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ORIGINAL ARTICLE

On MHD flow of an incompressible viscous fluid

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MAPLE 13

Abstract In this paper, we apply Homotopy Perturbation Method (HPM) to find the analytical solutions of nonlinear MHD flow of an incompressible viscous fluid through convergent or divergent channels in presence of a high magnetic field. The flow of an incompressible electrically conducting viscous fluid in convergent or divergent channels under the influence of an externally applied homogeneous magnetic field is studied both analytically and numerically. The graphs are presented to reveal the physical characteristics of flow by changing angles of the channel, Hartmann and Reynolds numbers.

MATHEMATICS SUBJECT CLASSIFICATION: 35K99; 35P05; 35P99

1. Introduction

The incompressible viscous fluid flow through convergent or divergent channels is one of the most applicable cases in many applications such as aerospace, chemical, civil, environmental, mechanical, and biomechanical engineering as well as in understanding rivers and canals. Jeffery [1] and Hamel [2] have carried out the mathematical formulations of this problem in 1915 and 1916, respectively. If we simplify Navier–Stokes equations in the particular case of two-dimensional flow through a channel with inclined walls, finally we can reach Jeffery–Hamel problem [3–6]. Jeffery–Hamel flows have been extensively studied by several authors and discussed in many textbooks, for example [7–11], and so forth. The study of electrically conducting viscous fluid that flows through convergent or divergent channels under the influence of an external magnetic field not only is fascinating theoretically but also finds applications in mathematical modeling of several industrial and biological systems. A possible practical application of the theory we envisage is in the field of industrial metal casting, the control of molten metal flows. Another area in which the theoretical study may be of interest is in the motion of liquid metals or alloys in the cooling systems of advanced nuclear reactors [12]. Clearly, the motion in the region with intersecting walls may represent a local transition between two parallel channels with different cross-sections, a widening or a contraction of the flow. The first recorded use of the word magnetohydrodynamics (MHD) is by Bansal [13]. The theory of MHD is inducing current in a moving conductive fluid in the presence of magnetic field which creates force on electrons of the conductive fluid and also changes the magnetic field itself. A survey of magnetohydrodynamics studies in the mentioned technological field can be found in [14]. The problem is basically an extension of classical Jeffery–Hamel flows of ordinary fluid mechanics to MHD. In the MHD solution an external magnetic field acts as a control parameter for both convergent

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Table 1 Values of α for $Re = 100$ and $\alpha = -2.5°$.

H	0	1000	2000	4000
a	−1.117418863	−0.9644432630	−0.8367326331	−0.6392026162

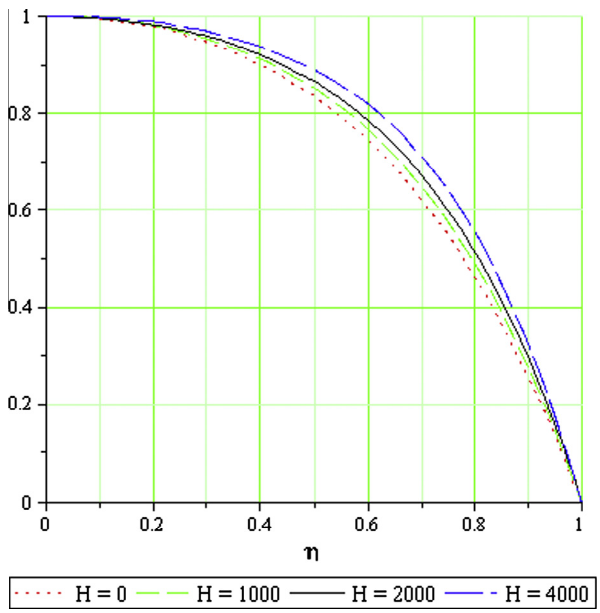

Figure 1 HPM solution for velocity is convergent channel for $Re = 100$ and $\alpha = -2.5°$.

Table 2 Values of α for $Re = 100$ and $\alpha = 2.5°$.

H	0	1000	2000	4000
a	−3.512069452	−3.011764524	−2.583264460	−1.912965835

and divergent channel flows. Here, besides the flow Reynolds number and the channel angular widths, at least an additional dimensionless parameter appears, namely, the Hartman number. Hence, a much larger variety of solutions than in the classical problem are expected. The inspiration of this paper is the extension of a relatively new technique which is called Homotopy Perturbation Method [15–17] to investigate the MHD flow through convergent or divergent channels in presence of a high magnetic field. The governing highly nonlinear equation of this problem is also solved numerically by shooting method, coupled with fourth-order Runge–Kutta scheme.

2. Mathematical formulation

Consider a system of cylindrical polar coordinates (r, θ, z), where the steady two-dimensional flow of an incompressible conducting viscous fluid from a source or sink at channel walls lie in planes and intersect in z-axis. The schematic diagram of problem is illustrated in [18]. Now we assumed that $u_\theta = 0$; it means that there are no changes with respect to z direction; thus the motion is purely in radial direction and merely depends on r and θ and there is no magnetic field along z-axis. The polar form of equation of continuity, Navier–Stokes and Maxwell's in reduce form is given as follows:

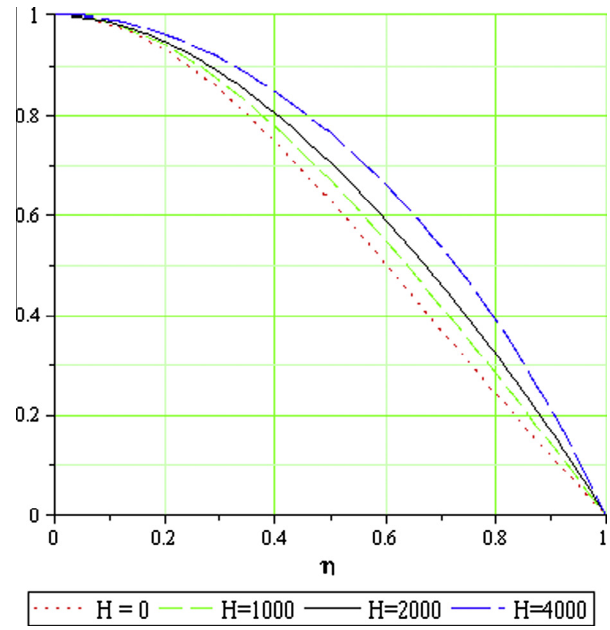

Figure 2 HPM solution for velocity is convergent channel for $Re = 100$ and $\alpha = 2.5°$.

$$\rho\frac{\partial}{r\partial r}[ru(r,\theta)] = 0, \tag{1}$$

$$u(r,\theta)\frac{\partial u(r,\theta)}{\partial r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + v\left[\frac{\partial^2 u(r,\theta)}{\partial r^2} + \frac{1}{r}\frac{\partial u(r,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u(r,\theta)}{\partial \theta^2} - \frac{u(r,\theta)}{r^2}\right] - \frac{\sigma B_0^2}{\rho r^2}u(r,\theta), \tag{2}$$

$$\frac{1}{\rho r}\frac{\partial P}{\partial \theta} - \frac{2v}{r^2}\frac{\partial u(r,\theta)}{\partial \theta} = 0, \tag{3}$$

where B_0 is the electromagnetic induction strength, σ the conductivity of the fluid, u the velocity along radial direction, P the fluid pressure, v the coefficient of kinematic viscosity, and ρ the fluid density.

Now from Eq. (1), we have

$$f(\theta) = ru(r,\theta). \tag{4}$$

Using $\eta = \frac{\theta}{\alpha}$, i.e., the dimensionless parameters, where α is the semiangle between the inclined walls

$$f(\eta) = \frac{f(\theta)}{f_{max}}, \tag{5}$$

substituting Eq. (5) into Eq. (2) and Eq. (3), we have

$$f'''(\eta) + 2\alpha Re f(\eta)f'(\eta) + (4 - H)\alpha^2 f'(\eta) = 0, \tag{6}$$

where $H = \sqrt{\frac{\sigma B_0^2}{\rho v}}$ is Hartmann number and Re is the Reynolds number is

$$Re = \frac{f_{max}\alpha}{v}\begin{cases} \text{divergent} - \text{channel} : \alpha > 0, & f_{max} > 0, \\ \text{convergent} - \text{channel} : \alpha < 0, & f_{max} < 0. \end{cases}$$

So we have the BCs

$$f(0) = 1, \quad f'(0) = 0, \quad f(1) = 0. \tag{7}$$

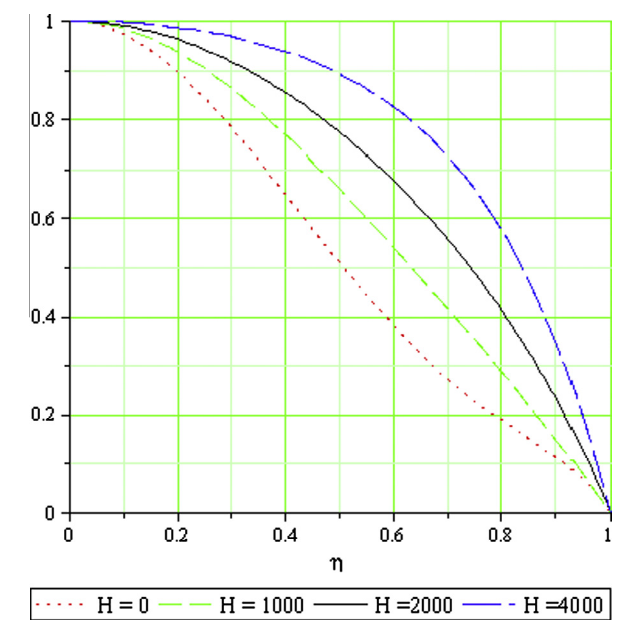

Figure 3 HPM solution for velocity is convergent channel for $Re = 100$ and $\alpha = 5°$.

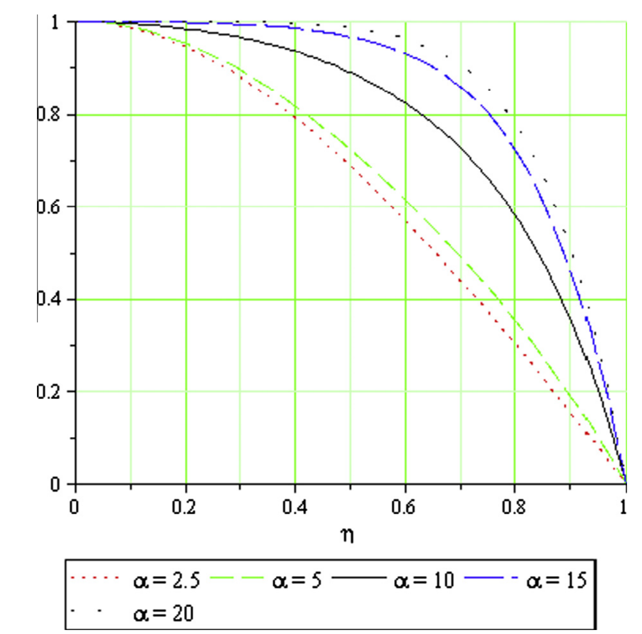

Figure 4 HPM solution for velocity is convergent channel for a positive, $Re = 100$ and $H = 1500$.

3. Analysis of Homotopy Perturbation Method (HPM)

To illustrate the basic concept of Homotopy Perturbation Method, consider the following nonlinear functional equation:

$$A(u) = f(r), \quad r \in \Omega, \tag{8}$$

with the boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r\partial\Omega \tag{9}$$

where A is the general functional operator, B the boundary operator, $f(r)$ the known analytic function, and $\partial\Omega$ is the boundary of the domain Ω.

The operator A is decomposed as $A = L + N$, where L is the linear and N is the nonlinear operator. Hence Eq. (8) can be written as follows:

$$L(u) + N(u) - f(r) = 0, \quad r \in \Omega. \tag{10}$$

We construct a Homotopy $v(r,p):\Omega \times [0,1] \to R$ satisfying

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] \\ = 0, \quad p[0,1], \quad r \in \Omega. \tag{11}$$

Hence

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \tag{12}$$

where u_0 is an initial approximation for the solution of (10). As

$$H(v,0) = L(v) - L(u_0) \text{ and } H(v,1) = A(v) - f(r), \tag{13}$$

It shows that $H(v,p)$ continuously traces an implicitly defined curve from a starting point $H(u_0,0)$ to a solution $H(v,1)$. The embedding parameter p increases monotonously from zero to one as the trivial linear part $L(u) = 0$ deforms continuously to the original problem $A(u) = f(r)$. The embedding parameter $p \in [0,1]$ can be considered as an expanding parameter to obtain

$$v = v_0 + pv_1 + p^2 v_2 + \cdots. \tag{14}$$

Table 3 Values of α for $Re = 100$ and $\alpha = 5°$.

H	0	1000	2000	4000
a	−5.458959462	−3.262595571	−1.854117794	−0.6557546191

Table 4 Values of a for α positive, $Re = 100$ and $H = 1500$.

α	2.5°	5°	10°	15°	20°
a	−2.788876071	−2.458506932	−0.6923394340	−0.1033788301	−0.01711080411

Table 5 Values of a for α negative, $Re = 100$ and $H = 1500$.

α	−2.5°	−5°	−10°	−15°	−20°
a	−0.8977494512	−0.306202513	−0.03917948224	−0.007564057560	−0.00201446237

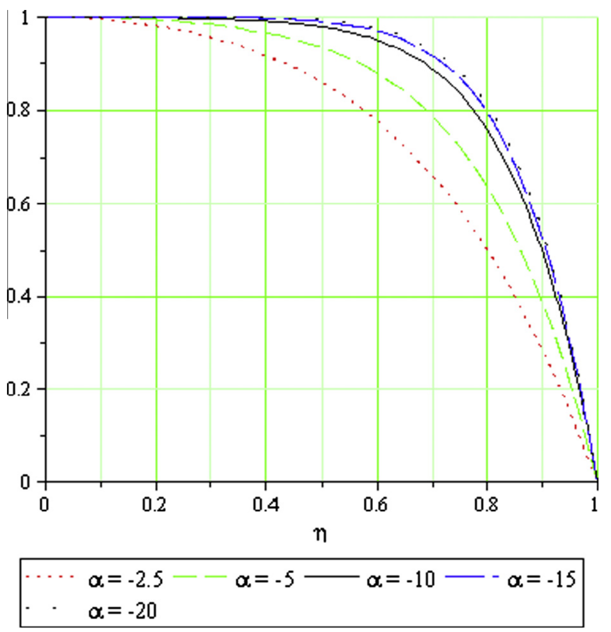

Figure 5 HPM solution for velocity is convergent channel for a negative, $Re = 100$ and $H = 1500$.

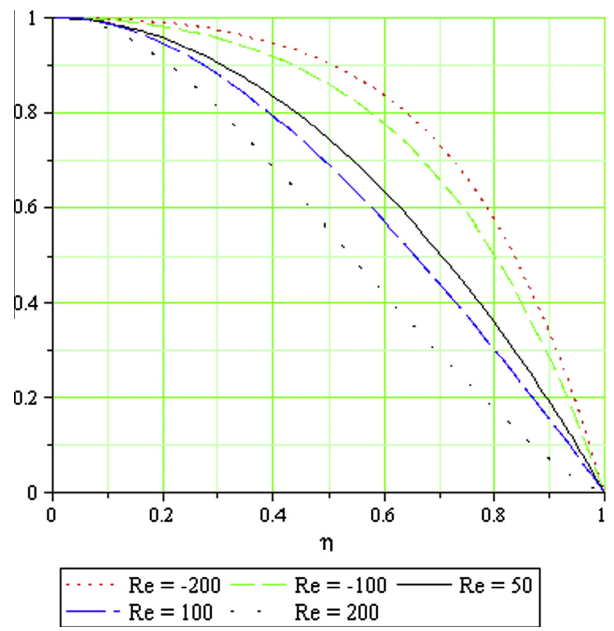

Figure 6 HPM solution for velocity is convergent channel for $\alpha = 2.5°$ and $H = 1500$.

The solution is obtained by taking the limit as p tends 0 to 1 in Eq. (14). Hence

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots. \tag{15}$$

4. Numerical simulation

In this section we compute an accurate analytical solution to the problem (6–7) using Homotopy Perturbation Method. Assuming $f = v$ the Eq. (6) we have

$$v'''(\eta) + 2\alpha Re v(\eta)v'(\eta) + (4 - H)\alpha^2 v'(\eta) = 0. \tag{16}$$

According to the Homotopy Perturbation Method we construct the homotopy in the form:

$$\begin{aligned} H(v,p) = {} & (1-p)[v'''(\eta) - f_0'''(\eta)] \\ & - p[v'''(\eta) + 2\alpha Re v(\eta)v'(\eta) + (4 - H)\alpha^2 v'(\eta)]. \end{aligned} \tag{17}$$

Substituting Eq. (14) into Eq. (16) and rearranging based on powers of p-terms, it is reduced to

$$v_0'''(\eta) - f_0'''(\eta) = 0,$$
$$v_1'''(\eta) + 2\alpha Re v_0(\eta)v_0'(\eta) + (4 - H)\alpha^2 v_0'(\eta) = 0,$$
$$\vdots$$
$$\vdots$$
$$\vdots$$

Solving the above equations

$$v_0(\eta) = 1 + a\frac{\eta^2}{2!},$$
$$v_1 = -\frac{1}{120}Re\alpha a^2\eta^6 - \frac{1}{12}Re\alpha a\eta^4 - \frac{1}{6}\alpha^2 a\eta^4 - \frac{1}{120}H\alpha^2 a\eta^4,$$
$$\vdots$$
$$\vdots$$
$$\vdots$$

As before, the solution of Eq. (6) when $p \to 1$ is as follows:

$$f(\eta) = v_0(\eta) + v_1(\eta) + v_2(\eta) + \cdots.$$

$$\begin{aligned} f(\eta) = {} & 1 + a\frac{\eta^2}{2!} - \frac{1}{120}Re\alpha a^2\eta^6 - \frac{1}{12}Re\alpha a\eta^4 - \frac{1}{6}\alpha^2 a\eta^4 \\ & + -\frac{1}{120}H\alpha^2 a\eta^4 + \frac{1}{10800}Re^2\alpha^2 a^3\eta^{10} + \frac{1}{560}Re^2\alpha^2 a^2\eta^8 \\ & + \cdots. \end{aligned}$$

5. Results and discussion

The objective of the present study was to apply the Homotopy Perturbation Method to obtain an explicit analytic solution of MHD flow through convergent or divergent channels in the presence of a high magnetic field. The magnetic field plays its role in no dimensional parameter, namely, the Hartmann number. If we fix Re number that reveals the fact that by increasing magnetic field the velocity profile becomes flat and

Table 6 Values of α for $\alpha = 2.5°$ and $H = 1500$.

Re	−200	−100	50	100	200
a	−0.5181038551	−0.8977494510	−2.108830241	−2.788876073	−4.704765922

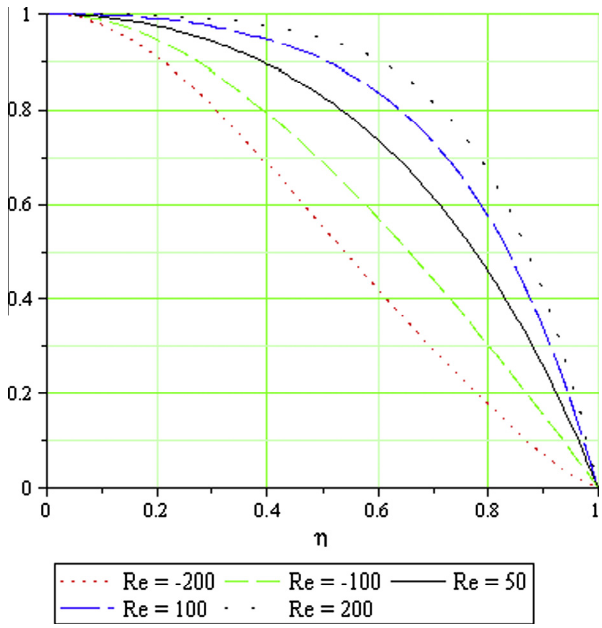

Figure 7 HPM solution for velocity is convergent channel for $\alpha = -2.5°$ and $H = 1500$.

thickness of boundary layer decreases. In fact magnetic field induces a force in opposite of the momentum's direction that stabilizes the velocity profile. We have Table 1.

Now we take inverse case of Fig. 1 in which we see by decreasing Hartman number, the velocity profile becomes flat and thickness of boundary layer decreases, and we have Table 2 (see Figs. 2 and 3).

We take $\alpha = 5°$ instead of $\alpha = 2.5°$ as discuss in Fig. 1, notice that velocity profile is more clear (see Table 3).

We examine if we increases α the effect of walls on fluid flow decreases when we move away from them which lead to an increase in velocity and the velocity profile in divergent channels, we have Tables 4 and 5).

If we decrease α the behavior in velocity profile is overturned as in Fig. 4. The velocity profile in convergent channels and we have (see Fig. 5).

Now we examine the behavior of velocity profile if *Re* varies, and fixed Hartmann numbers we obtained (see Table 6).

The reverse behavior of velocity profile is notice when we increase *Re*. We can infer from Figs. 6 and 7 for inflow regime, back flow is prevented in the case of convergent channels but is possible for large Reynolds numbers in the case of divergent channels. Figs. 6 and 7 show that there is a reverse condition for outflow regime.

The comparison between the numerical results and HPM solution (1st, 2nd, 3rd and 4th order approximate) for velocity when $Re = 100$ and $H = 1500$ is shown in Tables 7a and 7b.

6. Discussion

Homotopy Perturbation Method applied to find the approximate solutions of nonlinear MHD Flow of an incompressible viscous fluid through convergent or divergent channels in the presence of a high magnetic field. Tables 7a and 7b show the comparison of numerical method and the proposed technique

Table 7 Values of α for $\alpha = -2.5°$ and $H = 1500$.

Re	−200	−100	50	100	200
a	−4.704765923	−2.788876073	−1.192205819	−0.5181038541	−0.1989016484

Table 7a The comparison between the numerical results and HPM solution for velocity when $\alpha = 2.5°$, $Re = 100$ and $H = 1500$.

X	0th Order approximants	1st Order approximants	2nd Order approximants	3rd Order approximants	4th Order approximants	Numerical solution
0.00	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.05	0.99650	0.99650	0.99648	0.99652	0.99652	0.99652
0.10	0.98599	0.98606	0.98598	0.98612	0.98612	0.98612
0.15	0.96848	0.96883	0.96865	0.96897	0.96895	0.96895
0.20	0.94397	0.94505	0.94473	0.94529	0.94525	0.94525
0.25	0.91245	0.91506	0.91456	0.91542	0.91536	0.91536
0.30	0.87393	0.87928	0.87852	0.87974	0.87966	0.87966
0.35	0.82840	0.83817	0.83706	0.83871	0.83860	0.83860
0.40	0.77587	0.79227	0.79067	0.79279	0.79269	0.79265
0.45	0.71633	0.74210	0.73983	0.74247	0.74232	0.74230
0.50	0.64980	0.68822	0.68503	0.68824	0.68809	0.68804
0.55	0.57625	0.63115	0.62674	0.63056	0.63038	0.63032
0.60	0.49571	0.57132	0.56538	0.56984	0.56956	0.56956
0.65	0.40815	0.50912	0.50135	0.50643	0.50620	0.50612
0.70	0.31360	0.44478	0.43498	0.44063	0.44029	0.44029
0.75	0.21204	0.37837	0.36655	0.37264	0.37240	0.37229
0.80	0.10348	0.30974	0.29631	0.30257	0.30225	0.30224
0.85	−0.01209	0.23848	0.22443	0.23042	0.23012	0.23013
0.90	−0.13466	0.16389	0.15105	0.15609	0.15602	0.15589
0.95	−0.26424	0.08490	0.07625	0.07939	0.07939	0.07930
1.00	−0.40082	0.00000	−0.00000	0.00000	−0.00000	−0.00000

Table 7b The comparison between the numerical results and HPM solution for velocity when $\alpha = -2.5°$, $Re = 100$ and $H = 1500$.

x	0th Order approximants	1st Order approximants	2nd Order approximants	3rd Order approximants	4th Order approximants	Numerical solution
0.00	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.05	0.99868	0.99867	0.99886	0.99888	0.99887	0.99887
0.01	0.99470	0.99465	0.99539	0.99547	0.99546	0.99545
0.15	0.98808	0.98782	0.98951	0.98968	0.98966	0.98965
0.20	0.97881	0.97799	0.98103	0.98134	0.98131	0.98129
0.25	0.96689	0.96490	0.96972	0.97022	0.97016	0.97013
0.30	0.95232	0.94821	0.95526	0.95599	0.95591	0.95586
0.35	0.93510	0.92751	0.93723	0.93826	0.93808	0.93808
0.40	0.91523	0.90232	0.91515	0.91654	0.91638	0.91629
0.45	0.89272	0.87210	0.88843	0.89021	0.89001	0.88989
0.50	0.86755	0.83625	0.85634	0.85859	0.85832	0.85817
0.55	0.83974	0.79410	0.81809	0.82085	0.82030	0.82031
0.60	0.80927	0.74495	0.77274	0.77601	0.77558	0.77534
0.65	0.77616	0.68802	0.71923	0.72299	0.72213	0.72215
0.70	0.74040	0.62250	0.65636	0.66052	0.65986	0.65951
0.75	0.70199	0.54756	0.58281	0.58723	0.58600	0.58601
0.80	0.66093	0.46231	0.49714	0.50155	0.50062	0.50014
0.85	0.61722	0.36586	0.39776	0.40182	0.40028	0.40028
0.90	0.57086	0.25728	0.28299	0.28624	0.28523	0.28473
0.95	0.52186	0.13563	0.15103	0.15294	0.15219	0.15283
1.00	0.47020	0.00000	−0.00000	−0.00000	0.00000	0.00000

for different values of η, α and $Re = 100$. Table 7 shows the accuracy of the Homotopy Perturbation Method. In this investigation, the flow of an incompressible electrically conducting viscous fluid in convergent or divergent channels under the influence of an externally applied homogeneous magnetic field is studied both numerically and analytically. The behavior of the HPM solution is in good agreement with the numerical simulation. Graphical results are presented to investigate the influence of the angles of the channel, Hartmann number and Reynolds number on the velocity profiles. In addition, the reliability of the method and the reduction in the size of computational domain give this method a wider applicability.

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