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Almost complex structures on $(n-1)$-connected $2n$-manifolds

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ABSTRACT

Let M be a closed $(n-1)$-connected $2n$-dimensional smooth manifold with $n \geqslant 3$. In terms of the system of invariants for such manifolds introduced by Wall, we obtain necessary and sufficient conditions for M to admit an almost complex structure.

1. Introduction

First we introduce some notations. For a topological space X, let $Vect_{\mathbb{C}}(X)$ (resp. $Vect_{\mathbb{R}}(X)$) be the set of isomorphic classes of complex (resp. real) vector bundles on X, and let $r: Vect_{\mathbb{C}}(X) \to Vect_{\mathbb{R}}(X)$ be the real reduction, which induces the real reduction homomorphism $\tilde{r}: \widetilde{K}(X) \to \widetilde{KO}(X)$ from the reduced *KU*-group to the reduced *KO*-group of X. For a map $f: X \to Y$ between topological spaces X and Y, denote by $f_u^*: \widetilde{K}(Y) \to \widetilde{K}(X)$ and $f_o^*: \widetilde{KO}(Y) \to \widetilde{KO}(X)$ the induced homomorphisms. We will denote by $\tilde{\xi} \in \widetilde{K}(X)$ (resp. $\widetilde{KO}(X)$) the stable class of $\xi \in Vect_{\mathbb{C}}(X)$ (resp. $Vect_{\mathbb{R}}(X)$) (cf. Hilton [9, p. 62]).

Let M be a $2n$-dimensional smooth manifold with tangent bundle TM. We say that M admits an *almost complex structure* (resp. a *stable almost complex structure*) if $TM \in \operatorname{Im} r$ (resp. $\widetilde{TM} \in \operatorname{Im} \tilde{r}$). Clearly, M admits an almost complex structure implies that M admits a stable almost complex structure. It is a classical topic in geometry to determine which M admits an almost complex structure. See for instance Wu [20], Ehresmann [6], Dessai [4], Heaps [8], Müller and Geiges [13], Thomas [18], Sutherland [17], etc. In this paper we determine those closed $(n-1)$-connected $2n$-dimensional smooth manifolds M with $n \geqslant 3$ that admit an almost complex structure.

Throughout this paper, M will be a closed oriented $(n-1)$-connected $2n$-dimensional smooth manifold with $n \geqslant 3$. In [19], C.T.C. Wall assigned to each M a system of invariants as follows.

1) $H = H^n(M;\mathbb{Z}) \cong Hom(H_n(M;\mathbb{Z});\mathbb{Z}) \cong \bigoplus_{j=1}^k \mathbb{Z}$, the cohomology group of M, with k the n-th Betti number of M.

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2) $I: H \times H \to \mathbb{Z}$, the intersection form of M which is unimodular and n-symmetric, defined by

$$I(x, y) = \langle x \cup y, [M] \rangle,$$

where the homology class $[M]$ is the orientation class of M.

3) A map $\alpha: H_n(M; \mathbb{Z}) \to \pi_{n-1}(SO_n)$ that assigns each element $x \in H_n(M; \mathbb{Z})$ to the characteristic map $\alpha(x)$ for the normal bundle of the embedded n-sphere S^n_x representing x.

These invariants satisfy the relation (cf. Wall [19, Lemma 2])

$$\alpha(x + y) = \alpha(x) + \alpha(y) + I(x, y)\partial\iota_n, \tag{1.1}$$

where ∂ is the boundary homomorphism in the exact sequence

$$\cdots \to \pi_n\big(S^n\big) \xrightarrow{\partial} \pi_{n-1}(SO_n) \xrightarrow{S} \pi_{n-1}(SO_{n+1}) \to \cdots \tag{1.2}$$

of the fiber bundle $SO_n \hookrightarrow SO_{n+1} \to S^n$, and $\iota_n \in \pi_n(S^n)$ is the class of the identity map.

Denote by $\nu = S \circ \alpha: H_n(M; \mathbb{Z}) \to \pi_{n-1}(SO_{n+1}) \cong \widetilde{KO}(S^n)$ the composition map, then from (1.1) and (1.2)

$$\nu = S \circ \alpha \in H^n\big(M; \widetilde{KO}\big(S^n\big)\big) = Hom\big(H_n(M; \mathbb{Z}); \widetilde{KO}\big(S^n\big)\big) \tag{1.3}$$

can be viewed as an n-dimensional cohomology class of M, with coefficient in $\widetilde{KO}(S^n)$. It follows from Kervaire [11, Lemma 1.1] and Hirzebruch index theorem [10, p. 86] that the Pontrjagin classes $p_j(M) \in H^{4j}(M; \mathbb{Z})$ of M can be expressed in terms of the cohomology class ν and the index τ of the intersection form I (when n is even) as follows (cf. Wall [19, pp. 179–180]).

Lemma 1.1. *Let M be a closed oriented $(n-1)$-connected $2n$-dimensional smooth manifold with $n \geqslant 3$. Then*

1) *if* $n \equiv 2 \pmod 4$

$$p_{n/2}(M) = \frac{n!}{2^n(2^{n-1} - 1)B_{n/2}}\tau,$$

2) *if* $n \equiv 0 \pmod 4$

$$p_j(M) = \begin{cases} \pm a_{n/4}(n/2 - 1)!\nu, & j = n/4, \\ \frac{a^2_{n/4}}{2}((n/2 - 1)!)^2\{1 - \frac{(2^{n/2-1}-1)^2}{2^{n-1}-1}\binom{n}{n/2}\frac{B^2_{n/4}}{B_{n/2}}\}I(\nu, \nu) + \frac{n!}{2^n(2^{n-1}-1)B_{n/2}}\tau, & j = n/2, \end{cases}$$

where

$$a_{n/4} = \begin{cases} 1, & n \equiv 0 \pmod 8, \\ 2, & n \equiv 4 \pmod 8, \end{cases}$$

B_m is the m-th Bernoulli number.

Now we can state the main results as follows.

Theorem 1. *Let M be a closed oriented $(n-1)$-connected $2n$-dimensional smooth manifold with $n \geqslant 3$, ν be the cohomology class defined in* (1.3), *τ be the index of the intersection form I (when n is even). Then the necessary and sufficient conditions for M to admit a stable almost complex structure are*:

1) $n \equiv 2, 3, 5, 6, 7 \pmod 8$, *or*
2) *if* $n \equiv 0 \pmod 8$: $\nu \equiv 0 \pmod 2$ *and* $\frac{B_{n/2} - B_{n/4}}{B_{n/2}B_{n/4}} \cdot \frac{n\tau}{2^n} \equiv 0 \pmod 2$,
3) *if* $n \equiv 4 \pmod 8$: $\frac{B_{n/2} + B_{n/4}}{B_{n/2}B_{n/4}} \cdot \frac{\tau}{2^{n-2}} \equiv 0 \pmod 2$,
4) *if* $n \equiv 1 \pmod 8$: $\nu = 0$.

Theorem 2. *Let M be a closed oriented $(n-1)$-connected $2n$-dimensional smooth manifold with $n \geqslant 3$, ν be the cohomology class defined in* (1.3), *k be the n-th Betti number, I be the intersection form, and $p_j(M)$ be the Pontrjagin classes of M as in Lemma* 1.1. *Then M admits an almost complex structure if and only if M admits a stable almost complex structure and one of the following conditions are satisfied*:

1) *if* $n \equiv 0 \pmod 4$: $4p_{n/2}(M) - I(p_{n/4}(M), p_{n/4}(M)) = 8(k + 2)$,
2) *if* $n \equiv 2 \pmod 8$: *there exists an element* $x \in H^n(M; \mathbb{Z})$ *such that* $x \equiv \nu \pmod 2$ *and*

$$I(x, x) = \frac{2(k + 2) + p_{n/2}(M)}{((n/2 - 1)!)^2},$$

Table 1
Real reduction $\tilde{r}: \widetilde{K}(S^m) \to \widetilde{KO}(S^m)$.

m (mod 8)	$\widetilde{K}(S^m)$	$\widetilde{KO}(S^m)$	$\tilde{r}: \widetilde{K}(S^m) \to \widetilde{KO}(S^m)$
0	$\mathbb{Z}\tilde{\omega}^m_{\mathbb{C}}$	$\mathbb{Z}\tilde{\omega}^m_{\mathbb{R}}$	$\tilde{r}(\tilde{\omega}^m_{\mathbb{C}}) = 2\tilde{\omega}^m_{\mathbb{R}}$
1	0	$(\mathbb{Z}/2)\tilde{\omega}^m_{\mathbb{R}}$	$\tilde{r} = 0$
2	$\mathbb{Z}\tilde{\omega}^m_{\mathbb{C}}$	$(\mathbb{Z}/2)\tilde{\omega}^m_{\mathbb{R}}$	$\tilde{r}(\tilde{\omega}^m_{\mathbb{C}}) = \tilde{\omega}^m_{\mathbb{R}}$
4	$\mathbb{Z}\tilde{\omega}^m_{\mathbb{C}}$	$\mathbb{Z}\tilde{\omega}^m_{\mathbb{R}}$	$\tilde{r}(\tilde{\omega}^m_{\mathbb{C}}) = \tilde{\omega}^m_{\mathbb{R}}$
6	$\mathbb{Z}\tilde{\omega}^m_{\mathbb{C}}$	0	$\tilde{r} = 0$
3, 5, 7	0	0	$\tilde{r} = 0$

Table 2
Real reduction $\tilde{r}: \widetilde{K}(M) \to \widetilde{KO}(M)$.

n (mod 8)	$\widetilde{K}(M)$	$\widetilde{KO}(M)$	$\tilde{r}: \widetilde{K}(M) \to \widetilde{KO}(M)$
0	$\mathbb{Z}\tilde{\xi} \oplus \bigoplus_{j=1}^k \mathbb{Z}\tilde{\eta}_j$	$\mathbb{Z}\tilde{\gamma} \oplus \bigoplus_{j=1}^k \mathbb{Z}\tilde{\zeta}_j$	$\tilde{r}(\tilde{\xi}) = 2\tilde{\gamma}$, $\tilde{r}(\tilde{\eta}_j) = 2\tilde{\zeta}_j$
1	$\mathbb{Z}\tilde{\xi}$	$(\mathbb{Z}/2)\tilde{\gamma} \oplus \bigoplus_{j=1}^k (\mathbb{Z}/2)\tilde{\zeta}_j$	$\tilde{r}(\tilde{\xi}) = \tilde{\gamma}$
2	$\mathbb{Z}\tilde{\xi} \oplus \bigoplus_{j=1}^k \mathbb{Z}\tilde{\eta}_j$	$\mathbb{Z}\tilde{\gamma} \oplus \bigoplus_{j=1}^k (\mathbb{Z}/2)\tilde{\zeta}_j$	$\tilde{r}(\tilde{\xi}) = \tilde{\gamma}$, $\tilde{r}(\tilde{\eta}_j) = \tilde{\zeta}_j$
4	$\mathbb{Z}\tilde{\xi} \oplus \bigoplus_{j=1}^k \mathbb{Z}\tilde{\eta}_j$	$\mathbb{Z}\tilde{\gamma} \oplus \bigoplus_{j=1}^k \mathbb{Z}\tilde{\zeta}_j$	$\tilde{r}(\tilde{\xi}) = 2\tilde{\gamma}$, $\tilde{r}(\tilde{\eta}_j) = \tilde{\zeta}_j$
5	$\mathbb{Z}\tilde{\xi}$	$(\mathbb{Z}/2)\tilde{\gamma}$	$\tilde{r}(\tilde{\xi}) = \tilde{\gamma}$
6	$\mathbb{Z}\tilde{\xi} \oplus \bigoplus_{j=1}^k \mathbb{Z}\tilde{\eta}_j$	$\mathbb{Z}\tilde{\gamma}$	$\tilde{r}(\tilde{\xi}) = \tilde{\gamma}$, $\tilde{r}(\tilde{\eta}_j) = 0$
3, 7	$\mathbb{Z}\tilde{\xi}$	0	$\tilde{r} = 0$

3) *if* $n \equiv 6 \pmod 8$: *there exists an element* $x \in H^n(M;\mathbb{Z})$ *such that* $I(x,x) = (2(k+2) + p_{n/2}(M))/((n/2-1)!)^2$,
4) *if* $n \equiv 1 \pmod 4$: $2((n-1)!) \mid (2-k)$,
5) *if* $n \equiv 3 \pmod 4$: $(n-1)! \mid (2-k)$.

Remark 1.2. i) Since the rational numbers $\frac{B_{n/2}-B_{n/4}}{B_{n/2}B_{n/4}} \cdot \frac{n\tau}{2^n}$ and $\frac{B_{n/2}+B_{n/4}}{B_{n/2}B_{n/4}} \cdot \frac{\tau}{2^{n-2}}$ in Theorem 1 can be viewed as 2-adic integers (see the proof of Theorem 1), it makes sense to take congruent classes modulo 2.

ii) In the cases 2) and 3) of Theorem 2, when the conditions are satisfied, the almost complex structure on M depends on the choice of x.

This paper is arranged as follows. In Section 2 we obtain presentations for the groups $\widetilde{KO}(M)$, $\widetilde{K}(M)$ and determine the real reduction $\tilde{r}: \widetilde{K}(M) \to \widetilde{KO}(M)$ accordingly. In Section 3 we determine the expression of $\widetilde{TM} \in \widetilde{KO}(M)$ with respect to the presentation of $\widetilde{KO}(M)$ obtained in Section 2. With these preliminary results, Theorems 1 and 2 are established in Section 4.

2. The real reduction $\tilde{r}: \widetilde{K}(M) \to \widetilde{KO}(M)$

According to Wall [19], M is homotopic to a CW complex $(\bigvee_{\lambda=1}^k S^n_\lambda) \cup_f \mathbb{D}^{2n}$, where k is the n-th Betti number of M, $\bigvee_{\lambda=1}^k S^n_\lambda$ is the wedge sum of n-spheres which is the n-skeleton of M and $f \in \pi_{2n-1}(\bigvee_{\lambda=1}^k S^n_\lambda)$ is the attaching map of $\mathbb{D}^{2n}$ which is determined by the intersection form I and the map α (cf. Duan and Wang [5, Lemma 3]). We will denote by $i: \bigvee_{\lambda=1}^k S^n_\lambda \to M$ the inclusion map of the n-skeleton of M, $p: M \to S^{2n}$ the map collapsing the n-skeleton $\bigvee_{\lambda=1}^k S^n_\lambda$ to the base point and $t_j: \bigvee_{\lambda=1}^k S^n_\lambda \to S^n_j$ the map collapsing $\bigvee_{\lambda \neq j} S^n_\lambda$ to the base point.

Let $\mathbb{Z}\beta$ (resp. $(\mathbb{Z}/2)\beta$) be the infinite cyclic group (resp. finite cyclic group of order 2) generated by β. Recall that the generators $\tilde{\omega}^m_{\mathbb{C}}$ (resp. $\tilde{\omega}^m_{\mathbb{R}}$) of the cyclic group $\widetilde{K}(S^m)$ (resp. $\widetilde{KO}(S^m)$) with $m > 0$ can be so chosen such that the real reduction $\tilde{r}: \widetilde{K}(S^m) \to \widetilde{KO}(S^m)$ can be summarized as in Table 1 (cf. Mimura and Toda [16, Theorem 6.1, p. 211]).

Then we get that:

Lemma 2.1. *Let M be a closed oriented $(n-1)$-connected $2n$-dimensional smooth manifold with $n \geqslant 3$. Then the presentations of the groups $\widetilde{K}(M)$ and $\widetilde{KO}(M)$ as well as the real reduction $\tilde{r}: \widetilde{K}(M) \to \widetilde{KO}(M)$ can be given as in Table 2, where the generators $\tilde{\xi}$, $\tilde{\eta}_j$, $\tilde{\gamma}$, $\tilde{\zeta}_j$, $1 \leqslant j \leqslant k$, satisfy*:

$$\begin{cases} \tilde{\xi} = p_u^*(\tilde{\omega}^{2n}_{\mathbb{C}}), & i_u^*(\tilde{\eta}_j) = t_{ju}^*(\tilde{\omega}^n_{\mathbb{C}}); \\ \tilde{\gamma} = p_o^*(\tilde{\omega}^{2n}_{\mathbb{R}}), & i_o^*(\tilde{\zeta}_j) = t_{jo}^*(\tilde{\omega}^n_{\mathbb{R}}). \end{cases}$$

Proof. By the naturality of the Puppe sequence, for any $h \in \mathbb{Z}$, we have the exact ladder (2.1), where the horizontal homomorphisms $\Sigma^h p_u^*$, $\Sigma^h p_o^*$, $\Sigma^h i_u^*$, $\Sigma^h i_o^*$ and $\Sigma^h f_u^*$, $\Sigma^h f_o^*$ are induced by $\Sigma^h p$, $\Sigma^h i$ and $\Sigma^h f$ respectively, and where Σ denotes the suspension.

$$\begin{array}{ccccccccccc} \cdots \longrightarrow & \widetilde{K}(S^{2n+h}) & \xrightarrow{\Sigma^h p_u^*} & \widetilde{K}(\Sigma^h M) & \xrightarrow{\Sigma^h i_u^*} & \widetilde{K}(\bigvee_{\lambda=1}^k S_\lambda^{n+h}) & \xrightarrow{\Sigma^h f_u^*} & \widetilde{K}(S^{2n+h-1}) & \longrightarrow \cdots \\ & \downarrow{\scriptstyle \tilde{r}} & & \downarrow{\scriptstyle \tilde{r}} & & \downarrow{\scriptstyle \tilde{r}} & & \downarrow{\scriptstyle \tilde{r}} & \\ \cdots \longrightarrow & \widetilde{KO}(S^{2n+h}) & \xrightarrow{\Sigma^h p_o^*} & \widetilde{KO}(\Sigma^h M) & \xrightarrow{\Sigma^h i_o^*} & \widetilde{KO}(\bigvee_{\lambda=1}^k S_\lambda^{n+h}) & \xrightarrow{\Sigma^h f_o^*} & \widetilde{KO}(S^{2n+h-1}) & \longrightarrow \cdots \end{array} \tag{2.1}$$

Recall that the group $\pi_{2n-1}(\bigvee_{j=1}^k S_j^n)$ can be decomposed as (cf. the Hilton–Milnor theorem [21, p. 511]):

$$\pi_{2n-1}\left(\bigvee_{j=1}^k S_j^n\right) \cong \bigoplus_{j=1}^k \pi_{2n-1}\left(S_j^n\right) \bigoplus_{1 \leqslant i<j \leqslant k} \pi_{2n-1}\left(S_{ij}^{2n-1}\right),$$

where $S_{ij}^{2n-1} = S^{2n-1}$, the group $\pi_{2n-1}(S_j^n)$ is embedded in $\pi_{2n-1}(\bigvee_{j=1}^k S_j^n)$ by the natural inclusion, and the group $\pi_{2n-1}(S_{ij}^{2n-1})$ is embedded by composition with the Whitehead product of certain elements in $\pi_n(\bigvee_{j=1}^k S_j^n)$. Hence by Duan and Wang [5, Lemma 3], the attaching map f can be decomposed accordingly as:

$$f = \sum_{j=1}^k f_j + g,$$

where

$$f_j \in \operatorname{Im} J \subset \pi_{2n-1}\left(S^n\right)$$

J being the J-homomorphism (cf. Whitehead [21, p. 504]) and

$$g \in \bigoplus_{1 \leqslant i<j \leqslant k} \pi_{2n-1}(S_{ij}^{2n-1}).$$

Note that $\tilde{r} \circ c = 2 : \widetilde{KO}(X) \to \widetilde{KO}(X)$, where $c : \widetilde{KO}(X) \to \widetilde{K}(X)$ is the complexification. Then by the exact ladder (2.1) and the Bott periodicity theorem [3], the fact of Table 2 when $n \not\equiv 2 \pmod 8$ follows from Table 1 and Adams [1, Proposition 7.1] while the fact of Table 2 when $n \equiv 2 \pmod 8$ follows from Table 1, Adams [1, Propositions 7.1, 7.19] and the Bott sequence (cf. Kishimoto [12, Proposition 4.1]). □

Remark 2.2. All the KU-groups and KO-groups of M can be deduced easily from the exact ladder (2.1).

Remark 2.3. Since the induced homomorphisms $i^* : H^n(M;\mathbb{Z}) \to H^n(\bigvee_{\lambda=1}^k S_\lambda^n;\mathbb{Z})$ and $p^* : H^{2n}(S^{2n};\mathbb{Z}) \to H^{2n}(M;\mathbb{Z})$ are both isomorphisms, and the generator $\tilde{\omega}_{\mathbb{C}}^{2n} \in \widetilde{K}(S^{2n})$ can be chosen such that its n-th Chern class $c_n(\tilde{\omega}_{\mathbb{C}}^{2n}) = (n-1)!$ (cf. Hatcher [7, p. 101]), from the naturality of the Chern class, we get

$$c_n(\tilde{\xi}) = (n-1)!, \qquad c_{n/2}(\tilde{\xi}) = 0.$$

Similarly, when n is even, $\tilde{\eta}_j$, $1 \leqslant j \leqslant k$, can be chosen such that

$$c_{n/2}\left(\sum_{j=1}^k x_j \tilde{\eta}_j\right) = (n/2-1)!(x_1, x_2, \dots, x_k) \in H^n(M;\mathbb{Z}),$$

where $x_j \in \mathbb{Z}$ for all $1 \leqslant j \leqslant k$ (since $H^n(M;\mathbb{Z}) \cong \bigoplus_{j=1}^k \mathbb{Z}$, we can write an element $x \in H^n(M;\mathbb{Z})$, under the isomorphism i^*, as the form $(x_1, x_2, \dots, x_k)$).

Remark 2.4. As in Remark 2.3, if we write the cohomology class ν as $(\nu_1, \dots, \nu_k) \in H^n(M; \widetilde{KO}(S^n))$, where

$$\nu_j \in \widetilde{KO}\left(S^n\right) \cong \begin{cases} \mathbb{Z}, & n \equiv 0 \pmod 4, \\ \mathbb{Z}/2, & n \equiv 1, 2 \pmod 8, \\ 0, & \text{others}, \end{cases}$$

then since the tangent bundle of sphere is stably trivial, it follows that

$$i_o^*(\widetilde{TM}) = \sum_{j=1}^k \nu_j t_{jo}^*\left(\tilde{\omega}_{\mathbb{R}}^n\right).$$

3. The stable tangent bundle of *M*

Denote by $\dim_{\mathbb{C}}\alpha$ the dimension of $\alpha \in Vect_{\mathbb{C}}(M)$. When $n \equiv 0 \pmod 4$, we set

$$\hat{A}(M) = \langle \hat{\mathfrak{A}}(M), [M] \rangle,$$
$$\hat{A}_{\mathbb{C}}(M) = \langle ch(TM \otimes \mathbb{C}) \cdot \hat{\mathfrak{A}}(M), [M] \rangle,$$
$$\hat{A}_{\nu}(M) = \left\langle ch\left(\sum_{j=1}^{k} \nu_j \eta_j\right) \cdot \hat{\mathfrak{A}}(M), [M] \right\rangle,$$

where ch denotes the Chern character, and $\hat{\mathfrak{A}}(M)$ is the $\mathfrak{A}$-class of M (cf. Atiyah and Hirzebruch [2]). It follows from the differentiable Riemann–Roch theorem (cf. Atiyah and Hirzebruch [2]) that $\hat{A}(M)$, $\hat{A}_{\mathbb{C}}(M)$ and $\hat{A}_{\nu}(M)$ are all integers. In particular, $\hat{A}_{\nu}(M)$ is even when $\nu \equiv 0 \pmod 2$.

Using the notation above, we get

Lemma 3.1. *Let M be a closed oriented $(n-1)$-connected $2n$-dimensional smooth manifold with $n \geqslant 3$. Then the stable tangent bundle $\widetilde{TM}$ of M can be expressed by the generators $\tilde{\gamma}$, $\tilde{\zeta}_j$, $1 \leqslant j \leqslant k$, of $\widetilde{KO}(M)$ as follows*:

$$\widetilde{TM} = \begin{cases} \ell\tilde{\gamma} + \sum_{j=1}^{k} \nu_j \tilde{\zeta}_j, & n \equiv 0, 1, 2, 4 \pmod 8, \\ \ell\tilde{\gamma}, & n \equiv 6 \pmod 8, \\ 0, & n \equiv 3, 5, 7 \pmod 8, \end{cases}$$

where

$$\ell \begin{cases} = \hat{A}_{\mathbb{C}}(M) + (\sum_{j=1}^{k} a_{n/4}\nu_j \dim_{\mathbb{C}} \eta_j - 2n)\hat{A}(M) - a_{n/4}\hat{A}_{\nu}(M), & n \equiv 0 \pmod 4, \\ = -\frac{1}{2} p_{n/2}(M)/(n-1)!, & n \equiv 2 \pmod 4, \\ \in \mathbb{Z}/2, & n \equiv 1 \pmod 8. \end{cases}$$

Remark 3.2. In the case $n \equiv 1 \pmod 8$, the generators of $\widetilde{KO}(M)$ can be good chosen such that $\ell = 0 \in \mathbb{Z}/2$ (see the proof below).

Proof. Case $n \equiv 0 \pmod 8$. By Remark 2.4, we may suppose that

$$\widetilde{TM} = \ell\tilde{\gamma} + \sum_{j=1}^{k} \nu_j \tilde{\zeta}_j \in \widetilde{KO}(M),$$

where $\ell \in \mathbb{Z}$. Hence from $\tilde{r} \circ c = 2$ and Table 2, we have

$$c(\widetilde{TM}) = \widetilde{TM \otimes \mathbb{C}} = \ell\tilde{\xi} + \sum_{j=1}^{k} \nu_j \tilde{\eta}_j \in \widetilde{K}(M).$$

Then by the definition of stable equivalence, we have

$$TM \otimes \mathbb{C} \oplus \varepsilon^s \cong \ell\xi \oplus \bigoplus_{j=1}^{k} \nu_j \eta_j \oplus \varepsilon^t,$$

for some $s, t \in \mathbb{Z}$ satisfying

$$s - t = \ell \cdot \dim_{\mathbb{C}} \xi + \sum_{j=1}^{k} \nu_j \dim_{\mathbb{C}} \eta_j - 2n,$$

where ε^j is the trivial complex vector bundle of dimension j. Thus we have

$$\hat{A}_{\mathbb{C}}(M) = -\left(\ell \cdot \dim_{\mathbb{C}} \xi + \sum_{j=1}^{k} \nu_j \dim_{\mathbb{C}} \eta_j - 2n\right)\hat{A}(M) + \left\langle ch\left(\ell\xi + \sum_{j=1}^{k} \nu_j \eta_j\right) \cdot \hat{\mathfrak{A}}(M), [M] \right\rangle,$$

that is

$$\ell = \hat{A}_{\mathbb{C}}(M) + \left(\sum_{j=1}^{k} \nu_j \dim_{\mathbb{C}} \eta_j - 2n\right)\hat{A}(M) - \hat{A}_{\nu}(M).$$

Cases $n \equiv 2, 4, 6 \pmod 8$ can be proved by the same way as above.

Case $n \equiv 1 \pmod 4$. From Milnor and Kervaire [14, Lemma 1] and Adams [1, Theorem 1.3], we get that $\nu = 0$ implies $\widetilde{TM} = 0 \in \widetilde{KO}(M)$. Then the proof of this case and Remark 3.2 is trivial.

Case $n \equiv 3 \pmod 4$ is trivial. □

4. Almost complex structure on *M*

We are now ready to prove Theorem 1 and Theorem 2.

Proof of Theorem 1. Cases 1) and 2) $n \equiv 0 \pmod 4$. In these cases, we get that (cf. Wall [19, pp. 179–180])

$$\hat{A}(M) = -\frac{B_{n/2}}{2(n!)} p_{n/2}(M) + \frac{1}{2}\left\{\frac{B_{n/4}^2}{4((n/2)!)^2} + \frac{B_{n/2}}{2(n!)}\right\} I\big(p_{n/4}(M), p_{n/4}(M)\big),$$
$$\hat{\mathfrak{A}}(M) = 1 - \frac{B_{n/4}}{2((n/2)!)} p_{n/4}(M) + \hat{A}(M),$$
$$ch(TM \otimes \mathbb{C}) = 2n + (-1)^{n/4+1} \frac{p_{n/4}(M)}{(n/2-1)!} + \frac{I(p_{n/4}(M), p_{n/4}(M)) - 2p_{n/2}(M)}{2((n-1)!)}.$$

Hence by Lemma 1.1 we have

$$\hat{A}_{\mathbb{C}}(M) = 2n\left\{1 + \frac{1}{B_{n/2}} + \frac{(2^{n-1}-1)}{(2^{n/2}-1)^2} \cdot \frac{(-1)^{n/4}B_{n/2} - B_{n/4}}{B_{n/2}B_{n/4}}\right\}\hat{A}(M) + \frac{1}{(2^{n/2}-1)^2} \cdot \frac{(-1)^{n/4}B_{n/2} - B_{n/4}}{B_{n/2}B_{n/4}} \cdot \frac{n\tau}{2^n}. \tag{4.1}$$

Moreover since B_m can be written as the form $B_m = b_m/(2c_m)$ (cf. Milnor [15, p. 284]), where c_m and b_m are odd integers. Then multiply each side of (4.1) by $(2^{n/2}-1)^2 \cdot b_{n/2} \cdot b_{n/4}$, we get that

$$\begin{aligned}\big(2^{n/2}-1\big)^2 b_{n/2} b_{n/4} \hat{A}_{\mathbb{C}}(M) = 2n\big\{&\big(2^{n/2}-1\big)^2 \cdot b_{n/2} \cdot b_{n/4} + 2\big(2^{n/2}-1\big)^2 b_{n/4} c_{n/2} \\ &+ 2\big(2^{n-1}-1\big)\big((-1)^{n/4} b_{n/2} c_{n/4} - b_{n/4} c_{n/2}\big)\big\}\hat{A}(M) \\ &+ 2\big((-1)^{n/4} b_{n/2} c_{n/4} - b_{n/4} c_{n/2}\big)\frac{n\tau}{2^n}.\end{aligned}$$

Since $\hat{A}_{\mathbb{C}}(M)$ and $\hat{A}(M)$ are integers and $(2^{n/2}-1)^2 \cdot b_{n/2} \cdot b_{n/4}$ is an odd integer, it follows that $\frac{(-1)^{n/4}B_{n/2} - B_{n/4}}{B_{n/2}B_{n/4}} \cdot \frac{n\tau}{2^n}$ is a 2-adic integer, and hence

$$\hat{A}_{\mathbb{C}}(M) \equiv 0 \pmod 2 \quad \Longleftrightarrow \quad \frac{(-1)^{n/4}B_{n/2} - B_{n/4}}{B_{n/2}B_{n/4}} \cdot \frac{n\tau}{2^n} \equiv 0 \pmod 2.$$

Then by combining these facts with Lemmas 2.1 and 3.1, one verifies the results in these cases.

Cases 3) and 4) $n \not\equiv 0 \pmod 4$ are trivial. □

To prove Theorem 2, we need the following lemma (cf. Sutherland [17, Theorem 1.1] or Thomas [18, Theorem 1.7]).

Lemma 4.1. *Let N be a closed smooth $2n$-manifold. Then N admits an almost complex structure if and only if it admits a stable almost complex structure $\tilde{\beta}$ satisfying $c_n(\tilde{\beta}) = e(N)$, where $e(N)$ is the Euler class of N.*

Proof of Theorem 2. Firstly, it follows from Lemma 4.1 that M admits an almost complex structure if and only if there exists an element $\tilde{\beta} \in \widetilde{K}(M)$ such that

$$\begin{cases} \tilde{r}(\tilde{\beta}) = \widetilde{TM} \in \widetilde{KO}(M), \\ c_n(\tilde{\beta}) = e(M). \end{cases} \tag{4.2}$$

Secondly, if there exists an element $\tilde{\beta} \in \widetilde{K}(M)$ such that $\tilde{r}(\tilde{\beta}) = \widetilde{TM} \in \widetilde{KO}(M)$, then we have the following identity (cf. Milnor [15, p. 177]):

$$\left(\sum_j (-1)^j c_j(\tilde{\beta})\right) \cdot \left(\sum_j c_j(\tilde{\beta})\right) = \sum_j (-1)^j p_j(M). \tag{4.3}$$

Now we prove Theorem 2 case by case.

Case 1) $n \equiv 0 \pmod 4$. In this case $e(M) = k+2$. From Lemma 4.1 we know that M admits an almost complex structure if and only if there exists an element $\tilde{\beta} \in \widetilde{K}(M)$ such that (4.2) is satisfied. Now (4.3) becomes

$$\left(1 + c_{n/2}(\tilde{\beta}) + c_n(\tilde{\beta})\right) \cdot \left(1 + c_{n/2}(\tilde{\beta}) + c_n(\tilde{\beta})\right) = 1 + (-1)^{n/4} p_{n/4}(M) + p_{n/2}(M),$$

it follows that

$$c_{n/2}(\tilde{\beta}) = (-1)^{n/4} p_{n/4}(M)/2,$$

hence

$$c_n(\tilde{\beta}) = p_{n/2}(M)/2 - I\left(p_{n/4}(M), p_{n/4}(M)\right)/8.$$

Therefore from (4.2) we get that, M admits an almost complex structure if and only if M admits a stable almost complex structure and satisfies

$$4p_{n/2}(M) - I\left(p_{n/4}(M), p_{n/4}(M)\right) = 8(k+2).$$

Case 2) $n \equiv 2 \pmod 8$. In this case $e(M) = k+2$. Set $\tilde{\beta} = \ell\tilde{\xi} + \sum_{j=1}^{k} x_j \tilde{\eta}_j \in \widetilde{K}(M)$ where $\ell \in \mathbb{Z}$ is the integer as in Lemma 3.1 and $x_j \in \mathbb{Z}$, $1 \leqslant j \leqslant k$, are the integers such that $x_j \equiv \nu_j \pmod 2$. Then from Lemma 2.1, we know that $\tilde{r}(\tilde{\beta}) = \widetilde{TM}$. Hence by (4.2), we see that M admits an almost complex structure if and only if

$$\begin{cases} \tilde{\beta} = \ell\tilde{\xi} + \displaystyle\sum_{j=1}^{k} x_j \tilde{\eta}_j \in \widetilde{K}(M), \\ c_n(\tilde{\beta}) = e(M). \end{cases}$$

Let $x = (x_1, x_2, \ldots, x_k) \in H^n(M;\mathbb{Z})$. Then by Remark 2.3

$$c_{n/2}(\tilde{\beta}) = (n/2-1)!x.$$

Now (4.3) is

$$\left(1 - c_{n/2}(\tilde{\beta}) + c_n(\tilde{\beta})\right) \cdot \left(1 + c_{n/2}(\tilde{\beta}) + c_n(\tilde{\beta})\right) = 1 - p_{n/2}(M),$$

therefore

$$c_n(\tilde{\beta}) = \left(I\left(c_{n/2}(\tilde{\beta}), c_{n/2}(\tilde{\beta})\right) - p_{n/2}(M)\right)/2 = \left\{\left((n/2-1)!\right)^2 I(x,x) - p_{n/2}(M)\right\}/2.$$

Thus it follows from (4.2) that M admits an almost complex structure if and only if there exists an element $x \in H^n(M;\mathbb{Z})$ such that

$$\begin{cases} x \equiv \nu \pmod 2, \\ I(x,x) = \left(2(k+2) + p_{n/2}(M)\right)/\left((n/2-1)!\right)^2. \end{cases}$$

Case 3) $n \equiv 6 \pmod 8$. The proof is similar to the proof of case 2).

Case 4) $n \equiv 1 \pmod 4$. Now $e(M) = 2-k$. From (4.2), Lemmas 2.1, 3.1, Remarks 3.2 and 2.3, we see that M admits an almost complex structure if and only if

$$\begin{cases} \nu = 0, \\ \tilde{\beta} = 2a\tilde{\xi}, \\ 2a(n-1)! = 2-k, \end{cases}$$

for some $a \in \mathbb{Z}$. Hence by Lemmas 2.1 and 3.1, M admits an almost complex structure if and only if M admits a stable almost complex structure and

$$2(n-1)! \mid (2-k).$$

Case 5) $n \equiv 3 \pmod 4$. The proof is similar to the proof of case 4). □

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References

[1] J.F. Adams, On the groups $J(X)$, IV, Topology 5 (1966) 21–71.
[2] M.F. Atiyah, F. Hirzebruch, Riemann–Roch theorems for differentiable manifolds, Bull. Amer. Math. Soc. 65 (1959) 276–281.
[3] R. Bott, The stable homotopy of the classical groups, Ann. of Math. 70 (1959) 313–337.
[4] A. Dessai, Some remarks on almost and stable almost complex manifolds, Math. Nachr. 192 (1998) 159–172.
[5] H. Duan, S. Wang, The degrees of maps between manifolds, Math. Z. 244 (2003) 67–89.
[6] C. Ehresmann, Sur les variétés presque complexes, in: Proc. Int. Cong. Math., vol. II, 1950, pp. 412–419.
[7] A. Hatcher, Vector bundles and K-theory, Version 2.1, available http://www.math.cornell.edu/~hatcher/VBKT/VBpage.html, 2009.
[8] T. Heaps, Almost complex structures on eight- and ten-dimensional manifolds, Topology 9 (1970) 111–119.
[9] P. Hilton, General Cohomology Theory and K-Theory, London Math. Soc. Lecture Note Ser., Cambridge University Press, London, 1971.
[10] F. Hirzebruch, Topological Methods in Algebraic Geometry, Classics Math., Springer-Verlag, Berlin, Heidelberg, New York, 1978.
[11] M.A. Kervaire, A note on obstructions and characteristic classes, Amer. J. Math. 81 (1959) 773–784.
[12] D. Kishimoto, *KO*-theory of complex Stiefel manifolds, J. Math. Kyoto Univ. 44 (3) (2004) 669–674.
[13] S. Müller, H. Geiges, Almost complex structures on 8-manifolds, Enseign. Math. 46 (2000) 95–107.
[14] J. Milnor, M. Kervaire, Bernoulli numbers, homotopy groups, and a theorem of Rohlin, in: Milnor, Collected Papers of John Milnor, III, American Mathematical Society, Providence, RI, 2007, pp. 243–247.
[15] J. Milnor, J.D. Stasheff, Characteristic Classes, Ann. of Math. Stud., vol. 76, Princeton University Press, Princeton, 1974.
[16] M. Mimura, H. Toda, Topology of Lie Groups, I and II, American Mathematical Society, Providence, RI, 1991.
[17] W.A. Sutherland, A note on almost complex and weakly complex structures, J. Lond. Math. Soc. 40 (1965) 705–712.
[18] E. Thomas, Complex structures on real vector bundles, Amer. J. Math. 89 (1967) 887–908.
[19] C.T.C. Wall, Classification of $(n-1)$-connected $2n$-manifolds, Ann. of Math. 75 (1962) 163–189.
[20] W.T. Wu, Sur les espaces fibres, Actual. Scient. Ind., No. 1183, 1952.
[21] G.W. Whitehead, Elements of Homotopy Theory, Grad. Texts in Math., vol. 61, Springer-Verlag, Berlin, Heidelberg, New York, 1978.