence $f = 0$ (mod $\operatorname{Im} d_5$), and $\operatorname{Ker} d_4 = \operatorname{Im} d_5$, that is, (5.1) is exact at $\mathcal{A}(-9)^{\oplus 2}$.

Notice that $H \neq 0$. In fact, if $H = 0$, then $Dy^2(x^2y + l_2xyx + l_5yx^2) + Kyx(xy^2 + pyxy + qy^2x) = 0$, i.e., $(D - Kl_5)y^2(x^2y) + (Dl_2 - Kl_3)y^2(xy)x + (Dl_5 - Kl_4)y^3x^2 + K(p - l_2)y(xy)^2 + K(q - l_5)y(xy^2)x = 0$. It follows from $p - l_2 \neq 0$ that $K = 0$. Then $Dy^2(x^2y + l_2xyx + l_5yx^2) = 0$. This contradicts $D \neq 0$.

Suppose that $(f, g, h)^T \in \operatorname{Ker} d_3$. Writing h in the standard form, by modulo $\operatorname{Im} d_4$ we may assume that no monomial appearing in h starts with xy^2 or y^3. Then $h = yh_1 + y^2h_2 + (xy)^lh_3$ ($l \geqslant 0$), with no monomial appearing in h_1 or h_2 starts with y, and no monomial appearing in h_3 starts with y or xy^2. Since $Dy^2g + (Hxy + Kyx)h = 0$, $Hxy^2h_1 + H(xy)^{l+1}h_3 = yz$ for some $z \in \mathcal{A}$. It follows that $h_1 = h_3 = 0$. So $h = y^2h_2$. Then

$$0 = Dy^2g + (Hxy + Kyx)h = Dy^2g + Hxy^3h_2 + Kyxy^2h_2,$$

which implies

$$Dy^2g + (K - pH)yxy^2h_2 = Hqy^2xyh_2 + Hry^3xh_2.$$

Writing the terms in the above equation in the standard form, it follows from $K - pH \neq 0$ that $h_2 = 0$. So $h = 0$. It follows from $Dy^2g = 0$ and $D \neq 0$ that $g = 0$. Then $Ay^2f = 0$, which implies that $f = 0$ as $A \neq 0$. So $(f, g, h)^T \in \operatorname{Im} d_4$ and $\operatorname{Ker} d_3 = \operatorname{Im} d_4$, i.e., (5.1) is exact at $\mathcal{A}(-6)^{\oplus 3}$.

So the complex (5.1) is a minimal projective resolution of the trivial module k.

Applying $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{A})$ to this projective resolution, we get a complex of left $\mathcal{A}$-modules

$$0 \to \mathcal{A} \xrightarrow{d_1^*} \mathcal{A}(1)^{\oplus 2} \xrightarrow{d_2^*} \mathcal{A}(4)^{\oplus 3} \xrightarrow{d_3^*} \mathcal{A}(6)^{\oplus 3} \xrightarrow{d_4^*} \mathcal{A}(9)^{\oplus 2} \xrightarrow{d_5^*} \mathcal{A}(10) \to 0, \tag{5.2}$$

where each d_i^* is given by the right multiplication of the corresponding matrix. The complex (5.2) is exact at $\mathcal{A}$ since x is not a right zero-divisor. It is also exact at $\mathcal{A}(9)^{\oplus 2}$ again by [AS, (1.4)] and the dimension of the homology group at $\mathcal{A}(10)$ is 1. Similarly, to show the exactness of (5.2) at all other positions, it suffices to check the exactness of (5.2) at $\mathcal{A}(1)^{\oplus 2}$ and $\mathcal{A}(4)^{\oplus 3}$.

Suppose $(f,g) \in \mathrm{Ker}\, d_2^*$. By modulo $\mathrm{Im}\, d_1^*$ we may assume that no monomial appearing in f ends with x. Since $f(x^2y+pxyx+qyx^2)+rgx^3=0$, which implies that the monomials in fx^2y would end with x, then $f=0$. Since $r\neq 0$, $g=0$. So $\mathrm{Ker}\, d_2^* = \mathrm{Im}\, d_1^*$, i.e., (5.2) is exact at $\mathcal{A}(1)^{\oplus 2}$.

Suppose $(f,g,h) \in \mathrm{Ker}\, d_3^*$. By modulo $\mathrm{Im}\, d_2^*$ we may assume that no monomial appearing in f ends with x^2y or x^3. Writing f as $f=f_1x+f_2x^2+f_3(xy)^s$ $(s\geqslant 0)$ with that no monomial appearing in f_1 or f_2 ends with x, and no monomial appearing in f_3 ends with x or x^2y. Since $f(Hxy+Kyx)+Lgx^2 = (f_1x+f_2x^2+f_3(xy)^s)(Hxy+Kyx)+Lgx^2=0$,

$$Hf_1x^2y+Hf_3(xy)^{s+1} = Hf_2\big(px^2yx+qxyx^2+ryx^3\big) - K\big(f_1x+f_2x^2+f_3(xy)^s\big)yx - Lgx^2.$$

Writing the right-hand side in standard form, it follows from $H\neq 0$ that $f_1=f_3=0$. So $f=f_2x^2$ and

$$0 = f_2x^2(Hxy+Kyx)+Lgx^2 = (K-pH)f_2x^2yx - Hf_2\big(qxyx^2+ryx^3\big)+Lgx^2.$$

Since $K-pH\neq 0$, $f_2=0$. So $f=0$. Then $Lgx^2=0$, which implies $g=0$ as $L\neq 0$. It follows from $Ghx^2=0$ that $h=0$ as $G\neq 0$. Hence $\mathrm{Ker}\, d_3^* = \mathrm{Im}\, d_2^*$, i.e. (5.2) is exact at $\mathcal{A}(4)^{\oplus 3}$.

Therefore $\mathcal{A}$ satisfies the Gorenstein condition with $\mathrm{gldim}\,\mathcal{A} = \mathrm{GK\text{-}dim}\,\mathcal{A} = 5$, i.e., $\mathcal{A}$ is a 5-dimensional AS-regular algebra. □

Now we can prove the regularity for the algebras **A**, **B** and **F**.

Theorem 5.2. *Algebras* **A**, **B** *and* **F** *are all AS-regular.*

Proof. It suffices to list the suitable parameters satisfying the conditions of Lemma 5.1.

For algebra **A**, take

$$A=-t^6,\quad B=-t^9,\quad C=-t^9l_2,\quad D=1,\quad E=F=0,$$
$$G=-t^6,\quad H=-l_2t^{-4},\quad K=-t^{-2},\quad L=1.$$

For algebra **B**, take

$$A=p^6,\quad B=0,\quad C=-p^{10},\quad D=1,\quad E=p^2,$$
$$F=p^3,\quad G=p^6,\quad H=-p^{-3},\quad K=0,\quad L=1.$$

For algebra **F**, take

$$A=p^6,\quad B=l_2p^8,\quad C=(l_2-p)p^9,\quad D=1,$$
$$E=p^2,\quad F=p^3,\quad G=p^6,\quad H=(l_2-p)p^{-4}$$

and $K=l_2p^{-3}$, $L=1$. □

To prove other homological properties, let $A(l,t) = \mathbf{A} = k\langle x, y\rangle/(f_1, f_2, f_3)$, where

$$\begin{aligned} f_1 &= x^3y + t^3yx^3, \\ f_2 &= x^2y^2 + lxyxy - t^2lyxyx - t^4y^2x^2, \\ f_3 &= xy^3 + t^3y^3x, \quad t, l \in k \text{ and } tl \neq 0. \end{aligned}$$

Lemma 5.3. *The algebra $A(l,t)$ is graded twist-equivalent* [Zh1] *to $A(l^2/t^2, l/t)$.*

Proof. Let $\sigma : A(l,t) \to A(l,t)$, $\sigma(x) = t^2x$, $\sigma(y) = ly$. Then $A^\sigma \cong A(l^2/t^2, l/t)$. $\square$

Theorem 5.4. *Algebra* **A** *is strongly noetherian, Auslander regular and Cohen–Macaulay.*

Proof. It suffices to prove the properties for $A(t^2,t)$ for some $t \neq 0$ by [Zh1, Theorem 1.3] under the condition that $A(t^2,t)$ is noetherian. Now

$$A\big(t^2,t\big) = k\langle x, y\rangle/\big(x^3y + t^3yx^3, x^2y^2 + t^2xyxy - t^4yxyx - t^4y^2x^2, xy^3 + t^3y^3x\big).$$

Note that $\{x^3, y^3, x^2y^2 - t^4yxyx\}$ is a sequence of normal regular elements of $A(t^2,t)$. By [ASZ, Proposition 4.9] and [Le, Theorem 5.10] it is enough to show that $A(t^2,t)/(x^3, y^3, x^2y^2 - t^4yxyx)$ is strongly noetherian, Auslander–Gorenstein and Cohen–Macaulay.

Let $A_1 = A(t^2,t)/(x^3, y^3, x^2y^2 - t^4yxyx) \cong k\langle x, y\rangle/(x^3, y^3, x^2y^2 - t^4yxyx, y^2x^2 - t^{-2}xyxy)$.

Now twisting A_1 by the graded automorphism

$$\sigma : A_1 \to A_1, \qquad \sigma(x) = x, \qquad \sigma(y) = t^{-1}y,$$

we get a new algebra

$$A_2 = (A_1)^\sigma = k\langle x, y\rangle/\big(x^3, y^3, x^2y^2 - tyxyx, xyxy - t^{-1}y^2x^2\big).$$

By [Zh1, Theorem 1.3] it suffices to show that A_2 is strongly noetherian, Auslander–Gorenstein and Cohen–Macaulay.

Let

$$\Omega_1 = xy^2xyx + yxyx^2y + ty^2xyx^2 \quad \text{and} \quad \Omega_2 = xy^2x^2y + t^{-1}yxy^2x^2 + t^{-1}y^2x^2yx.$$

Then Ω_1 and Ω_2 are normal elements of A_2 such that $\Omega_1\Omega_2 = \Omega_2\Omega_1 = 0$.

Let $A_3 = A_2/(\Omega_1, \Omega_2)$, then

$$A_3 \cong k\langle x, y\rangle/\big(x^3, y^3, \Omega_1, \Omega_2, x^2y^2 - tyxyx, xyxy - t^{-1}y^2x^2\big).$$

Similarly, we can find two normal elements

$$\omega_1 = \big(xy^2\big)^3 + y\big(xy^2\big)^2xy + y^2\big(xy^2\big)^2x \quad \text{and} \quad \omega_2 = \big(x^2y\big)^3 + xy\big(x^2y\big)^2x + y\big(x^2y\big)^2x^2$$

of A_3 such that $\omega_1\omega_2 = \omega_2\omega_1 = 0$.

Let $A_4 = A_3/(\omega_1, \omega_2)$. Then

$$A_4 \cong k\langle x, y\rangle/\big(x^3, y^3, \Omega_1, \Omega_2, \omega_1, \omega_2, x^2y^2 - tyxyx, xyxy - t^{-1}y^2x^2\big)$$

is a finite-dimensional algebra. So A_4 is strongly noetherian. It follows that A_2 is also strongly noetherian by [ASZ, Proposition 4.9].

Since $\{\Omega_1, \Omega_2, \omega_1, \omega_2\}$ is a sequence of normal elements of A_2, A_2 has enough normal elements. So it is Auslander–Gorenstein and Cohen–Macaulay by [Zh, Theorem 1] which ends the proof. □

Theorem 5.5. *The algebras* **B** *and* **F** *are strongly noetherian, Auslander regular and Cohen–Macaulay.*

Proof. If we set $l_2 = 0$ in algebra **F**, then **F** reduces to **B**. By [Zh1, Theorem 1.3], it suffices to prove the conclusion for the twisted algebra $\mathbf{F}^\sigma$ where σ is the automorphism defined by $\sigma(x) = x$ and $\sigma(y) = p^{-1}y$. Or equivalently, we may assume $p = 1$ in **F**. Then x^4, y^4, $\Omega_1 = (x^2y - yx^2)^2$, $\Omega_2 = (xy^2 - y^2x)^2$ and $\Omega_3 = (xy + yx)^4$ are central regular elements of **F**.

Let $F' = \mathbf{F}/(x^4, y^4, \Omega_1, \Omega_2, \Omega_3)$ be the quotient algebra. Then F' is a finite-dimensional algebra with a basis $\{y^i(xy^2)^j(xy)^k(x^2y)^lx^m \mid 0 \leqslant i, k, m \leqslant 3,\ 0 \leqslant j, l \leqslant 1\}$. Since F' is strongly noetherian, Cohen–Macaulay and has an Auslander dualizing complex, by [YZ, Theorem 5.1] **F** is strongly noetherian, Auslander regular and Cohen–Macaulay. □

5.2. Algebras **D** *and* **G**

Recall that a ring B is an Ore extension $A[z; \sigma, \delta]$ of a ring A, for some endomorphism σ of A and σ-derivation δ, if and only if that $B = \bigoplus_{i \geqslant 0} Az^i$ as a free A-module with $zA \subseteq Az + A$ [GW,MR]. Graded version of Ore extensions is defined accordingly. We show in this subsection that the algebras **D** and **G** are given by iterated Ore extensions.

Let A be the graded polynomial ring $k[y]$ over k with $\deg y = 1$. We proceed to construct an algebra A_4 from A by an iterated Ore extension in the following four steps.

Step 1: Let z_1 be a new variable of degree 3 and $A_1 = A[z_1; \sigma_1]$ be the graded Ore extension of A, where σ_1 is the automorphism of A given by

$$\sigma_1(y) = ay$$

for a fixed $0 \neq a \in k$.

Step 2: Let z_2 be a new variable of degree 2 and let $0 \neq b \in k$ and

$$A_2 = k\langle y, z_1, z_2\rangle/(z_1y = ayz_1, z_2y = byz_2 + z_1, z_2z_1 = az_1z_2).$$

It follows from the diamond lemma [Be] that $A_2 = \bigoplus_{i \geqslant 0} A_1z_2^i$ as a free A_1-module. Obviously, $z_2A_1 \subseteq A_1z_2 + A_1$. So $A_2 = A_1[z_2; \sigma_2, \delta_2]$ is a graded Ore extension of A_1, with σ_2 and δ_2 defined by

$$\sigma_2(y) = by, \qquad \sigma_2(z_1) = az_1;$$
$$\delta_2(y) = z_1, \qquad \delta_2(z_1) = 0.$$

Step 3: Let z_3 be a new variable of degree 3 and let

$$A_3 = k\langle y, z_1, z_2, z_3\rangle \Big/ \begin{pmatrix} z_1y = ayz_1, & z_2y = byz_2 + z_1, & z_3z_1 = b^3z_1z_3 + (a-b)z_2^3, \\ z_2z_1 = az_1z_2, & z_3z_2 = az_2z_3, & z_3y = b^3a^{-1}yz_3 + z_2^2 \end{pmatrix}.$$

Again by the diamond lemma, $A_3 = \bigoplus_{i \geqslant 0} A_2z_3^i$ as a free A_2-module. It follows from $z_3A_2 \subseteq A_2z_3 + A_2$ that $A_3 = A_2[z_3; \sigma_3, \delta_3]$ is a graded Ore extension of A_2, with σ_3 and δ_3 defined by

$$\sigma_3(y) = b^3a^{-1}y, \qquad \sigma_3(z_1) = b^3z_1, \qquad \sigma_3(z_2) = az_2;$$
$$\delta_3(y) = z_2^2, \qquad \delta_3(z_1) = (a-b)z_2^3, \qquad \delta_3(z_2) = 0.$$

Step 4: Let x be a new variable of degree 1. Suppose that $a \neq -b$. Let

$$A_4 = k\langle y, z_1, z_2, z_3, x\rangle \Big/ \begin{pmatrix} z_1 y = a y z_1, & z_2 z_1 = a z_1 z_2, & z_3 z_2 = a z_2 z_3, \\ z_2 y = b y z_2 + z_1, & z_3 z_1 = b^3 z_1 z_3 + (a-b) z_2^3, & x z_3 = a z_3 x, \\ z_3 y = b^3 a^{-1} y z_3 + z_2^2, & x z_1 = b^3 a^{-1} z_1 x + (a^3 - b^3)(a^2+ab)^{-1} z_2^2, & \\ xy = b^2 a^{-1} y x + z_2, & x z_2 = b z_2 x + (a^3 - b^3)(a^2+ab)^{-1} z_3 & \end{pmatrix}.$$

Similarly, $A_4 = \bigoplus_{i \geqslant 0} A_3 x^i$ as a free A_3-module and $xA_3 \subseteq A_3 x + A_3$ which implies that $A_4 = A_3[x; \sigma_4, \delta_4]$ is a graded Ore extension of A_3, with σ_4 and δ_4 defined by

$$\sigma_4(y) = b^2 a^{-1} y, \qquad \sigma_4(z_1) = b^3 a^{-1} z_1, \qquad \sigma_4(z_2) = b z_2, \qquad \sigma_4(z_3) = a z_3;$$
$$\delta_4(y) = z_2, \qquad \delta_4(z_1) = \frac{a^3 - b^3}{a(a+b)} z_2^2, \qquad \delta_4(z_2) = \frac{a^3 - b^3}{a(a+b)} z_3, \qquad \delta_4(z_3) = 0.$$

Lemma 5.6. *Given $a, b \in k$ such that $ab(a+b) \neq 0$. Then the algebra A_4 is an AS-regular algebra of dimension* 5 *with Hilbert series* $(1-t)^{-2}(1-t^2)^{-1}(1-t^3)^{-2}$.

Proof. By [ZZ2, Lemma 5.3], A_4 is 5-dimensional AS-regular. By the definition of graded Ore extensions, A_4 is a free left A-module, and

$$H_{A_4}(t) = H_A(t) \cdot \frac{1}{(1 - t^{\deg z_1})(1 - t^{\deg z_2})(1 - t^{\deg z_3})(1 - t^{\deg x})}$$
$$= \frac{1}{(1-t)^2(1-t^2)(1-t^3)^2}. \qquad \square$$

Now, let $\mathcal{A}(a,b) = k\langle x, y\rangle/(f_1, f_2, f_3)$, with the generating relations f_1, f_2 and f_3 as follows:

$$f_1 = x^3 y + p x^2 y x + q x y x^2 + r y x^3,$$
$$f_2 = x^2 y^2 + l_2 xyxy + l_3 yxyx + l_4 y^2 x^2 + l_5 x y^2 x + l_5 y x^2 y,$$
$$f_3 = x y^3 + p y x y^2 + q y^2 x y + r y^3 x$$

where

$$p = -\frac{ab + b^2 + a^2}{a}, \qquad q = \frac{b(ab + b^2 + a^2)}{a}, \qquad r = -b^3,$$
$$l_2 = -\frac{a^2 + ab + 2b^2}{a+b}, \qquad l_3 = \frac{b^5(2a^2 + ab + b^2)}{a^3(a+b)}, \qquad l_4 = -\frac{b^6}{a^2}, \qquad l_5 = \frac{b^2(a^3 - b^3)}{a^2(a+b)}. \tag{5.3}$$

Then we have the following proposition.

Proposition 5.7. *Given $a, b \in k$ such that $ab(a+b)(a^2+b^2)(a^3-b^3) \neq 0$. Then $\mathcal{A}(a,b)$ is isomorphic to A_4 as a graded algebra. So, $\mathcal{A}(a,b)$ is strongly noetherian, Auslander regular, AS-regular of dimension* 5 *and Cohen–Macaulay.*

Proof. By the construction, $A_4 = k\langle y, z_1, z_2, z_3, x\rangle/I$, where I is generated by the following ten relations:

$$z_1 y = a y z_1, \tag{5.4}$$

$$z_2 y = b y z_2 + z_1, \tag{5.5}$$

$$z_2 z_1 = a z_1 z_2, \tag{5.6}$$

$$z_3 y = b^3 a^{-1} y z_3 + z_2^2, \tag{5.7}$$

$$z_3 z_1 = b^3 z_1 z_3 + (a-b) z_2^3, \tag{5.8}$$

$$z_3 z_2 = a z_2 z_3, \tag{5.9}$$

$$xy = b^2 a^{-1} yx + z_2, \tag{5.10}$$

$$x z_1 = \frac{b^3}{a} z_1 x + \frac{a^3-b^3}{a(a+b)} z_2^2, \tag{5.11}$$

$$x z_2 = b z_2 x + \frac{a^3-b^3}{a(a+b)} z_3, \tag{5.12}$$

$$x z_3 = a z_3 x. \tag{5.13}$$

By (5.10) $z_2 = xy - b^2a^{-1}yx$, by (5.5) $z_1 = xy^2 - (b^2a^{-1}+b)yxy + b^3a^{-1}y^2x$, and by (5.12)

$$z_3 = \frac{a(a+b)x^2y - (a+b)^2bxyx + (a+b)b^3yx^2}{a^3-b^3}.$$

So A_4 is generated by x and y as a k-algebra. Moreover, replacing z_1, z_2 and z_3 with these expressions, the relations (5.4), (5.11) and (5.13) turn out to be the following three relations:

$$xy^3 + pyxy^2 + qy^2xy + ry^3x = 0,$$
$$x^2y^2 + l_2xyxy + l_3yxyx + l_4y^2x^2 + l_5xy^2x + l_5yx^2y = 0,$$
$$x^3y + px^2yx + qxyx^2 + ryx^3 = 0,$$

where the parameters are given in (5.3). The relations (5.6), (5.7), (5.8) and (5.9) can be derived from the above three relations by using $a^2+b^2 \neq 0$. So, $A_4 = \mathcal{A}(a,b)$.

It follows from [ASZ, Proposition 4.1] and [YZ, Theorem 5.1, Corollary 6.8] that $\mathcal{A}(a,b)$ is strongly noetherian, Auslander regular, AS-regular of dimension 5 and Cohen–Macaulay. □

Theorem 5.8. *Algebras* **D** *and* **G** *are strongly noetherian, Auslander regular, AS-regular of dimension* 5 *and Cohen–Macaulay.*

Proof. It is easy to check that

$$\mathbf{D} \cong \mathcal{A}(a,b) \quad \text{with } a = q^2/p^3,\ b = -q/p;$$
$$\mathbf{G} \cong \mathcal{A}(a,b) \quad \text{with } a = r^2/g,\ b = qr^3g/\big(r^5 + qrg^2 + g^3\big).$$

The conclusions follow from Proposition 5.7. □

5.3. *Algebras* **C**, **E**, **H** *and* **I**

In this subsection, we show the algebras **C**, **E**, **H** and **I** are normal extensions of some 4-dimensional AS-regular algebras given in [LPWZ2].

Theorem 5.9. *Algebras* **C**, **E**, **H** *and* **I** *are all AS-regular algebras of dimension* 5, *which are strongly noetherian, Auslander regular and Cohen–Macaulay.*

Proof. By the diamond lemma [Be], $xy^2+p^2y^2x$ is a normal regular element of **C** and $\mathbf{C}/(xy^2+p^2y^2x)$ is isomorphic to $D(0,p)$ [LPWZ2, Theorem A]. So **C** is a normal extension of $D(0,p)$.

Algebra **E** is a normal extension of $D(p-t,t)$ since $xy^2+(p-t)yxy+t^2y^2x$ is a normal regular element of **E** and $\mathbf{E}/(xy^2+(p-t)yxy+t^2y^2x)$ is isomorphic to $D(p-t,t)$ [LPWZ2, Theorem A].

Algebra **H** is a normal extension of $B(p)$ since $xy^2-ip^2y^2x$ is a normal regular element of **H** and $\mathbf{H}/(xy^2-ip^2y^2x)$ is isomorphic to $B(p)$ [LPWZ2, Theorem A].

Algebra **I** is a normal extension of $D(cg(1+g),cg^4)$ since $xy^2+cg(1+g)yxy+c^2g^3y^2x$ is a normal regular element of **I** and $\mathbf{I}/(xy^2+cg(1+g)yxy+c^2g^3y^2x)$ is isomorphic to $D(cg(1+g),cg^4)$.

These algebras $B(p)$, $D(0,p)$, $D(p-t,t)$ and $D(cg(1+g),cg^4)$ are strongly noetherian, Auslander regular, Cohen–Macaulay and AS-regular of dimension 4 as given in [LPWZ2, Theorem A].

So, all the algebras considered here are normal extensions of AS-regular algebras of dimension 4. They are all noetherian by [ATV, Lemma 8.2]. By [Le, Theorem 5.10] and [LPWZ2, Lemma 7.6], they are AS-regular of dimension 5.

It follows from [ASZ, Proposition 4.9] and [YZ, Theorem 5.1] that all these algebras are strongly noetherian, Auslander regular and Cohen–Macaulay. □

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References

[AS] M. Artin, W.F. Schelter, Graded algebras of dimension 3, Adv. Math. 66 (2) (1987) 171–216.
[ASZ] M. Artin, L.W. Small, J.J. Zhang, Generic flatness for strongly noetherian algebras, J. Algebra 221 (2) (1999) 579–610.
[ATV] M. Artin, J. Tate, M. Van den Bergh, Some algebras associated to automorphisms of elliptic curves, in: The Grothendieck Festschrift, vol. I, in: Progr. Math., vol. 86, Birkhäuser Boston, Boston, MA, 1990, pp. 33–85.
[AZ] M. Artin, J.J. Zhang, Noncommutative projective schemes, Adv. Math. 109 (2) (1994) 228–287.
[Be] G.M. Bergman, The diamond lemma for ring theory, Adv. Math. 29 (2) (1978) 178–218.
[FV] G. Floystad, J.E. Vatne, Artin–Schelter regular algebras of dimension five, in: Algebras, Geometry and Mathematical Physics, in: Banach Center Publ., vol. 93, 2011, pp. 19–39.
[GW] K.R. Goodearl, R.B. Warfield Jr., An Introduction to Noncommutative Noetherian Rings, second ed., London Math. Soc. Stud. Texts, vol. 61, Cambridge University Press, Cambridge, 2004.
[Ka] T.V. Kadeishvili, The algebraic structure in the homology of an A_∞-algebra, Soobshch. Akad. Nauk Gruzin. SSR 108 (2) (1982) 249–252 (in Russian).
[Ke] B. Keller, *A*-infinity algebras in representation theory, in: Proceedings of ICRA IX, Beijing, 2000.
[Le] T. Levasseur, Some properties of non-commutative regular graded rings, Glasg. Math. J. 34 (1992) 227–300.
[LPWZ1] D.-M. Lu, J.H. Palmieri, Q.-S. Wu, J.J. Zhang, A_∞-algebras for ring theorists, Algebra Colloq. 11 (1) (2004) 91–128.
[LPWZ2] D.-M. Lu, J.H. Palmieri, Q.-S. Wu, J.J. Zhang, Regular algebras of dimension 4 and their A_∞ Ext-algebras, Duke Math. J. 137 (3) (2007) 537–584.
[LPWZ3] D.-M. Lu, J.H. Palmieri, Q.-S. Wu, J.J. Zhang, *A*-infinity structure on Ext-algebras, J. Pure Appl. Algebra 213 (11) (2009) 2017–2037.
[LPWZ4] D.-M. Lu, J.H. Palmieri, Q.-S. Wu, J.J. Zhang, Koszul equivalences in A_∞-algebras, New York J. Math. 14 (2008) 1–54.
[LS] T. Levasseur, S.P. Smith, Modules over the 4-dimensional Sklyanin algebra, Bull. Soc. Math. France 121 (1) (1993) 35–90.
[LSV] L. Le Bruyn, S.P. Smith, M. Van den Bergh, Central extensions of three-dimensional Artin–Schelter regular algebras, Math. Z. 222 (2) (1996) 171–212.
[Me] S.A. Merkulov, Strong homotopy algebras of a Kähler manifold, Int. Math. Res. Not. 3 (1999) 153–164.
[MR] J.C. McConnell, J.C. Robson, Noncommutative Noetherian Rings, Wiley, Chichester, 1987.

[RZ] D. Rogalski, J.J. Zhang, Regular algebras of dimension 4 with 3 generators, in: Contemp. Math., vol. 562, Amer. Math. Soc., 2012, pp. 221–241.

[Sk1] E.K. Sklyanin, Some algebraic structures connected with the Yang–Baxter equation, Funktsional. Anal. i Prilozhen. 16 (4) (1982) 27–34 (in Russian).

[Sk2] E.K. Sklyanin, Some algebraic structures connected with the Yang–Baxter equation. Representations of a quantum algebra, Funktsional. Anal. i Prilozhen. 17 (4) (1983) 34–48 (in Russian).

[Sm] S.P. Smith, Some finite-dimensional algebras related to elliptic curves, in: Representation Theory of Algebras and Related Topics, Mexico City, 1994, in: CMS Conf. Proc., vol. 19, Amer. Math. Soc., Providence, RI, 1996, pp. 315–348.

[SV1] B. Shelton, M. Vancliff, Some quantum $\mathbb{P}^3$s with one point, Comm. Algebra 27 (3) (1999) 1429–1443.

[SV2] B. Shelton, M. Vancliff, Embedding a quantum rank three quadric in a quantum $\mathbb{P}^3$, Comm. Algebra 27 (6) (1999) 2877–2904.

[SS] S.P. Smith, J.T. Stafford, Regularity of the four-dimensional Sklyanin algebra, Compos. Math. 83 (3) (1992) 259–289.

[Va1] M. Vancliff, Quadratic algebras associated with the union of a quadric and a line in $\mathbb{P}^3$, J. Algebra 165 (1) (1994) 63–90.

[Va2] M. Vancliff, The defining relations of quantum $n \times n$ matrices, J. Lond. Math. Soc. (2) 52 (1995) 255–262.

[VV1] K. Van Rompay, M. Vancliff, Embedding a quantum nonsingular quadric in a quantum $\mathbb{P}^3$, J. Algebra 195 (1) (1997) 93–129.

[VV2] K. Van Rompay, M. Vancliff, Four-dimensional regular algebras with point scheme a nonsingular quadric in $\mathbb{P}^3$, Comm. Algebra 28 (5) (2000) 2211–2242.

[VVW] K. Van Rompay, M. Vancliff, L. Willaert, Some quantum $\mathbb{P}^3$s with finitely many points, Comm. Algebra 26 (4) (1998) 1193–1208.

[YZ] A. Yekutieli, J.J. Zhang, Rings with Auslander dualizing complexes, J. Algebra 213 (1) (1999) 1–51.

[Zh] J.J. Zhang, Connected graded Gorenstein algebras with enough normal elements, J. Algebra 189 (2) (1997) 390–405.

[Zh1] J.J. Zhang, Twisted graded algebras and equivalences of graded categories, Proc. Lond. Math. Soc. (3) 72 (1996) 281–311.

[ZZ1] J.J. Zhang, J. Zhang, Double Ore extension, J. Pure Appl. Algebra 212 (12) (2008) 2668–2690.

[ZZ2] J.J. Zhang, J. Zhang, Double extension regular algebras of type (14641), J. Algebra 322 (2) (2009) 373–409.