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A new approach for solving Duffing equations involving both integral and non-integral forcing terms

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Duffing equation; Legendre wavelet; Operational matrix of derivative; Integral and non-integral forcing terms

Abstract In this paper a Legendre wavelet operational matrix of derivative (LWOM) is used to solve the Duffing equation involving both integral and non-integral forcing terms with separated boundary conditions. This operational matrix method together with Gaussian quadrature formula converts the given Duffing equation into system of algebraic equations, which indeed makes computation of solution easier. The applicability and simplicity of the proposed method is demonstrated by some examples and comparison with other recent methods. It is to be noted that, to the best of our knowledge, no wavelet based method applied for solving Duffing equations so far.

1. Introduction
The general form of Duffing equation involving both integral and non-integral forcing terms with separated boundary conditions is given by

\[ u''(x) + \sigma u'(x) + f(x, u, u') + \int_0^x k(x, s, u(s))ds = 0, \quad 0 \leq x \leq 1 \]  

(1)

\[ p_0 u(0) - q_0 u'(0) = a, \quad p_1 u(1) + q_1 u'(1) = b \]  

(2)

where \( f: [0, 1] \times \mathbb{R}^2 \to \mathbb{R}, k: [0, 1] \times [0, 1] \times \mathbb{R} \to \mathbb{R} \) are continuous functions and \( \sigma \in \mathbb{R} - \{0\}, p_0, p_1, q_0, q_1, a, b \in \mathbb{R} \).

The Duffing equation is a well-known nonlinear equation of applied science, and its mathematical model is used as a powerful tool to describe many physical and practical phenomena such as classical oscillator in chaotic phenomena, periodic orbit extraction, nonlinear mechanical oscillators, and prediction of diseases [1–5]. Solutions of Duffing equation have been studied through variety of numerical methods by many researchers [6–10]. A generalized quasilinearization technique has been investigated in [11] for the solution of Duffing equation involving both integral and non-integral forcing terms with separated boundary conditions. The analytic approximation of the forced Duffing equation with integral boundary conditions has been investigated in [12] through quasilinearization technique. This method provides a sequence of approximate solutions, converges monotonically and quadratically to the unique solution. Yao [13] presented an iterative reproducing kernel method for solving Duffing equation given by Eqs. (1) and (2). In this method exact solution is represented...
in the form of series. Geng [14] developed an improved variational iteration method for solving (1) and (2). This improved method avoids unnecessary repeated computation of unknown parameters in the initial solution.

In the recent years, wavelets theory is one of the growing and predominantly new methods in the area of mathematical and engineering research. It has been applied in wide range of engineering sciences and mathematical sciences in thriving manner for solving variety of linear and non-linear differential and partial differential equations due to they build a connection with fast numerical algorithms [15–17], this is due to wavelets admit the exact representation of a variety of functions and operators. Recently, one can see the application of Legendre wavelets for solving variety of problems involving both engineering and applied sciences in [18–22].

In this paper, we shall present a method based on the Legendre wavelet operational matrix of derivative for solving Duffing equation involving both integral and non-integral forcing terms with separated boundary conditions. We have adopted this method to solve Duffing equation not only due to its emerging application of but also due to its greater convergence region. It is to be noted that, to the best of our knowledge, no wavelet based method applied for solving Duffing equations so far.

The rest of the paper is organized as follows: Properties of Legendre wavelets and its operational matrix of derivative are introduced in Section 2. In Section 3 applicability of the proposed method for solving dumping equations is described. The numerical experiments are presented in Sections 4 and 5 ends with brief conclusion.

2. Properties of Legendre wavelets and its operational matrix of derivative

A family of functions constituted by wavelets, constructed from dilation and translation of a single function called mother wavelet. When the parameters a of dilation and b of translation vary continuously, following are the family of continuous wavelets [23]

$$\psi_{a,b}(x) = |a|^{-1/2} \psi\left(\frac{x - b}{a}\right), \quad a, b \in \mathbb{R}, a \neq 0.$$  

If the parameters a and b are restricted to discrete values as $a = a_0^k$, $b = nb_0a_0^k$, $a_0 > 1$, $b_0 > 0$ and n, and k are positive integers, following are the family of discrete wavelets:

$$\psi_{k,n}(x) = |a_0|^{k/2} \psi(a_0^k x - nb_0).$$  

where $\psi_{k,n}(x)$ form a wavelet basis for $L^2(\mathbb{R})$. In particular, when $a_0 = 2$, and $b_0 = 1$, $\psi_{k,n}(x)$ forms an orthonormal basis [23].

Legendre wavelets $\psi_{n,m}(x) = \psi(k, \hat{n}, m, x)$ have four arguments; $\hat{n} = 2n - 1, n = 1, 2, 3, \ldots, 2^k - 1$, k can assume any positive integer, m is the order for Legendre polynomials and t is the normalized time. They are defined on the interval [0, 1] as

$$\psi_{n,m}(x) = \begin{cases} \sqrt{m + \frac{1}{2}} P_m(2^k x - \hat{n}), & \text{for } \frac{-1}{2} \leq t < \frac{1}{2}, \\ 0, & \text{otherwise,} \end{cases}$$  

where $m = 0, 1, \ldots, M - 1$ and $n = 1, 2, 3, \ldots, 2^k - 1$ [24,25]. The coefficient $\sqrt{m + \frac{1}{2}}$ is for orthonormality, the dilation parameter is $a = 2^{-k}$ and translation parameter is $b = n2^{-k}$.

$P_n(x)$ are the well-known Legendre polynomials of order m defined on the interval [−1, 1], and can be determined with the aid of the following recurrence formulae:

$$P_0(x) = 1, \quad P_1(x) = x,$$

$$P_{m+1}(x) = \left(\frac{2m+1}{m+1}\right)xP_m(x) - \left(\frac{m}{m+1}\right)P_{m-1}(x), \quad m = 1, 2, 3, \ldots.$$  

The Legendre wavelet series representation of the function $f(x)$ defined over [0, 1] is given by

$$f(x) = \sum_{a=1}^{2^L-1} \sum_{n=0}^{\infty} c_{an}\psi_{a,n}(x),$$  

where $c_{an} = \langle f(x), \psi_{a,n}(x) \rangle$, in which $\langle \cdot \rangle$ denotes the inner product. If the infinite series in Eq. (3) is truncated, then Eq. (3) can be written as

$$f(x) \approx \sum_{a=1}^{2^L-1} \sum_{n=0}^{M-1} c_{an}\psi_{a,n}(x) = C^T \Psi(x).$$  

where C and $\Psi(x)$ are $2^L \times 1$ matrices given by

$$C = [c_{10}, c_{11}, \ldots, c_{1M-1}, c_{20}, c_{21}, \ldots, c_{2M-1}, \ldots, c_{2^{l-1}0}, c_{2^{l-1}1}, \ldots, c_{2^{l-1}M-1}]$$

$$\Psi(x) = [\psi_{10}(x), \psi_{11}(x), \ldots, \psi_{1M-1}(x), \psi_{20}(x), \psi_{21}(x), \ldots, \psi_{2M-1}(x), \ldots, \psi_{2^L-10}(x), \psi_{2^L-11}(x), \ldots, \psi_{2^L-1M-1}(x)]^T.$$  

The derivative of the vector $\Psi(x)$ defined in [21] can be expressed by

$$\frac{d \Psi(x)}{dx} = D \Psi(x).$$  

where $D$ is the $2^k(M + 1) \times 2^k(M + 1)$ operational matrix of derivative given by

$$D = \begin{bmatrix} F & 0 & \cdots & 0 \\ 0 & F & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & F \end{bmatrix}$$  

in which $F$ is $(M + 1) \times (M + 1)$ matrix and its $(i,j)$th element is defined as follow

$$F_{ij} = \begin{cases} 2^{k-1} \sqrt{(2i-1)(2j-1)}, & \text{if } i = 1 \text{ and } (i+j) \text{ odd,} \\ 0, & \text{otherwise} \end{cases}$$  

By using Eq. (6) the operational matrix for $n$th derivative can be derived as

$$\frac{d^n \Psi(x)}{dx^n} = D^n \Psi(x).$$  

where $D^n$ is the $n$th power of matrix $D$ and this operational matrix of derivative has been successfully applied in [22] for optimal control in a convective-diffusive fluid problem.

For $M = 2$, $k = 0$, the operational matrix of derivative is given by
3. Solution of Duffing equations involving both integral and non-integral forcing terms

Consider the Duffing equation given in (1) and (2). Using Legendre wavelet approximations we have let

\[ u(x) = C^T \Psi(x) \]  

(10)

By using Eqs. (9) and (10), we have

\[ u(x) = C^T D^2 \Psi(x) \]  

(11)

Therefore Eq. (1) becomes

\[ C^T D^2 \Psi(x) + \sigma C^T D \Psi(x) + f(x, C^T \Psi(x), C^T D \Psi(x)) + \int_0^x k(x, s, C^T \Psi(s)) ds = 0 \]  

(12)

By means of the transformation

\[ s = \frac{x}{2} (z + 1) \]  

(13)

and by using Gaussian quadrature rule, we have

\[ \int_0^x k(x, s, C^T \Psi(s)) ds \approx \frac{x}{2} \sum_{i=1}^{N} \omega_i k\left(\frac{x}{2} (z + 1), C^T \Psi\left(\frac{x}{2} (z + 1)\right)\right) \]  

(14)

Here the weights \( \omega_i \) can be calculated with the help of the formula

\[ \omega_i = \int_0^1 \prod_{j \neq i} \frac{z_j - z_i}{z_i - z_j} \, dz \]  

(15)

Now apply Eqs. (14), (15) into (12), we have

\[ C^T D^2 \Psi(x) + \sigma C^T D \Psi(x) + f(x, C^T \Psi(x), C^T D \Psi(x)) + \frac{x}{2} \sum_{i=1}^{N} \omega_i k\left(\frac{x}{2} (z + 1), C^T \Psi\left(\frac{x}{2} (z + 1)\right)\right) = 0 \]  

(16)

Also, initial and boundary conditions from Eq. (2) yields

\[ p_0 C^T \Psi(0) - q_0 C^T D \Psi(0) = a, \]

\[ p_1 C^T \Psi(1) + q_1 C^T D \Psi(1) = b \]  

(17)

To find the solution \( u(x) \), we first collocate Eq. (16) into \( 2^k - 1 \) \( M \) points at \( x \), by taking suitable collocation points as following

\[ x_i = \cos\left(\frac{(2i + 1)\pi}{2^k M}\right), \quad i = 1, 2, \ldots, 2^k - 1 \]  

(18)

These equations together with Eq. (17) generate \( 2^k - 1 \) \( M + 2 \) nonlinear equations which can be solved using Newton’s iterative method. Consequently \( u(x) \) given in Eq. (10) can be calculated.

3.1. Convergence analyses

Let \( \psi_{k,n}(x) = |a_0|^{k-1} \psi(a_0^k x - nb_n) \), \( k, n = 1, 2, \ldots \), where \( \psi_{k,n}(x) \) form a wavelet basis for \( L^2(R) \). In particular, when \( a_0 = 2, \) and \( b_n = 1, \) \( \psi_{k,n}(x) \) forms an orthonormal basis [23].

By Eq. (10), let \( u(x) = \sum_{j=1}^{M} \epsilon_j \psi_{j}(x) \) be the solution of Eqs. (1) and (2) where \( \epsilon_j = \langle u(x), \psi_{j}(x) \rangle \), for \( k = 1 \) in which \( \langle \ldots \rangle \) denotes the inner product.

\[ u(x) = \sum_{j=1}^{n} \langle u(x), \psi_{j}(x) \rangle \psi_{j}(x) \]

Let \( \beta_j = \langle u(x), \psi(x) \rangle \) where \( \psi(x) = \psi_{1}(x) \)

Let \( s_n = \sum_{j=1}^{n} \beta_j \psi(x) \) be a sequence of partial sums. Then,

\[ \langle u(x), s_n \rangle = \sum_{j=1}^{n} \int_{-\infty}^{\infty} \overline{\beta_j} (u(x), \psi(x)) \]

\[ = \sum_{j=1}^{n} |\beta_j|^2 \]

Further

\[ \| s_n - s_m \|^2 = \left\| \sum_{j=m+1}^{n} \beta_j \psi(x) \right\|^2 \]

\[ = \left\{ \sum_{j=m+1}^{n} \beta_j \psi(x), \sum_{j=m+1}^{n} \beta_j \psi(x) \right\} \]

\[ = \sum_{j=m+1}^{n} |\beta_j|^2 \]

As \( n \to \infty \), from Bessel’s inequality, we have \( \sum_{j=m+1}^{n} |\beta_j|^2 \) is convergent.

It implies that \( s_n \) is a Cauchy sequence and it converges to \( s \) (say).

Also

\[ \langle s - u(x), \psi(x) \rangle = \langle s, \psi(x) \rangle - \langle u(x), \psi(x) \rangle \]

\[ = \left\langle Lt_{n \to \infty} s_n, \psi(x) \right\rangle - \beta_j \]

\[ = \left\langle Lt_{n \to \infty} \sum_{j=1}^{n} \beta_j \psi(x), \psi(x) \right\rangle - \beta_j \]

\[ = \beta - \beta_j \]

\[ = 0. \]

Which is possible only if \( u(x) = s \), i.e. both \( u(x) \) and \( s_n \) converges to the same value, which indeed give the guarantee of convergence of LWOM.

4. Numerical experiments

In this section in order to demonstrate the applicability of the proposed method, we have solved two Damping equations, studied in [13,14], involving both integral and non-integral forcing terms. The obtained results are compared with the corresponding experimental results obtained by the methods presented in [13,14].
Example 4.1. Consider the Duffing equation

\[
\begin{align*}
\ddot{u}(x) + u(x) + u(x)\dot{u}(x) + \int_0^x s^2 u(s)ds &= f(x), \quad 0 < x < 1 \\
2u(0) - u'(0) &= 0, \quad u(1) + u'(1) = 0.
\end{align*}
\]

where \( f(x) = \frac{4}{x^2} - \frac{x^3}{2} - \frac{x^4}{2} + 3x^3 - 2x^2 - 3x. \)

It can be easily seen that the exact solution is \( u(x) = 1 + x - x^3. \) We have solved this problem using the approach described in the previous section. Here the solution \( u(x) \) is approximated by \( u(x) = C^T\Psi(x) = c_{10}\psi_{10} + c_{11}\psi_{11} + c_{12}\psi_{12} \)

\[ C^T D^2 \Psi(x) + C^T D\Psi(x) + C^T D^3 \Psi(x) \]

\[
\approx (z + 1) C^T \Psi \left( \frac{x}{z + 1} \right) = f(x)
\]

where \( f(x) = \frac{4}{x^2} - \frac{x^3}{2} - \frac{x^4}{2} + 3x^3 - 2x^2 - 3x. \)

by using Eq. (15), the weights \( \omega_j \) of the above equation are calculated.

Considering \( M = 2 \) and \( k = 0 \), we get a system of equations involving three variables and this system of equations can be solved using Newton’s iterative method through MATLAB, we get the following values

\[
c_{10} = 0, \quad c_{11} = 0.16666667, \quad c_{12} = -0.0745359925.
\]

\[
u(x) = C^T\Psi(x) = c_{10}\psi_{10} + c_{11}\psi_{11} + c_{12}\psi_{12}
\]

\[
= 0\psi_{10} + 0.16666667\psi_{11} - 0.0745359925\psi_{12}
\]

\[
\approx 1 + x - x^3
\]

which is the exact solution. Fig. 1 shows that, for different values of \( x \), the obtained results of the proposed method nearly equal to the exact solution. Table 1 describes the efficiency of the proposed method by comparing with the methods in [13, 14] through their absolute error.

Example 4.2. Consider the another Duffing equation

\[
\begin{align*}
\ddot{u}'(x) + u'(x) + u(x)\dot{u}'(x) + \int_0^x s^2 u(s)ds &= f(x), \quad 0 < x < 1 \\
2u(0) - u'(0) &= 0, \quad \dot{u}(1) + u'(1) = 0.
\end{align*}
\]

where \( f(x) = \frac{1}{2k+1} [(-9x^4 + 16x^3 + 18x + 726x^{2x} - 132e^{x+1}(x^2 + 2x + 11) - 66e^{x}(2x^3 - 7x + 11) + 8e^{x}(3x^2 + 2x + 60x + 42) + e(15x^4 + 32x^3 - 228x + 540) + 204].
\]

![Figure 1](image1.png) The exact and LWOM solution for Example 4.1.

![Figure 2](image2.png) The exact and LWOM solution for Example 4.2.

Exact solution is \( u(x) = 2 - \frac{15x^4 + (3x - 8)x^2}{2(1+x)}. \)

Using the method described in Section 3, \( u(x) \) is approximated by

\[
u(x) = C^T\Psi(x)
\]

\[
= c_{10}\psi_{10} + c_{11}\psi_{11} + c_{12}\psi_{12} + c_{13}\psi_{13} + c_{14}\psi_{14} + c_{15}\psi_{15}
\]

<table>
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<tr>
<th>Node</th>
<th>Exact solution</th>
<th>Absolute error in [13]</th>
<th>Absolute error in [14]</th>
<th>LWOM ( M = 0, ) ( k = 2 )</th>
<th>LWOM ( M = 0, ) ( k = 3 )</th>
<th>LWOM ( M = 4, ) ( k = 3 )</th>
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Table 1 Numerical results for Example 4.1.
and considering \( M = 2 \) and \( k = 0 \), we get a system of equations with six variables. These equations are solved by Newton’s iterative method, we get the values

\[
\begin{align*}
c_{10} & = 7.73306612088 & c_{11} & = 8.38825114824 & c_{12} & = -3.26840807332 \\
c_{13} & = 1.26843577220 & c_{14} & = 0.11884432672 & c_{15} & = 0.00432765421
\end{align*}
\]

Hence

\[
u(x) = C^T \varphi(x) = c_{10} \varphi_{10} + c_{11} \varphi_{11} + c_{12} \varphi_{12} + c_{13} \varphi_{13} + c_{14} \varphi_{14} + c_{15} \varphi_{15} \\
\approx 2 - \frac{11e^{x} + (3 - 8e)x}{2(1 + e)}
\]

which is the exact solution. Fig. 2 shows that, for different values of \( x \), the obtained results of the proposed method nearly equal to the exact solution. Table 2 describes the efficiency of the proposed method by comparing with the methods in [13,14] through their absolute error.

**Table 2** Numerical results for Example 4.2.

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5. Conclusion

A generalized Legendre wavelet operational matrix of derivative is used to solve the Duffing equation involving both integral and non-integral forcing terms. This operational matrix method together with Gaussian quadrature formula reduces the given system into a system of algebraic equations. Solution obtained by this method and comparison with the recently proposed methods reveals that the proposed LWOM method is a very effective and convenient method to solve Duffing equations. Moreover the proposed method is very simple, easy to implement and is able to approximate the solution more accurate in the given interval.

References


S. Balaji received his B.Sc and M.Sc degree in Mathematics from Manonmaniam Sundaranar University in 1999 and 2001. In 2004 he received his M.Phil degree in Mathematics from Madurai Kamaraj University, India. He received his Ph.D in Mathematics from SASTRA University, India in 2012. He is currently an Assistant Professor in the Department of Mathematics, SASTRA University, India. His research interest include elegant design of graph theoretical based approximation algorithms for graph and combinatorial optimization problems, designing routing protocols for the ad hoc wireless networks and wavelet approximation for differential and partial differential equations. He has 12 research papers in the peer-reviewed international journals.