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Interpreting canonical tensor model in minisuperspace

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ABSTRACT

Canonical tensor model is a theory of dynamical fuzzy spaces in arbitrary space–time dimensions. Examining its simplest case, we find a connection to a special case of minisuperspace model of general relativity in arbitrary dimensions. This is a first step in interpreting variables in canonical tensor model based on the known language of general relativity.

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1. Introduction

Locality is an important issue in quantum gravity: due to the diffeomorphism invariance, one cannot define any well-defined local observables in general space–time,¹ which is also closely related to Bekenstein's entropy bounds [2,3]. This notion might suggest that space–time has a fundamental fuzziness and eventually acquires a smooth manifold structure at a long-distance scale. Since fuzzy spaces possess no notion of dimensionality and locality in general, their space–time dimension is not a parameter but something that ought to be determined through dynamics; additionally, as a result of dynamics, locality has to be favoured at a long-distance scale.

Canonical tensor model is one of such trials introduced by one of the current authors as a theory of dynamical fuzzy spaces [4–7]. The fundamental variable is a rank-three tensor, which specifies the structure of fuzzy spaces; the time evolution of fuzzy spaces can be determined by a Hamiltonian flow. Somewhat amazingly, the Hamiltonian can be uniquely fixed under some reasonable assumptions [5]. So far, it has been shown that locality is favoured as a result of dynamics at least when indices can take only two values [7].

The canonical tensor model may not be an isolated model and is expected to be related to other types of tensor models. Here taking an overlook at history of tensor models, let us introduce several cousins. Dating back to the original introduction of ten-

sor models [8–10], the first motivation was to construct a model of higher-dimensional simplicial quantum gravity as a natural extension of matrix models which describe two-dimensional simplicial quantum gravity. As far as a symmetric tensor is concerned, this program did not work [9,11]. However, tensor models with unsymmetric tensors called *coloured tensor models* have been proposed [12]; the newly introduced “colour” degrees of freedom turn out to fit together well with simplicial geometries and this line of work is still in progress [13]. On the other hand, apart from the interpretation as a simplicial quantum gravity, tensor models have developed into so-called *group field theories* letting indices be group-valued [14–18] and into the canonical tensor model as a theory of dynamical fuzzy spaces which we argue in this Letter.

Most importantly, as a theory of quantum gravity the canonical tensor model ought to be related to general relativity in arbitrary dimensions as well. Therefore, interpreting rank-three tensors in the canonical tensor model based on the established language of general relativity is absolutely imperative. However, this part is still veiled in mystery. The purpose of this Letter is to make progress in that direction: in a simple situation that indices can take only a single value ($N = 1$), we have identified the tensors as variables of general relativity in a special case of minisuperspace. The Letter is organised as follows. In Section 2, we examine the canonical tensor model with $N = 1$ and derive an effective action written by its degrees of freedom. In Section 3, we consider the Einstein–Hilbert action in arbitrary dimensions and reduce it by the special minisuperspace ansatz. As a result, we obtain a corresponding effective action, which is nothing but the effective action derived from the canonical tensor model. In Section 4, we summarise our results.

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E-mail addresses: sasakura@yukawa.kyoto-u.ac.jp (N. Sasakura), Yuki.Sato@wits.ac.za (Y. Sato).¹ In the AdS/CFT correspondence [1], such local observables can be defined on an infinite conformal boundary in an anti-deSitter space.

2. Canonical tensor model

The canonical tensor model has been developed by a series of works [4–7], and designed to describe a theory of dynamical fuzzy space in the canonical formalism. Since the fuzzy space itself does not necessarily include information of space dimensions a priori, the canonical tensor model might have a potential to describe quantum gravity in arbitrary dimensions. Physical degrees of freedom in the canonical tensor model are a rank-three tensor M_{abc} and its canonical conjugate P_{abc} ; its classical and even quantum dynamics can be completely determined by the unique Hamiltonian in principle. However, so far, the role of the rank-three tensors is unclear: we don't know how to interpret them in the standard language of gravitation. The main purpose of this Letter is to address this issue in the most simplest case.

We start with reviewing the basic concepts of the fuzzy space described by the canonical tensor model. The fuzzy space is a notion of space defined not by coordinates but by the algebra of linearly independent functions on the space; the product of such functions, f_a ($a = 1, \dots, N$), is characterised by a rank-three tensor:

$$f_a \star f_b = C_{ab}{}^c f_c. \quad (2.1)$$

To make contact with the canonical tensor model, we further impose two requirements [19]:

1. Reality conditions:

$$f_a^* = f_a, \quad (f_a \star f_b)^* = f_b \star f_a, \quad (2.2)$$

where $*$ stands for a complex conjugation;

2. A trace-like property:

$$\langle f_a | f_b \star f_c \rangle = \langle f_a \star f_b | f_c \rangle = \langle f_c \star f_a | f_b \rangle, \quad (2.3)$$

where the inner product $\langle f_a | f_b \rangle$ has been chosen to be real, symmetric and bilinear.

Since there exists a real linear transformation of f_a which does not spoil the two requirements above, without loss of generality one can choose the inner product as follows:

$$\langle f_a | f_b \rangle = \delta_{ab}, \quad (2.4)$$

if the inner product was set to be positive-definite as an initial condition. Using (2.4), the degrees of freedom of the fuzzy space can be solely expressed by the rank-three tensor:

$$C_{abc} = \langle f_a \star f_b | f_c \rangle = C_{ab}{}^d \langle f_d | f_c \rangle. \quad (2.5)$$

Since (2.4) is invariant under the orthogonal group transformation $O(N)$, the rank-three tensor C_{abc} has the following kinematical symmetry:

$$C'_{abc} = J_a{}^d J_b{}^e J_c{}^f C_{def}, \quad J \in O(N). \quad (2.6)$$

The two requirements on the function f_a are translated into the generalised Hermiticity condition of C_{abc} :

$$C_{abc} = C_{bca} = C_{cab} = C_{bac}^* = C_{acb}^* = C_{cba}^*. \quad (2.7)$$

To make contact with general relativity, the time evolution of the fuzzy space (in other words, the rank-three tensor) is presumably generated by a “local” generator which somehow corresponds to the Hamiltonian constraint in general relativity. In addition, the system ought to have the invariance under the orthogonal group transformation (2.6), which can be expected to correspond to the

spatial diffeomorphism in general relativity. Therefore, it is reasonable to define the Hamiltonian in such a way that the system becomes a constrained system with the generators of the time evolution and the orthogonal transformation as first-class constraints. In this way, the total Hamiltonian of the canonical tensor model can be given as

$$H = N_a \mathcal{H}_a + N_{[ab]} \mathcal{J}_{[ab]} + N \mathcal{D}, \quad (2.8)$$

where $a, b, c = 1, \dots, N$; $[ab]$ denotes that a and b are anti-symmetric; N_a , $N_{[ab]}$ and N are Lagrange multipliers;

$$\mathcal{H}_a = P_{a(bc)} P_{bde} M_{cde}; \quad (2.9)$$

$$\mathcal{J}_{[ab]} = \frac{1}{2} (P_{acd} M_{bcd} - P_{bcd} M_{acd}); \quad (2.10)$$

$$\mathcal{D} = -\frac{1}{3} M_{abc} P_{abc}; \quad (2.11)$$

$P_{a(bc)} = \frac{1}{2} (P_{abc} + P_{acb})$. As a convention, indices appearing repeatedly are summed from 1 to N . Here \mathcal{H}_a and $\mathcal{J}_{[ab]}$ are the generators of the time evolution and the orthogonal group transformation, respectively; additionally \mathcal{D} , the generator of the scale transformation, has been introduced in order to regulate divergent behaviours of dynamics [6]. The rank-three tensors satisfy the following Poisson bracket:

$$\{M_{abc}, P_{def}\} = \delta_{ad} \delta_{be} \delta_{cf} + (\text{cyclic permutations of } (d, e, f)); \quad (2.12)$$

the other brackets vanish. In fact, the form of the time-evolution generator \mathcal{H}_a can be uniquely fixed up to an overall sign, if one imposes several reasonable assumptions, i.e., (1) closed algebra, (2) cubic terms at most, (3) invariance under the time-reversal symmetry and (4) connectivity [5]. The constraints, \mathcal{H}_a , $\mathcal{J}_{[ab]}$ and \mathcal{D} , are first-class: they form a first-class constraint algebra:

$$\{H(T^1), H(T^2)\} = J([\tilde{T}^1, \tilde{T}^2]), \quad (2.13)$$

$$\{J(V), H(T)\} = H(VT), \quad (2.14)$$

$$\{J(V^1), J(V^2)\} = J([V^1, V^2]), \quad (2.15)$$

$$\{\mathcal{D}, H(T)\} = H(T), \quad (2.16)$$

$$\{\mathcal{D}, J(V)\} = 0, \quad (2.17)$$

where $H(T) = T_a \mathcal{H}_a$, $J(V) = V_{[ab]} \mathcal{J}_{[ab]}$ and $\tilde{T}_{ab} = P_{(ab)c} T_c$; $[\cdot, \cdot]$ denotes the matrix commutator. It has been pointed out [4] that this algebra has a close relationship with the Dirac algebra of general relativity [20–24].

From now we will examine a simple version of the canonical tensor model called *minimal canonical tensor model* [6]. In this minimal model, the rank-three tensors, M_{abc} and P_{abc} , are not Hermitian in the sense of (2.7) but totally symmetric tensors. In that case, one can consistently add a “cosmological constant” term λM_{abb} to \mathcal{H}_a , if the constraint \mathcal{D} is ignored, as was shown in the first part of [5]; the Hamiltonian becomes

$$H = N_a \mathcal{H}_a + N_{[ab]} \mathcal{J}_{[ab]}, \quad (2.18)$$

where $\mathcal{J}_{[ab]}$ is the same as (2.10), while \mathcal{H}_a is changed to

$$\mathcal{H}_a = P_{abc} P_{bde} M_{cde} - \lambda M_{abb}. \quad (2.19)$$

The constraint algebra still takes the form given by (2.13), (2.14) and (2.15), while the part containing \mathcal{D} , (2.16) and (2.17), are discarded in this setting.

In order to extract some information of geometry from the rank-three tensors, let us consider the minimal model with $N = 1$:

when $N = 1$, the whole fuzzy space at some time slice can be described by a single function f_1 . Here we rewrite the ingredients in (2.18) as follows:

$$L \equiv \frac{1}{3}M_{111}, \quad \Pi \equiv P_{111}, \quad N \equiv 3N_1, \quad \Lambda \equiv \lambda. \quad (2.20)$$

By this convention, the Hamiltonian (2.9) becomes

$$H = N(L\Pi^2 - \Lambda L), \quad (2.21)$$

with the Poisson bracket,

$$\{L, \Pi\} = 1. \quad (2.22)$$

By the standard Legendre transformation, the corresponding action turns out to be

$$S_{\text{CT}}(L, N) = \int dt \left(\frac{\dot{L}(t)^2}{4N(t)L(t)} + \Lambda N(t)L(t) \right), \quad (2.23)$$

where \dot{L} denotes the time derivative of L . What we will do in the next section is to compare this action with a special case of minisuperspace action of general relativity in $d + 1$ dimensions. (2.19) is unique up to an overall sign, as previously mentioned; when comparing the actions, this ambiguity becomes important. We will elaborate this issue in Section 3.

3. Minisuperspace in general relativity

In this section, for the purpose of interpreting the variable L in the action (2.23) based on some known language, we consider $(d + 1)$ -dimensional general relativity in a special case of minisuperspace. To begin with, remember the Einstein–Hilbert action with a cosmological constant Λ in $d + 1$ dimensions ($d > 1$):

$$S_{\text{EH}} = \frac{1}{16\pi G_N} \int d^{d+1}x (R^{(d+1)} - 2\Lambda), \quad (3.1)$$

where G_N and $R^{(d+1)}$ are the Newton constant and the Ricci scalar in $(d + 1)$ -dimensional space–time, respectively. It has been given that space–time can be decomposed into space and time without breaking symmetry by Arnowitt, Deser and Misner (ADM) [20]:

$$ds^2 = -N^2 dt^2 + h_{ij}(N^i dt + dx^i)(N^j dt + dx^j), \quad (3.2)$$

where N , N^i and h_{ij} are a lapse function, a shift vector and a spatial metric, respectively; the Latin indices run from 1 to d . By applying the ADM decomposition, the Einstein–Hilbert action with the cosmological constant can be rewritten in the following form up to total derivative terms:

$$S_{\text{EH}} = \frac{1}{16\pi G_N} \int dt d^d x \sqrt{h} N (K_{ij}K^{ij} - K^2 + R^{(d)} - 2\Lambda), \quad (3.3)$$

where h is the determinant of the spatial metric; K_{ij} is the extrinsic curvature defined as

$$K_{ij} = \frac{1}{2N} (\dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i); \quad (3.4)$$

K is a trace of the extrinsic curvature. Here ∇_i denotes the covariant derivative associated with the spatial metric.

Then we consider the following ansatz:

$$N = N(t), \quad N_i = 0, \quad h_{ij} = a(t)^2 \delta_{ij}, \quad (3.5)$$

where $a(t)$ is a scale factor of the Universe, and δ_{ij} is the Kronecker delta. Through the ansatz (3.5), we focus only on a zero mode of the metric. Plugging the metric ansatz (3.5), one obtains the effective action:

$$S_{\text{EH}}(a, N) = \frac{V}{16\pi G_N} \int dt a^d \left(d(1-d) \frac{\dot{a}^2}{Na^2} - 2\Lambda N \right), \quad (3.6)$$

where

$$V = \int d^d x. \quad (3.7)$$

When the effective action (3.6) is written based on a quantity invariant under spatial diffeomorphism,

$$L(t) = \int d^d x \sqrt{h(t)} = V a^d(t), \quad (3.8)$$

one recovers (2.23) up to some redefinitions of coupling constants and an overall sign:

$$S_{\text{EH}}(L, N) = \frac{\alpha}{16\pi G_N} \int dt \left(\frac{\dot{L}(t)^2}{4N(t)L(t)} + \frac{d\Lambda}{2(d-1)} N(t)L(t) \right), \quad (3.9)$$

where

$$\alpha = \frac{4(1-d)}{d}. \quad (3.10)$$

For $d > 1$, the overall sign (3.10) is different from the one of (2.23); this difference could be important when coupling matter fields. However, taking into account that there is no principle to choose the overall sign of the Hamiltonian in the $N = 1$ canonical tensor model (2.19), the sign difference for $d > 1$ is not essential. In other words, one can fix the overall sign of the Hamiltonian in the $N = 1$ canonical tensor model in such a way as to coincide with (3.9). Note that the effective action (3.9) is universal: the effect of the dimensionality appears merely as the redefinition of coupling constants and the overall factor. As is clear from the ansatz we made (3.5), even if one adds higher spatial derivative terms by breaking Lorentz symmetry explicitly, the action still remains the same up to some redefinitions of coupling constants and an overall sign. For instance, in the case of Hořava–Lifshitz gravity [25], one adds spatial derivative terms in such a way that the unitarity is preserved but the full space–time symmetry is broken down to the so-called *foliation-preserving diffeomorphism*:

$$S_{\text{HL}} = \frac{1}{\kappa} \int dt d^d x \sqrt{h} N \times (K_{ij}K^{ij} - \lambda K^2 + \gamma R^{(d)} - 2\Lambda + \eta b^i b_i + \dots), \quad (3.11)$$

where κ , λ , γ , Λ and η are coupling constants; b_i is a d -dimensional vector field [26]:

$$b_i = \frac{\partial_i N}{N}. \quad (3.12)$$

The dots in (3.11) mean higher spatial derivative terms. When taking the ansatz (3.5), one obtains

$$S_{\text{HL}}(L, N) = \frac{\alpha}{\kappa} \int dt \left(\frac{\dot{L}(t)^2}{4N(t)L(t)} + \frac{d\Lambda}{2(d\lambda - 1)} N(t)L(t) \right), \quad (3.13)$$

where

$$\alpha = \frac{4(1-d\lambda)}{d}. \quad (3.14)$$

Furthermore, the same form of the effective action has been obtained in the $(1 + 1)$ -dimensional setup of Causal Dynamical Triangulations (CDT) [27] and the projectable Hořava–Lifshitz gravity [28] without taking the ansatz like (3.5).²

² It has been shown that they are in the same universality class [28].

4. Summary and discussions

In Section 2, we firstly have examined the minimal canonical tensor model with $N = 1$ and obtained the effective action (2.23) described by two functions of time, $L(t)$ and $N(t)$. Secondly in Section 3, we have confirmed that the effective action (2.23) coincides with the special case of minisuperspace action of general relativity in arbitrary dimensions (3.6) if the overall sign of (2.23) is properly chosen.

Let us closely look at the coincidence. In the canonical tensor model, we have set all the tensor slots to 1; as a consequence, the generator of the orthogonal group transformation, $\mathcal{J}_{[ab]}$, vanishes. Therefore, it can be considered that the spatial “diffeomorphism” (orthogonal group transformation) is gauged via the manipulation, which is consistent with philosophy of our ansatz (3.5) in general relativity.

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