Generalizations of fixed point theorems and computation

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1. Introduction

As a powerful mechanism for mathematical analysis, fixed point theory has many applications in areas such as mechanics, physics, transportation, control, economics, and optimization. Fixed point theorems have been extensively studied and generalized in the past years (e.g., \cite{2,4–7,9–11,13–17,19} and references therein). When a mapping is twice continuously differentiable from a convex and compact set \(D\) to itself, Kellogg et al. \cite{8} gave in 1976 a constructive proof of Brouwer fixed point theorem for the mapping and presented a homotopy method for computing a fixed point of the mapping in \(D\). In 1978, Chow et al. \cite{3} constructed the following homotopy

\[ H(x, x(0), t) = (1 - t)(x - F(x)) + t(x - x(0)) = 0, \tag{1.1} \]

which becomes an important tool for computing fixed points of a mapping, solutions of a system of nonlinear equations, all solutions of a system of polynomial equations, and so on (e.g., \cite{6}). In \cite{3,8}, the convexity of \(D\) is a virtual condition for the constructive proofs.

In general, it is difficult to reduce or remove the convexity. Under the normal cone condition, Brouwer fixed point theorem was generalized to a class of nonconvex sets in \cite{18}. In this paper, the well-known Brouwer fixed point theorem and Kakutani fixed point theorem are generalized to a class of nonconvex sets and a globally convergent homotopy method is developed for computing fixed points on this class of nonconvex sets.
2. Generalization of fixed point theorems to a class of nonconvex sets

Let $C \subset \mathbb{R}^n$ be a nonempty, convex and compact set with $\text{int}(C) \neq \emptyset$. Let $\Lambda$ be a nonempty and connected subset of $C$ satisfying that, for each point $x \in \Lambda$, $\text{int}(N(x, \delta, \Lambda)) \neq \emptyset$ for any $\delta > 0$, where $N(x, \delta, \Lambda) = \{y \in \Lambda \mid \|y - x\| < \delta\}$. Let $D$ denote the closure of $C \setminus \Lambda$ and $\overline{\Lambda}$ the closure of $\Lambda$. Obviously, $C = D \cup \Lambda$. We assume that $\Lambda$ satisfies at least one of the following two properties:

**Property 1.** There exists a point $p \notin C$ such that any interior point of $D$ is not contained in $V(p) = \{\alpha p + (1 - \alpha)y \mid y \in D \cap \overline{\Lambda} \text{ and } 0 \leq \alpha \leq 1\}$ and $\Lambda \subset V(p)$.

**Property 2.** There exists a direction $d \neq 0$ such that, for any point $x \in D \cap \overline{\Lambda}$, the closed ray $\{x - \mu d \mid \mu \geq 0\}$ intersects $D$ and the open ray $\{x + \lambda d \mid \lambda > 0\}$ does not intersect $D$.

It is obvious that $D \cap \overline{\Lambda}$ is connected. As a direct result of Property 2, we have

**Lemma 2.1.** If Property 2 holds, $\Lambda \subset \{x + \lambda d \mid x \in D \cap \overline{\Lambda} \text{ and } 0 \leq \lambda\}$.

**Theorem 2.1.** Let $f$ be a mapping from $D$ to itself. If $f$ is continuous on $D$, there exists a point $x^* \in D$ such that $f(x^*) = x^*$.

**Proof.** Assume that Property 1 holds. Then, for each point $x \in \overline{\Lambda}$, there exists a unique point $q(x) \in D \cap \overline{\Lambda}$ and a unique number $0 \leq \alpha(x) < 1$ such that $x = \alpha(x)p + (1 - \alpha(x))q(x)$. For any point $x \in D$, let $q(x) = x$. We define $f(x) = f(q(x))$ for any point $x \in C$. Then $f(x) \in D$ for any point $x \in C$. In the following, we show that $f$ is continuous on $C$. For any point $\bar{x} \in \overline{\Lambda}$, let $x^k \in C$, $k = 1, 2, \ldots$, be a sequence converging to $\bar{x}$. Consider $q(x^k)$, $k = 1, 2, \ldots$. For any given $\epsilon > 0$, let $H(q(\bar{x}), \epsilon)$ be the union of $\{x \in D \mid \|x - q(\bar{x})\| < \epsilon\}$ and $\{y p + (1 - \gamma)y \mid y \in D \cap \overline{\Lambda}, \|y - q(\bar{x})\| < \epsilon, \text{ and } 0 \leq \gamma \leq 1\}$. Since $\Lambda \subset V(p)$, there exists a sufficiently small $\delta > 0$ such that

$$N(\bar{x}, \delta) = \{x \in C \mid \|x - \bar{x}\| < \delta\} \subset H(q(\bar{x}), \epsilon).$$

This implies that, for any sufficiently small $\delta > 0$, there exists a sufficiently small $\delta > 0$ satisfying that $\|q(x^k) - q(\bar{x})\| < \epsilon$ when $\|x^k - \bar{x}\| < \delta$. Thus, $q(x^k) \to q(\bar{x})$ as $k \to \infty$. Therefore, $f$ is continuous on $C$. From Brouwer’s fixed point theorem, we obtain that there exists a point $x^* \in C$ such that $f(x^*) = x^*$. Since $f(x^*) \in D$, hence, $x^* \in D$. The theorem follows.

Assume that Property 2 holds. Then, for each point $x \in \overline{\Lambda}$, there exists a unique point $h(x) \in D \cap \overline{\Lambda}$ and a unique number $\lambda(x) \geq 0$ such that $x = h(x) + \lambda(x)d$, where $h(x)$, in fact, is a unique solution of

$$\min_{y \in D \setminus \{x - \mu d \mid \mu \geq 0\}} \|x - y\|_2.$$

For any point $x \in D$, let $h(x) = x$. We define $f(x) = f(h(x))$ for any point $x \in C$. Then $f(x) \in D$ for any point $x \in C$. In the following, we show that $f$ is continuous on $C$. For any point $\bar{x} \in \overline{\Lambda}$, let $x^k \in C$, $k = 1, 2, \ldots$, be a sequence converging to $\bar{x}$. For any given $\epsilon > 0$, let $P(h(\bar{x}), \epsilon)$ be the union of $\{x \in D \mid \|x - h(\bar{x})\| < \epsilon\}$ and $\{y + \lambda d \mid y \in D \cap \overline{\Lambda}, \|y - h(\bar{x})\| < \epsilon, \text{ and } 0 \leq \lambda\}$. From Lemma 2.1, we obtain that there exists a sufficiently small $\delta > 0$ such that

$$N(\bar{x}, \delta) = \{x \in C \mid \|x - \bar{x}\| < \delta\} \subset P(h(\bar{x}), \epsilon).$$

This implies that, for any sufficiently small $\epsilon > 0$, there exists a sufficiently small $\delta > 0$ satisfying that $\|q(x^k) - q(\bar{x})\| < \epsilon$ when $\|x^k - \bar{x}\| < \delta$. Thus, $h(x^k) \to h(\bar{x})$ as $k \to \infty$. Therefore, $f$ is continuous on $C$. From Brouwer’s fixed point theorem, we obtain that there exists a point $x^* \in C$ such that $f(x^*) = x^*$. Since $f(x^*) \in D$, hence, $x^* \in D$. The theorem follows. □

**Theorem 2.2.** Let $F$ be a point-to-set mapping from $D$ to the set of nonempty convex subsets of $D$. If $F$ is upper semicontinuous on $D$, there exists a point $x^* \in D$ such that $F(x^*) = x^*$.

**Proof.** Following a similar argument to that in Theorem 2.1, one can easily derive the result of this theorem. □

Let $\Gamma = \bigcup_{i=1}^m A_i$, where $A_k$ is a nonempty and connected subset of $C$ satisfying that, for each point $x \in A_k$, $\dim(N(x, \delta, A_k)) = \dim(C)$ for any $\delta > 0$, and $A_i \cap A_j$ is either empty or contained in the boundary of $C$ for any $i \neq j$. Let $D$ denote the closure of $C \setminus \Gamma$ and $\overline{A_k}$ the closure of $A_k$. For $k = 1, 2, \ldots, m$, we assume that $A_k$ satisfies at least one of the following two properties.
Property 1a. There exists a point \( p^k \notin C \) such that any interior point of \( D \) is not contained in
\[
V(p^k) = \{ \alpha p^k + (1 - \alpha)y \mid y \in D \cap \overline{A}_k \text{ and } 0 \leq \alpha \leq 1 \}
\]
and \( \Lambda_k \subset V(p^k) \).

Property 2a. There exists a direction \( d^k \neq 0 \) such that, for any point \( x \in D \cap \overline{A}_k \), the closed ray \( \{x - \mu d^k \mid \mu \geq 0\} \) intersects \( D \) and the open ray \( \{x + \lambda d^k \mid \lambda > 0\} \) does not intersect \( D \).

As a corollary of Theorem 2.1, we have

**Corollary 2.2.** Let \( f \) be a mapping from \( D \) to itself. If \( f \) is continuous on \( D \), there exists a point \( x^* \in D \) such that \( f(x^*) = x^* \).

As a corollary of Theorem 2.2, we have

**Corollary 2.3.** Let \( F \) be a point-to-set mapping from \( D \) to the set of nonempty convex subsets of \( D \). If \( F \) is upper semicontinuous on \( D \), there exists a point \( x^* \in D \) such that \( x^* \in F(x^*) \).

Let \( D \) be a nonempty and connected subset of \( C \) satisfying that, for each point \( x \in \Lambda_k \), \( \dim(N(x, \delta, \Lambda_k)) = \dim(C) \) for any \( \delta > 0 \), and \( \Lambda_i \cap \Lambda_j \) is either empty or contained in the boundary of \( C \) for any \( i \neq j \). From Corollaries 2.2 and 2.3, we obtain

**Theorem 2.3.** Let \( f \) be a continuous mapping from \( D \) to itself. If \( \Lambda_k \) satisfies Property 1a or Property 2a for any \( k \), there exists a point \( x^* \in D \) such that \( f(x^*) = x^* \).

**Theorem 2.4.** Let \( F \) be an upper semicontinuous mapping from \( D \) to the set of nonempty convex subsets of \( D \). If \( \Lambda_k \) satisfies Property 1a or Property 2a for any \( k \), there exists a point \( x^* \in D \) such that \( x^* \in F(x^*) \).

### 3. Homotopy method for computing fixed points

Let \( D = \{ x \in \mathbb{R}^n \mid g_i(x) \leq 0, 1 \leq i \leq k_1 \} \) and \( D \cap \overline{A} = \{ x \in \mathbb{R}^n \mid h_j(x) \leq 0, 1 \leq j \leq l \} \), where \( g_i(x) \) is a convex function and \( h_j(x) \) is a nonlinear function. Since \( D = C \cap \Lambda \), hence, for convenience of the following discussions, we let \( \eta_i(x) = g_i(x) \) for \( 1 \leq i \leq k_1 \), and \( \eta_i + h_j(x) = h_j(x) \) for \( 1 \leq i \leq l \). Thus, \( D = \{ x \mid \eta_i(x) = 0, 1 \leq i \leq m \} \), where \( m = l + k_1 \). Let \( D^0 \) denote the interior of \( D \). Then, \( D^0 = \{ x \mid \eta(x) < 0 \} \), where \( \eta(x) = (\eta_1(x), \ldots, \eta_m(x)) \).

Let \( I_g(x) = \{ i \mid g_i(x) = 0, 1 \leq i \leq \{1, \ldots, k_1\} \}, I_h(x) = \{ i \mid h_i(x) = 0, i \in [1, \ldots, l]\} \), and \( I(x) = I_g(x) \cup I_h(x) \). Let \( \xi(x) = (\xi_1(x), \ldots, \xi_m(x)) \), where \( \xi_i(x) = \nabla \eta_i(x) \) as \( x \in D^0 \), and
\[
\xi_i(x) = \begin{cases} 
\frac{p - x}{\nabla g_i(x)}, & i \in I_g(x), \\
\frac{p - x}{\nabla h_i(x)}, & i \in I_h(x)
\end{cases}
\]
as \( x \in \partial D \).

We rewrite Property 1 as the following Condition 1.

**Condition 1.** For some \( p \notin C \), \( V(p) = \{ \alpha p + (1 - \alpha)x \mid x \in D \cap \overline{A} \text{ and } 0 \leq \alpha \leq 1 \} \) satisfies that \( D \cap \{ x + \lambda(p - x) \mid \lambda \geq 0 \} \), \( x \in D \cap \overline{A} \) = \{ x \}.

Given Condition 1, we have

**Lemma 3.1.** Let \( g_i(x) \) (\( 1 \leq i \leq k_1 \)) and \( h_j(x) \) (\( 1 \leq j \leq l \)) be sufficiently smooth functions from \( \mathbb{R}^n \) to \( R \), and \( F \) be a continuous mapping from \( D \subset \mathbb{R}^n \) to itself, i.e., \( F(D) \subset D \). Supposed that Condition 1 holds. Then, \( x^* \in D \) is a fixed point of \( F \) if and only if there exists a point \( y^* \in R^m_+ \) such that \( w^* = (x^*, y^*) \in D \times R^m_+ \) is a solution of the following system
\[
\begin{align*}
\begin{cases}
x - F(x) + \xi(x)y = 0,
Y \eta(x) = 0, \quad \eta(x) < 0, \quad y \geq 0,
\end{cases}
\end{align*}
\]
where \( Y = \text{diag}(y) \in R^{m \times m} \) and \( y \in R^m_+ \).

**Proof.** Suppose that \( w^* \) is a solution of (3.1).

(i) When \( x^* \in D^0, \eta(x^*) < 0 \), by the second equality of (3.1), we have \( y^* = 0 \). Hence, we get \( x^* = F(x^*) \).

(ii) When \( x^* \in \partial D \), we consider three cases: (1) \( i \in I_g(x^*) \); (2) \( i \in I_h(x^*) \); (3) \( i \in Ig(x^*) \cap Ih(x) \).
(1) If \( i \in I_g(x^*) \), then, by the definition of \( \xi_i(x) \) and the first equality of (3.1), we have
\[
x^* + \sum_{i \in I_g(x^*)} y_i^* \nabla g_i(x^*) = F(x^*).
\]
Since \( g_i(x) \) is a convex function and \( y_i^* \geq 0 \), hence, \( \{x^* + \sum_{i \in I_g(x^*)} y_i^* \nabla g_i(x^*)\} \cap (\partial C \setminus \partial A) = \{x^*\} \). Therefore, \( x^* = F(x^*) \in D \).

(2) If \( i \in I_h(x^*) \), then, by the definition of \( \xi_i(x) \) and the first equality of (3.1), we have
\[
x^* + \sum_{i \in I_h(x^*)} y_i^* (p - x^*) = F(x^*).
\]
Since \( p - x^* \neq 0 \) (because \( x^* \in C \) and \( p \notin C \)), \( F(D) \subset D \) and Condition 1, hence, \( y_i^* = 0 \). Therefore, \( x^* = F(x^*) \in D \).

(3) If \( i \in I_g(x) \cap I_h(x) \), i.e., \( g_i(x^*) = 0 \) and \( h_i(x^*) = 0 \) at \( x^* \), we have
\[
x^* + \sum_{i \in I(x^*)} y_i^* \xi_i(x^*) = F(x^*).
\]

Since \( D^0 \cap V(p) = \emptyset \) and \( F(D) \subset D \), hence, \( y_i^* = 0 \). Therefore, \( x^* = F(x^*) \in D \).

On the other hand, if \( x^* \) is a fixed point of \( F \) in \( D \), then, for \( y^* = 0 \), we have that \( (x^*, y^*) \) is a solution of (3.1). \( \square \)

In order to solve the system (3.1), we construct the following homotopy equation.
\[
H(w, w^0, t) = \left( (1 - t)(x - F(x) + \xi(x)y) + t(x - x^0), Y \eta(x) - tY \eta(x^0) \right) = 0, \tag{3.2}
\]
where \( w = (x, y) \in D \times R^m_+ \), \( w^0 = (x^0, y^0) \in D^0 \times X^m_+, \ Y^0 = \text{diag}(y^0) \in R^{m \times m} \), and \( \eta(x^0) = (\eta_1(x^0), \ldots, \eta_m(x^0))^T \in R^m \).

To prove the result of convergence in this paper, we also need the next condition.

**Condition 2.**

(a) \( F, g = (g_1, \ldots, g_k), \) and \( h = (h_1, \ldots, h_l) \) are sufficiently smooth mappings (\( F \) is at least twice continuously differentiable, and \( g, h \) are at least three times continuously differentiable);
(b) \( D^0 \) is nonempty and bounded;
(c) \( \{\xi_i(x) \mid i \in I(x)\} \) are linearly independent.

Let \( U \subset R^n \) be an open set, and \( \phi: U \rightarrow R^p \) be a \( C^\alpha (\alpha > \max\{0, n - p\}) \) mapping. We say that \( y \in R^p \) is a regular value of \( \phi \), if, for any \( x \in \phi^{-1}(y) \),
\[
\text{Rang}[\partial \phi(x)/\partial x] = R^p.
\]
As follows, we give a homotopy method for computing fixed points, based on Eq. (3.2).

**Theorem 3.2.** Suppose that Condition 1 and Condition 2 hold, and \( F(D) \subset D \). Then,

(1) (Existence) \( F(x) \) has a fixed point in \( D \).
(2) (Convergence) For almost all \( w^0 \in D^0 \times R^m_+ \), the homotopy system (3.2) determines a smooth curve \( \Gamma_{w^0} \subset D \times R^m \times (0, 1] \) starting from \((w^0, 1)\), the limit set \( T \times \emptyset \subset D \times R^m_+ \times \emptyset \) of \( \Gamma_{w^0} \) is nonempty, and the \( x \)-component of any point in \( T \) is a fixed point of \( F \) in \( D \).

For a given \( w^0 \in D \times R^m_+ \), we rewrite \( H(w, w^0, t) \) in (3.2) as \( H_{w^0}(w, t) \). Then, the zero point set of \( H_{w^0} \) is given by
\[
H_{w^0}^{-1}(0) = \{(w, t) \in D \times R^m \times (0, 1] \mid H_{w^0}(w, t) = 0\}. \tag{3.3}
\]
In the following, we first prove the following two results.

(1) For almost all \( w^0 \in D^0 \times R^m_+ \), \( 0 \) is a regular value of \( H_{w^0} : D \times R^m_+ \times (0, 1] \rightarrow R^{n+m} \).
(2) For a given \( w^0 \in D^0 \times R^m_+ \), if \( 0 \) is a regular value of \( H_{w^0} \), then \( \Gamma_{w^0} \) is a bounded curve in \( D \times R^m_+ \times (0, 1] \).

**Lemma 3.3.** If Condition 2 holds, then, for almost all \( w^0 \in D^0 \times R^m_+ \), \( 0 \) is a regular value of \( H_{w^0} : D \times R^m_+ \times (0, 1] \rightarrow R^{n+m} \), and \( H_{w^0}^{-1}(0) \) consists of some smooth curves, among which, there exists a smooth curve \( \Gamma_{w^0} \) that starts from \((w^0, 1)\).
Proof. Let $DH(w, w^0, t)$ denote the Jacobi matrix of $H(w, w^0, t)$, we have

$$DH(w, w^0, t) = \left( \frac{\partial H(w, w^0, t)}{\partial w}, \frac{\partial H(w, w^0, t)}{\partial w^0}, \frac{\partial H(w, w^0, t)}{\partial t} \right).$$

For any $w^0 \in D^0 \times R_m^{m+1}$ and $t \in (0, 1]$, we have that

$$\frac{\partial H(w, w^0, t)}{\partial w^0} = \begin{pmatrix} -tI & \eta'(x^0) \nabla y^0(x^0) \end{pmatrix},$$

where $I$ is an identity matrix, $G(x^0) = \text{diag}(\eta(x^0))$, and $\nabla \eta(x^0) = (\nabla \eta_1(x^0), \ldots, \nabla \eta_m(x^0))$. Then the determinant

$$\left| \frac{\partial H(w, w^0, t)}{\partial w^0} \right| = (-t)^{m+n} \prod_{i=1}^n \eta_i(x^0) \prod_{j=1}^l h_j(x^0) \neq 0.$$

Thus, $DH(w, w^0, \mu)$ is of the full row rank. By Parametric Sard’s Theory (see [1]) and Implicit Function Theory (see [12]), one can obtain that, for almost all $w^0 \in D^0 \times R_m^{m+1}$, $0$ is a regular value of $H_{w^0}(w, t)$ and $H_{w^0}^{-1}(0)$ consists of some smooth curves. Since

$$H_{w^0}(w^0, 1) = 0,$$

hence, there is a smooth curve $g_{w^0}$ in $H_{w^0}^{-1}(0)$ starting from $(w^0, 1)$. □

**Lemma 3.4.** Suppose that Condition 1 and Condition 2 hold. Then, for almost all given $w^0 \in D^0 \times R_m^{m+1}$, $g_{w^0}$ is a bounded curve in $D \times R_m^{m+1}$ if $0$ is a regular value of $H_{w^0}$. 

Proof. Assume that $g_{w^0} \subset D \times R_m^{m+1} \times (0, 1]$ is an unbounded curve. Since $D$ and $(0, 1]$ are bounded, hence there exist a sequence of points $(x^k, y^k, t_k) \in g_{w^0}$ and a nonempty index set $I^* = \{i \in [1, \ldots, m] \mid \lim_{k \to \infty} y^k_i = \infty\}$, where $y^k = (y^k_1, \ldots, y^k_m)^T$, such that $x^k \to x^*, y^k_i \to t^*,$ $x^k_i \to +\infty$ for $i \in I^*$. From the second equality of (3.2), i.e., $y^k_h(x^k) = t_k Y^0 \eta(x^k)$, we have $I^* = I(x^*)$ for all $t^* \in (0, 1)$ and $I^* \subset I(x^*)$ for $t^* = 0$. From the first equality of (3.2), we have

$$(1 - t_k)(x^k - F(x^k) + \xi(x^k)y^k) + t_k(x^k - x^0) = 0. \quad (3.4)$$

(1) When $t^* = 1$, rewrite (3.4) as

$$\sum_{i \in I^*} (1 - t_k)\xi_i(x^k)y^k_i + x^k - x^0 = (1 - t_k)\left[ -\sum_{i \notin I^*} y^k_i \xi_i(x^k) - (x^k - F(x^k)) + x^k - x^0 \right]. \quad (3.5)$$

Since $|x^k|$ and $|y^k_i|$ for $i \notin I^*$ are bounded, hence, as $k \to \infty$, by (3.5), we have

$$x^0 = x^* + \lim_{k \to \infty} \left[ \sum_{i \notin I^*} (1 - t_k) y^k_i \xi_i(x^k) \right]. \quad (3.6)$$

Thus, we come to the following three cases.

Case 1: If $i \in I^*_g(x^*)$, then, from the convexity of $g_i(x)$ and $\lim_{k \to \infty} (1 - t_k)y^k_i \geq 0$, we have that

$$x^0 = x^* + \sum_{i \in I^*_g(x^*)} \lim_{k \to \infty} (1 - t_k)y^k_i g_i(x^*)$$

is in the normal cone of $D$ at $x^*$. Therefore, $x^0 \in \partial D$. This contradicts $x^0 \in D^0$.

Case 2: If $i \in I^*_h(x^*)$, we have that

$$x^0 = x^* + \sum_{i \in I^*_h(x^*)} \lim_{k \to \infty} ((1 - t_k)y^k_i)(p - x^*). \quad (3.7)$$

From $\lim_{k \to \infty} (1 - t_k)y^k_i \geq 0$ and Condition 1, we have that $x^0 = x^*$. This is impossible, because $x^0 \in D^0$ and $x^* \in \partial D$.

Case 3: If $i \in I^*_g(x^*) \cap I^*_h(x^*)$, we have that

$$x^0 = x^* + \sum_{i \in I^*_g(x^*)} \left[ \lim_{k \to \infty} (1 - t_k)y^k_i \right] \nabla g_i(x^*) + \sum_{i \in I^*_h(x^*)} \left[ \lim_{k \to \infty} (1 - t_k)y^k_i \right] (p - x^*). \quad (3.8)$$

By Condition 1 and Condition 2(c), this contradicts $x^0 \in D^0$. 


(2) When $t^* \in [0, 1)$, rewrite (3.4) as
\[
(1 - t_k)\left(x^k - F(x^k) + \sum_{i \in I^t} \xi_i(x^k)y_i^k\right) + t_k(x^0 - x^0) = -(1 - t_k)\sum_{i \in I^t} \xi_i(x^k)y_i^k.
\]
Then, as $k \to +\infty$, the left-hand side of (3.9) is bounded and, from Condition 2, the right-hand side of (3.9) is unbounded. Thus, a contradiction occurs. Therefore, $T_{w^0}$ is bounded. □

**Proof of Theorem 3.2.** By Lemma 3.3, for almost all $w^0 \in D^0 \times R_{++}^m$, $0$ is a regular value of $H_{w^0}(w, t)$ and $H_{w^0}^{-1}(0)$ contains a smooth curve $\Gamma_{w^0}$ starting from $(w^0, 1)$. By the classification theory of one-dimensional smooth manifold (see [12]), $\Gamma_{w^0}$ must be diffeomorphic to a unit circle or a unit interval $(0, 1)$. Since the matrix
\[
\frac{\partial H_{w^0}(w^0, 1)}{\partial w} = \begin{pmatrix} I & Y^0\nabla \eta(x^0)^T \end{pmatrix}
\]
is nonsingular, $\Gamma_{w^0}$ must be diffeomorphic to $(0, 1)$. Let $(w^*, t^*)$ be an extreme point of $\Gamma_{w^0}$ other than $(w^0, 1)$. Then, $(w^*, t^*)$ must lie in $\partial (D \times R_{++}^m \times (0, 1))$, and one of the following four cases occurs:

(i) $(w^*, t^*) \in D^0 \times R_{++}^m \times \{1\}$;
(ii) $(w^*, t^*) \in \partial (D \times R_{++}^m \times (0, 1))$;
(iii) $(w^*, t^*) \in \partial (D \times R_{++}^m \times (0, 1))$;
(iv) $(w^*, t^*) \in D \times R_{++}^m \times \{0\}$.

Since the equation $H_{w^0}(w, t) = 0$ has only one solution $(w^0, 1)$ in $D^0 \times R_{++}^m \times \{1\}$, case (i) is impossible.

For cases (ii) and (iii), there must exist a subsequence of $(x^k, y^k, t_k) \subset \Gamma_{w^0}$ (without loss of generality, we assume that the sequence is convergent itself) such that, for some $1 \leq i \leq m$, $g_i(x^k) \to 0$, $h_i(x^k) \to 0$, or $\|y^k\| \to \infty$. When $\|y^k\| \to \infty$, this contradicts Lemma 3.4. When $g_i(x^k) \to 0$ or $h_i(x^k) \to 0$, by the second equation of (3.2), i.e., $Y^0\eta(x^k) = t_kY^0\eta(x^0)$, we have $\|y^k\| \to \infty$ (because $t_k \to t^* \neq 0$ and each element in $Y^0\eta(x^0)$ is nonzero), which also contradicts Lemma 3.4.

As a conclusion of the above results, only case (iv) occurs, and thus $w^*$ is a solution of (3.1). □

**Remark.**

(1) Under Property 1a, we will have the same theorem as Theorem 3.2 when $\Lambda$ is changed into $\Lambda_1$ and $p$ of $\xi_i(x)$ is changed into $p_i$.

(2) Our results are more general than one of [18]. For example, $D_1 = \{x \in R^2 \mid x_1^2 + x_2^2 \leq 49, \ 3 + x_1 - x_2^2 \leq 0\}$ and $D_2 = \{x \in R^2 \mid x_1^2 + x_2^2 \leq 49, \ (x_1 - 4)^2 + x_2^2 \geq 16, \ 3 + x_1 - x_2^2 \leq 0\}$ satisfy Property 1, and $D_3 = \{x \in R^2 \mid x_1^2 + x_2^2 \leq 49, \ (x_1 - 4)^2 + x_2^2 \geq 16, \ 3 + x_1 + x_2^2 \geq 0\}$ satisfies Property 2a. However, these sets do not satisfy the normal cone condition of the sets of [18].

4. Numerical examples

To numerically trace the homotopy pathway $\Gamma_{w^0}$ starting from $(w^0, 1)$ until $t \to 0$, we first parameterize $\Gamma_{w^0}$ as $\Gamma(s) = (w(s), t(s))^T$, and then differentiate $H_{w^0}(w(s), t(s)) = 0$ with the parameter $s$. By doing these, we come to

**Theorem 4.1.** Let Condition 2 hold and $(w(s), t(s))$ be a solution curve of the differentiable equation
\[
\frac{\partial H_{w^0}(w, t)}{\partial (w, t)} \begin{pmatrix} w(s) \\ t'(s) \end{pmatrix} = 0, \quad w(0) = w^0, \quad t(0) = 1.
\]
If there exists $s^* > 0$ such that $t(s^*) = 0$, then $x^* = x(s^*), y^* = y(s^*)$ is the solution of the system (3.2).

From Theorem 4.1, one can trace $\Gamma(s)$ by Euler–Newton method, and the standard procedure can be found in Allgower and Georg [1]. Applying the Euler–Newton procedure formulated in [1] with a slight modification, we have solved the following three examples. The behaviors of homotopy paths are shown in Figs. 1, 2 and 3. Computational results are given in Table 1, where $SP$ denotes the starting point, $IN$ the number of iterations, $V$ the value of $\|H_{w^0}(w^0, t_k)\|$ when the algorithm stops, and $x^*$ the final solution (i.e., fixed points).

**Example 4.1.** $F(x) = (x_1, -x_2)^T, \ D = \{x \in R^2 \mid x_1^2 + x_2^2 \leq 49, \ 3 + x_1 - x_2^2 \leq 0\}$.

**Example 4.2.** $F(x) = (x_1, -x_2)^T, \ D = \{x \in R^2 \mid x_1^2 + x_2^2 \leq 49, \ (x_1 - 4)^2 + x_2^2 \geq 16, \ 3 + x_1 - x_2^2 \leq 0\}$.

**Example 4.3.** $F(x) = (x_1, -x_2)^T, \ D = \{x \in R^2 \mid x_1^2 + x_2^2 \leq 49, \ (x_1 - 4)^2 + x_2^2 \geq 16, \ 3 + x_1 + x_2^2 \geq 0\}$.
Fig. 1. Two homotopy pathways $\Gamma_1$, $\Gamma_2$ from two initial points for Example 4.1.

Fig. 2. Three homotopy pathways $\Gamma_1$, $\Gamma_2$, $\Gamma_3$ from three initial points for Example 4.2.

Fig. 3. Three homotopy pathways $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ from three initial points for Example 4.3.
Table 1
Numerical results of Examples 4.1–4.3.

<table>
<thead>
<tr>
<th>Example</th>
<th>SP (\mathcal{V})</th>
<th>IN (V)</th>
<th>(x^*)</th>
<th>(f(x^*))</th>
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<td>2.608785e-11</td>
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References