Split Bregman iteration algorithm for total bounded variation regularization based image deblurring

Xinwu Liu, Lihong Huang

College of Mathematics and Econometrics, Hunan University, Changsha, Hunan 410082, PR China

1. Introduction

The problem of image restoration is considered. Given the observed image $f$, the objective of image reconstruction is to find the optimal solution $u$ based on the following model

$$f = Ku + n,$$  \hfill (1.1)

where $K$ is a bounded linear operator representing the blur (usually a convolution), $n$ denotes the additive noise, $u$ and $f$ represent the original image and the observed degraded image, respectively.

Solving (1.1) is an ill-posed inverse problem, mathematical techniques known as adding the regularization terms to the energy functional have been developed to deal with ill-posedness. The original idea introduced by Tikhonov and Arsenin [21] is given by

$$\min_u \int_\Omega |Du|^2 + \frac{\lambda}{2} \| Ku - f \|_2^2.$$  \hfill (1.2)

This method can reduce the oscillations caused by highly noisy data, but it does not preserve the edges and corners well because of its isotropic smoothing properties. To overcome this shortcoming, Rudin, Osher, and Fatemi [19] formulated a total variation (TV) based regularization scheme (the ROF model)

$$\min_u \int_\Omega |Du| + \frac{\lambda}{2} \| Ku - f \|_2^2.$$  \hfill (1.3)
where $\Omega \subseteq \mathbb{R}^2$ denotes a bounded open subset with Lipschitzian boundary, $u \in L^1(\Omega)$, and $Du$ represents the distributional derivative of $u$. As a result, the bounded variation ($BV$) seminorm is endowed with $\|u\|_{BV} = \|u\|_{L^1} + \int_{\Omega} |Du|$, then the Banach space $BV(\Omega)$ is essentially an extension of $W^{1,1}$. Illustrations demonstrated that the ROF method is very efficient in edge-preserving image reconstruction.

Many computational methods for solving (1.3) sprung up in recent years. In [19], Rudin et al. proposed a time marching scheme based on the explicit gradient descent method. An improved time-marching scheme based on affine scaling algorithm described in [16]. Another best known method is the lagged diffusivity fixed-point iteration algorithm (see [2,3,9,23]), which solves the stationary Euler–Lagrange equations efficiently due to its stability and convergence. Furthermore, Chan et al. [8] employed the primal-dual variable strategy based on Newton’s method to manipulate the optimization problem (1.3). The second contribution of this paper is proposing the extended split Bregman iteration for solving the optimum solution of (1.4). Furthermore, we have provided a proof of the convergence for the extended split Bregman iterative method introduced in [17] has been shown to be particularly efficient for image restoration based on the model (1.3). This iterative method has several advantages over the traditional methods, such as fast convergence rate, flexibility of parameters $\lambda$, and prominent stability, etc. The linearized Bregman iteration was proposed in Refs. [12,13] to solve constrained optimization problems for $L^1$ sparse reconstruction. Motivated by these iterative algorithms, authors in [14,27] advanced the split Bregman iterative method, which can be used to solve a wide variety of constrained optimization problems (including the magnetic resonance imaging reconstruction, etc.).

Liu and Tai [26] employed a technique related to the augmented Lagrangian method to solve the ROF model based problems. Additionally, they showed that the proposed method can be extended to vectorial TV and high order models. Ref. [29] described the Bregmanized Operator splitting (BOS) algorithm, where the nonlocal regulated optimization problems were effectively solved by employing the proximal forward–backward splitting iteration.

Regularized methods mentioned above solve the inverse problem (1.1) by using the regularized seminorm in $BV(\Omega)$ (i.e., $L^1$ norm). However, we concentrate especially on the full norm $\|u\|_{L^1} + \rho \|u\|_2^2$ for regularization in this paper, and formulate the following variational model

$$\min_{u} \int_{\Omega} |Du| + \frac{\alpha}{2} \|u\|_2^2 + \frac{\beta}{2} \|Ku - f\|_2^2, \quad (1.4)$$

where $\Omega$ denotes a bounded domain in $\mathbb{R}^N$ with Lipschitzian boundary, $\alpha \geq 0$, $\beta > 0$ are the given parameters, $K \in L(L^2(\Omega) \cap BV(\Omega))$, and $K \cdot 1 \neq 0$. Let $X = L^2(\Omega) \cap BV(\Omega)$, then we derive the following facts: (i) The space $X$ equipped with the full norm is also a Banach space; (ii) $K \cdot 1 \neq 0$ ensures that (1.4) is coercive on $X$.

In this paper, we have two main contributions focusing on the image deblurring problem based on the model (1.4). Firstly, we introduce a total bounded variation norm regularization based image restoration scheme, and study the issue of existence and uniqueness of this model. The second contribution of this paper is proposing the extended split Bregman iteration for solving the optimum solution of (1.4). Furthermore, we have provided a proof of the convergence for the proposed iterative algorithm.

The remainder of this paper is arranged as follows. In Section 2, we describe the necessary definitions and preliminaries about the proposed model. In Section 3, we give a brief overview of some related iterative algorithms. Section 4 elaborates on the analysis of the proposed extended split Bregman iteration method. And its corresponding convergence analysis is displayed in Section 5. Numerical experiments intended for demonstrating the proposed method are provided in Section 6. Finally, conclusions are made in Section 7.

2. Preliminaries

Here, we restate the proposed model

$$\min_{u} \int_{\Omega} |Du| + \frac{\alpha}{2} \|u\|_2^2 + \frac{\beta}{2} \|Ku - f\|_2^2, \quad (2.1)$$

which was introduced in [10] and demonstrated the well-posedness of the solution. Ref. [15] considered two semismooth Newton methods for solving the Fenchel predual problems based on this model.

Comparing (2.1) with (1.3), we immediately discover that a quadratic regularization term is utilized in (2.1) additionally. Obviously, it is the standard ROF model when $\alpha = 0$. There are two advantages of this modification. First of all, it serves as the coercive term (when $\alpha > 0$) for the subspace of constant functions which belong to the kernel functions of the gradient operator. The second (and perhaps the most significant) advantage of the quadratic regularization term is that it provides the probability to discriminate the structure of stability results from that of the nonquadratic $BV$ term.

In order to show the efficiency of model (2.1) for image reconstruction, we must firstly include the necessary preparations and notations. We start with the definition of $\sum_{\Omega} |Du|$ for function $u \in L^1(\Omega)$.

**Definition 2.1.** Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open domain. Let $u \in L^1_{\text{loc}}(\Omega).$ Then the total variation of $u$ is defined by

$$\int_{\Omega} |Du| = \sup_{\phi \in C_0^1(\Omega, \mathbb{R}^N)} \left\{ \int_{\Omega} u(\phi) \cdot \nabla \phi + \lambda \left( \|u\|_{L^1(\Omega)} - \|u\|_{L^1_{\text{loc}}(\Omega)} \right) \right\}.$$
Proposition 2.1 (Lower semicontinuity). Suppose that \( \{u_i\}_{i=1}^{\infty} \subset BV(\Omega) \) and \( \tilde{u} \in L^1(\Omega) \) is such that \( u_i \to \tilde{u} \) in \( L^1(\Omega) \), then
\[
\int_\Omega |D\tilde{u}| \leq \liminf_{i \to \infty} \int_\Omega |Du_i|.
\]

Proposition 2.2. If \( \tilde{u} \in L^2(\Omega) \cap BV(\Omega) \), then there exists a minimizing sequence \( \{u_i\}_{i=1}^{\infty} \subset BV(\Omega) \) such that
\[
\lim_{i \to \infty} |u_i - \tilde{u}|_{L^1(\Omega)} = 0,
\]
and
\[
\lim_{i \to \infty} \int_\Omega |Du_i| = \int_\Omega |D\tilde{u}|.
\]

We are now in the position to present the existence and uniqueness of the optimization problem (2.1).

Theorem 2.1. The problem (2.1) has a unique solution in \( L^2(\Omega) \cap BV(\Omega) \).

Proof. Let \( \{u_i\}_{i=1}^{\infty} \) stands for a minimizing sequence. By the Kondrachov compactness theorem, the sequence \( \{u_i\}_{i=1}^{\infty} \) is precompact in \( L^2(\Omega) \cap BV(\Omega) \). Namely, there exists a function \( \tilde{u} \) satisfying \( u_i \to \tilde{u} \) a.e. Since the function is convex and coercive in \( L^2(\Omega) \cap BV(\Omega) \), the lower semicontinuity property is satisfied, then we have \( \liminf_{i \to \infty} \|u_i\|_{BV} \geq \|\tilde{u}\|_{BV} \), \( \|u_i\|_{L^2(\Omega)} \geq \|\tilde{u}\|_{L^2(\Omega)} \), and
\[
\inf \left\{ \int_\Omega |Du| + \frac{\alpha}{2} \|u\|_2^2 + \frac{\beta}{2} \|Ku - f\|_2^2 \right\} \geq \liminf_{i \to \infty} \left\{ \int_\Omega |Du_i| + \frac{\alpha}{2} \|u_i\|_2^2 + \frac{\beta}{2} \|Ku_i - f\|_2^2 \right\}
\]
\[
\geq \int_\Omega |D\tilde{u}| + \frac{\alpha}{2} \|\tilde{u}\|_2^2 + \frac{\beta}{2} \|K\tilde{u} - f\|_2^2.
\]

Which concludes that \( \tilde{u} \) is minimum point of (2.1). Next, we turn to the proof of the uniqueness. Let \( \tilde{u} \) and \( \tilde{v} \) be two minima of (2.1). From its convexity we easily obtain that \( D\tilde{u} = D\tilde{v} \), which means that \( \tilde{u} = \tilde{v} + c \). Additionally, considering that \( F(u) = \frac{\alpha}{2} \|u\|_2^2 + \frac{\beta}{2} \|Ku - f\|_2^2 \) is strictly convex, we conclude that \( K\tilde{u} = K\tilde{v} \), and therefore \( KC = 0 \). Note that \( K \) is linear and the function (2.1) is coercive in \( L^2(\Omega) \cap BV(\Omega) \), we deduce that \( c = 0 \) and \( \tilde{u} = \tilde{v} \). This completes the proof. \( \square \)

3. Related iterative methods

In this section, we elaborate on the relatedly computational methods for solving the inverse problem (1.1). These include the Bregman iteration, linearized Bregman iteration, and split Bregman iteration algorithms. We introduce them below.

3.1. Bregman iteration

As we have mentioned above, Bregman iteration was initially introduced and studied in image processing by Osher et al. in [17]. Where their core ideas were firstly to transform a constrained optimization problem into an unconstrained problem, then by means of the Bregman distance to perform variable separation iteration quickly.

More explicitly, they considered the generalized form of regularization model formulated as
\[
\min_u J(u) + H(u, f),
\]
where \( J(u) \) denotes a convex nonnegative regularization functional of \( u \), and the fidelity term \( H(u, f) \) is a smooth convex nonnegative function with respect to \( u \) for given \( f \). Then the Bregman distance associated with the convex function \( J \) at the point \( v \) is defined by
\[
D_J^v(u, v) = J(u) - J(v) - \langle p, u - v \rangle,
\]
where \( \langle \cdot, \cdot \rangle \) stands for the standard duality product, and \( p \) is in the subdifferential of \( J \) at \( v \) with
\[
\partial J(v) := \{ p \in BV(\Omega)^* \mid J(v) \geq J(u) + \langle p, u - v \rangle \}.
\]
Remark. Generally, $D^p_f(u, v)$ is not a distance in the usual sense, since it satisfies neither the symmetry nor the triangle inequality. However, it can serve as a measure for the strictly convex functionals. In view of the continuously differentiable property of $f(u)$, its subdifferential $\partial f(u)$ and Bregman distance both have unique value for each fixed $v$.

To solve the unconstrained optimization problem (3.1), authors in [17,24] employed the Bregman iteration method effectively. Detailedly, initialization $u^0 = 0$ and $p^0 = 0$, they iteratively solve

$$
\begin{align*}
  u^{k+1} & = \arg\min_u D^p_f(u, u^k) + \frac{\lambda}{2} \| Ku - f \|_2^2, \\
  p^{k+1} & = p^k - \lambda K^T(Ku^{k+1} - f).
\end{align*}
$$

As shown in [17,27], when $K$ is linear, the iteration (3.4) and (3.5) can be reformulated into a compact form

$$
\begin{align*}
  f^{k+1} & = f^k + (f - Ku^k), \\
  u^{k+1} & = \arg\min_u J(u) + \frac{\lambda}{2} \| Ku - f^{k+1} \|_2^2.
\end{align*}
$$

This Bregman iteration was first proposed in [17] for TV based image denoising, and then successfully used for image reconstruction (see [12,24,27]) owing to its high-efficiency and robustness. Meanwhile, the rigorous convergence and error analysis of this iteration can be found in [13,17,20].

3.2. Linearized Bregman iteration

Combining the Bregman iteration with the operator splitting method [11,22], Darbon and Osher [12] and Yin et al. [27] advanced the linearized Bregman iteration to solve the constrained optimization problem

$$
\begin{align*}
  \min_u J(u) \quad \text{s.t.} \quad Ku = f.
\end{align*}
$$

Where they approximated the term $\frac{1}{2} \| Ku - f \|_2^2$ in (3.4) by the sum of its first-order approximation at $u^k$ and $L^2$-proximity term at $u^k$, and obtained the following iterative scheme

$$
\begin{align*}
  u^{k+1} & = \arg\min_u D^p_f(u, u^k) + \frac{\lambda}{2} \| u - (u^k - \delta K^T(Ku^k - f)) \|_2^2, \\
  p^{k+1} & = p^k - \frac{\lambda}{\delta} (u^{k+1} - u^k) - \lambda K^T(Ku^k - f).
\end{align*}
$$

More properties and convergence analysis of this iteration described in [5,6,28] detailedly. Also its broad applications for solving the compressed sensing problem can be found in [5,18] and frame based image deblurring displayed in [3].

3.3. Split Bregman iteration

Inspired by the Bregman iteration and the linearized Bregman iteration, Goldstein and Osher [14] introduced the split Bregman iteration to solve more general $L^1$ regularized optimization problem (3.1). Based on the split formulation presented in [25], they firstly converted the constrained optimization problem

$$
\begin{align*}
  \min_{u, d} \| d \|_1 + H(u) \quad \text{s.t.} \quad d = J(u),
\end{align*}
$$

into an unconstrained optimization problem

$$
\begin{align*}
  \min_{u, d} \| d \|_1 + H(u) + \frac{\lambda}{2} \| d - J(u) \|_2^2.
\end{align*}
$$

Then the problem (3.12) was enforced with the simplified two-phase iterative algorithm

$$
\begin{align*}
  (u^{k+1}, d^{k+1}) & = \arg\min_{u, d} \| d \|_1 + H(u) + \frac{\lambda}{2} \| d - J(u) - b^k \|_2^2, \\
  b^{k+1} & = b^k + (J(u^{k+1}) - d^{k+1}).
\end{align*}
$$

where $J(u)$ and $H(u)$ stand for the convex functions. The distinctly difference between split Bregman iteration and two iterative algorithms anteriorly is that an intermediate variable $d$ is introduced additionally, which satisfies $d = J(u)$. This is an extremely fast algorithm, very simple to program. So it has been studied widely by researchers. For example, Refs. [4,14] described its further applications in image processing, and the convergence analysis was also given there.
4. The proposed algorithm

For the proposed model, we take the full norm $\|u\|_{BV} + \frac{\alpha}{2} \|u\|_2^2$ for regularization instead of the seminorm $\|u\|_{BV}$. In this case, we wish to solve

$$\min_u \|Du\|_1 + \frac{\alpha}{2} \|u\|_2^2 + \frac{\beta}{2} \|Ku - f\|_2^2.$$  \hspace{1cm} (4.1)

To solve the problem (4.1), the split Bregman iteration is preferred. We firstly take effective replacement $Du \rightarrow d$. This yields a constrained optimization problem

$$\min_{u,d} \|d\|_1 + \frac{\alpha}{2} \|u\|_2^2 + \frac{\beta}{2} \|Ku - f\|_2^2, \quad \text{s.t.} \quad d = Du.$$  \hspace{1cm} (4.2)

Rather than considering (4.2), we will consider an unconstrained optimization problem

$$\min_{u,d} \|d\|_1 + \frac{\alpha}{2} \|u\|_2^2 + \frac{\beta}{2} \|Ku - f\|_2^2 + \frac{\lambda}{2} \|d - Du\|_2^2.$$  \hspace{1cm} (4.3)

Concretely, the extended split Bregman iterative algorithm for solving (4.3) is depicted as

$$u^{k+1} = \arg \min_u \frac{\alpha}{2} \|u\|_2^2 + \frac{\beta}{2} \|Ku - f\|_2^2 + \frac{\lambda}{2} \|Du - d^k + b^k\|_2^2,$$ \hspace{1cm} (4.4)
$$d^{k+1} = \arg \min_d \|d\|_1 + \frac{\lambda}{2} \|d - Du^{k+1} - b^k\|_2^2,$$ \hspace{1cm} (4.5)

with the update formula for $b^{k+1}$

$$b^{k+1} = b^k + (Du^{k+1} - d^{k+1}).$$ \hspace{1cm} (4.6)

More precisely, given $u^0 = 0$, and $d^0 = b^0 = 0$, we derive an unconstrained split Bregman iterative algorithm

$$u^{k+1} = \arg \min_u \frac{\alpha}{2} \|u\|_2^2 + \frac{\beta}{2} \|Ku - f\|_2^2 + \frac{\lambda}{2} \|Du - d^k + b^k\|_2^2,$$ \hspace{1cm} (4.7)
$$d^{k+1} = \arg \min_d \|d\|_1 + \frac{\lambda}{2} \|d - Du^{k+1} - b^k\|_2^2,$$
$$b^{k+1} = b^k + (Du^{k+1} - d^{k+1}).$$

This yields the following decoupled subproblems. Solving for the $u$ subproblem, we derive the optimality condition

$$0 = \alpha u^{k+1} + \beta K^T (Ku^{k+1} - f) + \lambda D^T (Du^{k+1} - d^k + b^k),$$ \hspace{1cm} (4.8)

which means that

$$u^{k+1} = ((\alpha I + \beta K^T K - \lambda \Delta)^{-1} (\beta K^T f + \lambda D^T (d^k - b^k))).$$ \hspace{1cm} (4.9)

where $D^T = - \text{div}$ represents the adjoint of $D$ and $\Delta = -D^T D$. Considering that the system (4.9) is strictly diagonally dominant, we can solve the subproblem for $u$ by the Gauss-Seidel iteration efficiently. Similarly to [14,25], the subproblem for $d$ can be solved by applying the generalized shrinkage formula, namely,

$$d^{k+1} = \text{shrink} \left( Du^{k+1} + b^k, \frac{1}{\lambda} \right).$$ \hspace{1cm} (4.10)

where

$$\text{shrink}(x, \gamma) = \max(\|x\|_2 - \gamma, 0) \frac{x}{\|x\|_2}.$$ \hspace{1cm} (4.11)

Involving only matrix multiplication and scalar shrinkage, this yields a concisely fast iterative algorithm. By this means, we can solve the minimization problem for $d$ quickly.

This alternating minimization method contains two loops of iterations, in which each outer iteration corresponds to a fixed $\lambda$ value. Since the parameter $\lambda$ is arbitrary, one can select an optimal $\lambda$ such that the system (4.7) reaches the optimized status. One significant advantages of this algorithm is its extremely fast convergence. Another one is that this method is easy to code. As we shall see, this alternative method can be more effective for image deblurring by a small number of internal iterations and finite external iterations.
5. Convergence analysis

In this section, we elaborate on the rigorous convergence proof of the proposed iterative algorithm. Our analysis below is motivated by that of Ref. [4], where the design and analysis of the unconstrained split Bregman iteration were presented detailed.

The subproblems involved in (4.7) are convex and differentiable, so the first-order optimality conditions for \( u^{k+1} \) and \( d^{k+1} \) are easily derived. By differentiating the first two equations of (4.7) bilaterally with respect to \( u \) and \( d \) and valuing at \( u^k \) and \( d^{k+1} \) separately, we can obtain the following conclusion

\[
\begin{align*}
0 &= \alpha u^{k+1} + q^{k+1} + \lambda D^T (Du^{k+1} - d^k + b^k), \\
0 &= p^{k+1} + \lambda (d^{k+1} - Du^{k+1} - b^k), \\
b^{k+1} &= b^k + (Du^{k+1} - d^{k+1}),
\end{align*}
\]

(5.1)

where the identities \( H(u) = \frac{\lambda}{2} \| Ku - f \|_2^2, p^k \in \partial \|d^k\|_1 \), and \( q^k \in \partial H(u^k) \). The condition (5.1) will be used for analyzing the convergence properties of the proposed scheme in the subsequent section.

**Theorem 5.1.** Assume that the weight \( \lambda > 0 \). Then the sequence \( \{u^k\}_{k \in \mathbb{N}} \) generated by algorithm (4.7) converges to the unique solution \( u^* \) of the problem (4.1).

**Proof.** As shown in Theorem 2.1, there exists a unique solution \( u^* \) of the problem (4.1). The first-order necessary condition gives the following fact

\[
0 = D^T Du^* + \alpha u^* + q^*,
\]

(5.2)

where \( d^* = Du^* \) and \( q^* \in \partial H(u^*) \). Subsequently let \( b^* = \frac{1}{\lambda} Du^* \), hence we have

\[
\begin{align*}
0 &= \alpha u^* + q^* + \lambda D^T (Du^* - d^* + b^*), \\
0 &= p^* + \lambda (d^* - Du^* - b^*), \\
b^* &= b^* + (Du^* - d^*),
\end{align*}
\]

(5.3)

with \( p^* \in \partial \|d^*\|_1 \). As a result, it demonstrates that \( u^*, d^*, b^* \) is a fixed point of (4.7).

Assume that the sequence \( \{u^k, d^k, b^k\}_{k \in \mathbb{N}} \) is generated by algorithm (4.7). For convenience of notations, we let \( u^k = u^k - u^*, d^k = d^k - d^*, b^k = b^k - b^* \) represent the errors. All equations of (5.1) correspondingly subtracted from that of (5.3) give that

\[
\begin{align*}
0 &= \alpha u^{k+1}_e + (d^{k+1}_e - q^*), \\
0 &= (p^{k+1} - p^*), \\
b^{k+1}_e &= b^k_e + (Du^{k+1}_e - d^{k+1}_e).
\end{align*}
\]

(5.4)

As for (5.4), we take the duality product bilaterally of the first two equations with respect to \( u^{k+1}_e \) and \( d^{k+1}_e \) separately, and square both sides of the third one. This yields

\[
\begin{align*}
0 &= \alpha \|u^{k+1}_e\|_2^2 + (q^{k+1}_e - q^*, u^{k+1}_e) \\
&\quad + \lambda (D^T Du^{k+1}_e, u^{k+1}_e) + \lambda (u^{k+1}_e, D^T b^k_e - D^T d^{k+1}_e), \\
0 &= (p^{k+1} - p^*, d^{k+1}_e) + \lambda \|d^{k+1}_e\|_2^2 - \lambda (d^{k+1}_e, Du^{k+1}_e + b^k_e), \\
\|b^{k+1}_e\|_2^2 &= \|b^k_e\|_2^2 + \|Du^{k+1}_e - d^{k+1}_e\|_2^2 + 2(b^k_e, Du^{k+1}_e - d^{k+1}_e).
\end{align*}
\]

(5.5)

Equivalently, the third equation of (5.5) can be rewritten as

\[
\frac{\lambda}{2} \|b^{k+1}_e\|_2^2 = -\frac{\lambda}{2} \|Du^{k+1}_e - d^{k+1}_e\|_2^2 - \lambda (b^k_e, Du^{k+1}_e - d^{k+1}_e).
\]

(5.6)

Combining (5.5) with (5.6), we have

\[
\frac{\lambda}{2} \|b^{k+1}_e\|_2^2 = (\|b^k_e\|_2^2 + \|d^{k+1}_e\|_2^2 + \|d^{k+1}_e\|_2^2) = \alpha \|u^{k+1}_e\|_2^2 + (q^{k+1}_e - q^*, u^{k+1}_e) + \frac{\lambda}{2} \|Du^{k+1}_e - d^{k+1}_e\|_2^2
\]

(5.7)

Then, by summing the equality (5.7) bilaterally from 0 to \( N \), we obtain
Fig. 1. Objective evaluation of our method. (a) Original image. (b) Degraded image by Gaussian blur of size $7 \times 7$ with standard deviation 3. (c) Restored image by the ROF model. (d) Restored image by the proposed model.

\[
\frac{\lambda}{2} \left( \|b_0\|^2 + \|d_0\|^2 - \|b^{N+1}_0\|^2 - \|d^{N+1}_0\|^2 \right) = \alpha \sum_{k=0}^{N-1} \|u^{k+1}_{e}\|^2 + \sum_{k=0}^{N-1} \langle q^{k+1} - q^*, u^{k+1}_{e}\rangle \\
+ \frac{\lambda}{2} \sum_{k=0}^{N-1} \|D u^{k+1}_{e} - d^{k}_{e}\|^2 + \sum_{k=0}^{N-1} \langle p^{k+1} - p^*, d^{k}_{e}\rangle. \quad (5.8)
\]

Noting that all terms involved in (5.8) are nonnegative, we derive that

\[
\frac{\lambda}{2} \left( \|b_0\|^2 + \|d_0\|^2 \right) \geq \sum_{k=0}^{N-1} \langle p^{k+1} - p^*, d^{k}_{e}\rangle + \alpha \sum_{k=0}^{N-1} \|u^{k+1}_{e}\|^2 + \sum_{k=0}^{N-1} \langle q^{k+1} - q^*, u^{k+1}_{e}\rangle. \quad (5.9)
\]

Moreover, we have

\[
\sum_{k=0}^{N-1} \langle p^{k+1} - p^*, d^{k}_{e}\rangle < +\infty. \quad (5.10)
\]

From (5.10) it implies that

\[
\lim_{k \to \infty} \langle p^k - p^*, d^k - d^*\rangle = 0. \quad (5.11)
\]

Associated it with the Bregman distance, the following expressions hold

\[
\lim_{k \to \infty} \|d^k\|_1 - \|d^*\|_1 - \langle p^*, d^k - d^*\rangle = 0, \quad \text{and} \quad \lim_{k \to \infty} \|D u^{k+1} - d^k\|_2 = 0. \quad (5.12)
\]

Furthermore, (5.12) leads to

\[
\lim_{k \to \infty} \|D u^k\|_1 - \|D u^*\|_1 - \langle D^T Du^*, D u^k - D u^*\rangle = 0. \quad (5.13)
\]
Analogously, we have
\[ \lim_{k \to \infty} H(u^k) - H(u^*) - \langle q^*, u^k - u^* \rangle = 0, \quad \text{and} \quad \lim_{k \to \infty} \| u^k - u^* \|_2 = 0. \] (5.14)

Conditions (5.13) and (5.14) imply that
\[ \lim_{k \to \infty} \left( \| Du^k \|_1 + \frac{\alpha}{2} \| u^k \|_2^2 + H(u^k) \right) - \left( \| Du^* \|_1 + \frac{\alpha}{2} \| u^* \|_2^2 + H(u^*) \right) - \langle D^T Du^* + \alpha u^* + q^*, u^k - u^* \rangle = 0. \] (5.15)

Finally, from (5.15) and (5.2), we obtain the main results
\[ \lim_{k \to \infty} \| Du^k \|_1 + \frac{\alpha}{2} \| u^k \|_2^2 + H(u^k) = \| Du^* \|_1 + \frac{\alpha}{2} \| u^* \|_2^2 + H(u^*), \quad \text{and} \quad \lim_{k \to \infty} \| u^k - u^* \|_2 = 0. \] (5.16)

Which completes the proof of Theorem 5.1. \( \square \)

6. Numerical results

We present three numerical results in this section to validate the efficiency and feasibility of the proposed method. Our algorithm is compared with the ROF model, and the computations are performed in MATLAB. All images are processed by our new method with equivalent parameters \( \alpha = 10, \beta = 3 \times 10^3 \) and \( \lambda = 5 \). Moreover, the criterion for stopping the iteration is that the relative difference between the consecutive iteration of the reconstructed images should satisfy the following inequality
\[ \frac{\| u^{k+1} - u^k \|_2}{\| u^{k+1} \|_2} < 10^{-3}. \] (6.1)

In the first two experiments, the original images are Lena and satellite shown on the top left of Figs. 1 and 2 respectively, which are of size 256 \( \times \) 256 pixels wide with 8-bit gray levels. Where we assume that the blur operator is the 2D...
Gaussian point spread function (PSF) defined as
\[ h(x, y) = \frac{1}{2\pi \sigma^2} \exp \left( -\frac{x^2 + y^2}{2\sigma^2} \right). \] (6.2)

In the third example, the clean color image rose (Fig. 3(a)) sized 250 by 303 is contaminated by motion blur. We apply the extended split Bregman iteration to perform total bounded variation norm based image deblurring. For simplicity, we chose the inner iterations to be one. Meanwhile, we flexibly change the number of outer iterations to improve the quality of the recovered image.

**Example 1.** The first example, a Gaussian-shaped PSF extending over 7 × 7 pixels with variance \(\sigma^2 = 3\) is used to corrupt the original image. And the available degraded image is displayed in Fig. 1(b). Simulation results for the ROF method and the proposed method are shown in Figs. 1(c) and 1(d), respectively. We get our results by externally iterating 18 times. In comparison, it shows that the reconstruction output image of the proposed strategy is very satisfactory. Precisely, under the same conditions, distinct contour and details such as *Lenna*'s face and hat recovered by our novel model are more clearly than that of the ROF scheme.

**Example 2.** The second example, we further evaluate the performance of the proposed algorithm. The given *satellite* image is blurred by Gaussian kernel of size 10 × 10 with standard deviation 5. Fig. 2 shows the comparison of the restoration results for the ROF method and the proposed method. The bottom left image is the outcome of the ROF model. And the bottom right image is the reconstruction obtained using the proposed method. Its performance is exemplified in Fig. 2(d) with the number of the outer iterations to be 16. Where the small details of the *satellite*, such as the antenna located at the top of image can be found clearly. Comparison results show again that the proposed method is very efficient to perform image deblurring than the standard ROF model.
Example 3. In this experiment, we verify the efficiency of our novel algorithm to perform color image deblurring. Following the foundational theory of the vectorial total variation minimization algorithm described in [2] by Bresson and Chan, we also compare the numerical results of the proposed algorithm with that of the ROF method. Here a motion blurring kernel of motion distance \( \text{len} = 10 \) and angle \( \theta = 135 \) is applied to the rose image. The degraded image is shown in Fig. 3(b). Figs. 3(c) and 3(d) display the results of the ROF model and our method, respectively. Where the number of iterations to reach the solution is 10. Intuitively, it can seen that the proposed method yields a more distinct image than that of the ROF model. This demonstrates that the competitive performance of our method in color image deblurring.

7. Conclusion

A novel total bounded variation regularization based image deblurring model has been presented in this paper. Existence, and uniqueness of the model are also proved there. Based on this model, we have introduced the extended split Bregman iteration, and provided the rigorous convergence analysis of this iterative algorithm. The results obtained by our method are very promising particularly from the preserved details point of view compared with the standard ROF method. It is believed that the proposed model and algorithm can be extend to further application in image processing and computer vision.

Acknowledgments

We would like to thank the anonymous reviewers for their helpful comments and valuable suggestions.

References