JOURNAL OF COMBINATORIAL THEORY, Series A 24, 241–246 (1978)

Note

On Small $\{k; q\}$-arcs in Planes of Order q^2

M. BARNABEI,* D. SEARBY,† AND C. ZUCCHINI‡

Università di Bologna, Bologna, Italy

Communicated by A. Barlotti

Received December 6, 1976

In this paper we examine some properties of complete $\{k; q\}$-arcs in projective planes of order q^2. In particular, we derive a lower bound for k, and we exhibit a family of arcs having low values of k which exist in every such plane having a Baer subplane. In addition we resolve the existence problem for complete $\{k; 3\}$-arcs in $PG(2, 9)$.

1. INTRODUCTION

A $\{k; n\}$-arc in a finite projective plane π is a set K of k points such that no more than n are collinear. A line of the plane containing precisely m points of K is called an m-secant, and K is said to be complete if every point of $\pi - K$ lies on at least one n-secant. An obvious upper bound for k is $k \leqslant (n-1)(q+1)+1$, and such maximal arcs have been extensively studied (see, for example, [3, 4, 5, 6]). Of particular interest are results of Cossu [2] and Barlotti [1] which show that the desarguesian plane of order 9 does not contain a (maximal) $\{21; 3\}$-arc, or even a $\{20; 3\}$-arc. In the last section we shall give examples of complete $\{k; 3\}$-arcs in this plane for every possible value of k. Our main result is the derivation of a lower bound for k for complete $\{k; q\}$-arcs in planes of order q^2. We present also a family of complete $\{k; q\}$-arcs in planes of order q^2 containing a Baer subplane of order q, for "small" values of k.

2. A LOWER BOUND FOR k

THEOREM 1. *Let π be a finite projective plane of order q^2 containing a Baer subplane π_0 (of order q). Then π contains complete $\{q^2 + kq; q\}$-arcs for every value of k, $1 \leqslant k \leqslant q - 1$.*

* Borsista del Consiglio Nazionale delle Riceerche.
† Visiting professor, CNR.
‡ Borsista del CNR.

0097-3165/78/0242–0241$02.00/0

Proof. Let l and m be two distinct lines of the plane, each containing $q + 1$ points of π_0, $K = \{K_i \mid i = 1,\dots, k - 1\}$ a set of $k - 1$ points on $m \cap A$, where $A = \pi_0 - l$, $L = \{L_i \mid i = 1,\dots, k - 1\}$ a set of $k - 1$ points on $m - \pi_0$, and $p_1, \dots, p_{k-1}$ a set of $k - 1$ lines through $P = l \cap m$ containing no further point of π_0. Through each point K_i there pass just q lines of $\pi_0 - m$; define $\{L_{ij} \mid j = 1,\dots, q\}$ to be the set of intersections of these q lines and p_i. Let $Q = \{Q_i \mid i = 1,\dots, q\}$ be any set of q points on $l - (B \cup \pi_0)$, where $B = \{L_i L_{1j} \mid i = 1,\dots, k - 1,\ j = 1,\dots, q\}$. Such a set exists because $|B \cap l| \leqslant (k - 1)q \leqslant q^2 - 2q$, and $l - \pi_0$ contains $q^2 - q$ points. Define $J = (A - K) \cup L \cup Q \cup \{L_{ij} \mid i = 1,\dots, k - 1, j = 1,\dots, q\}$. Now $|J| = (q^2 - (k - 1)) + (k - 1) + q + (k - 1)q = q^2 + kq$. Every point of the plane lies on a q-secant of J because every point lies on a line which meets π_0 in $q + 1$ points, and such a line must be either

(i) l, in which case it contains Q but no other point of J,

(ii) m, in which case it contains $q + 1 - |K| + |L| - |l \cap m| = q$ points of J,

(iii) a line through one of the K_i, in which case it contains $q - 1$ points in $\pi_0 - K - l$ and precisely one point L_{ij}, or

(iv) any other line of π_0, which contains q points of $\pi_0 - l$.

In fact, J is a $\{q^2 + kq; q\}$-arc, because any line of the plane either meets π_0 in $q + 1$ points, and by the above is a q-secant, or is one of the lines p_i, and by the construction contains just q points L_{ij}, or is any other line. But if such a line were to contain more than q points it would have to contain $q - 2$ points amongst the L_{ij}'s (and hence $k = q - 1$), one point of L, one point of $A - K$ and one point of Q-but this is impossible by the definition of Q. J is thus a complete $\{q^2 + kq; q\}$-arc.

Arcs constructed in this way are remarkable in that $q^2 + kq$ is (for $q > 3$) very much less than the upper bound of $q^3 - q^2 + q$; in fact, $q^2 + q$ is only one greater than the lower bound we are about to prove. Even so, these arcs are in no sense unique: as long as $k < q - 1$ the points of Q can be chosen arbitrarily on $l - \pi_0$, so that it is possible for two such arcs to differ by just one point, moreover there exist examples of $\{q^2 + kq; q\}$-arcs which are not isomorphic to arcs constructed using Theorem 1. We now prove our main theorem.

THEOREM 2. *If K is a complete $\{k; q\}$-arc in a plane of order q^2, then* $k \geqslant q^2 + q - 1$.

Proof. The result is obvious for $q = 2$ and a simple computation for $q = 3$. Moreover, for $q \geqslant 3$ it is easy to show that $k > q^2$. So, assume that K is a complete $\{q^2 + h; q\}$-arc, $1 \leqslant h \leqslant q - 2$ in a plane of order q^2, $q > 3$.

A point of K lies on at most $q + 1$ q-secants. If t_q is the number of q-secants,

$$t_q \leqslant \frac{(q^2 + h)(q + 1)}{q} = q^2 + q + h + \frac{h}{q} \to t_q \leqslant q^2 + q + h$$

Now there exists an external line, since any external point lies on at least one q-secant and the remaining $q^2 + h - q < q^2$ arc points cannot account for the remaining q^2 lines through the point. As every point of such a line must lie on a q-secant,

$$t_q \geqslant q^2 + 1$$

If every point of K lay on at most $q - 1$ q-secants, $t_q \leqslant (q^2 + h)(q - 1)/q < q^2 + 1$ so some point of K lies on at least q q-secants. As such a point lies on a tangent, we obtain

$$t_q \geqslant q^2 + q$$

Again, if every point of K lay on at most q q-secants, $t_q \leqslant (q^2 + h)q/q < q^2 + q$ so some point R of K lies on at least $q + 1$ q-secants, and hence

$$t_q = q^2 + q + t \qquad \text{for some} \quad t, \qquad 1 \leqslant t \leqslant h.$$

Now let P_j, $0 \leqslant j \leqslant q + 1$, be the set of points of K which lie on precisely j q-secants. Let $r_1, \ldots, r_s$ be the lines through $R \in P_{q+1}$ containing at least one, but fewer than q, further points of K, and $r_{s+1}, \ldots, r_{q+s+1}$ the q-secants through R. A line r_i, $1 \leqslant i \leqslant s$, can contain at most one point of P_q or P_{q+1} apart from R, for, suppose $S, T \in P_q \cup P_{q+1} \cap r_i - \{R\}$, and let $l \leqslant h - 2$ be the number of other points of $K \cap r_i$. The remaining $q^2 + 1 - 3 - l$ external points have to be covered by at most $q^2 + q + t - (q + 1) - 2q = q^2 - 2q + t - 1$ q-secants, and this is clearly impossible. We can assume that $\{r_i \mid i = 1, \ldots, m\}$ is the set of lines which contain just one further point of $P_q \cup P_{q+1}$.

Now let γ be the number of points of $r_i \cap K$ $(i = 1, \ldots, m)$ which contain no q-secants. If $S \in r_i \cap P_q$ the remaining $q^2 + q + t - (q + 1) - q = q^2 - q + t - 1$ q-secants have to cover the remaining $q^2 + 1 - (2 + \gamma) = q^2 - 1 - \gamma$ points on r_i which implies $\gamma \geqslant q - t$. Similarly, if $S \in (r_i - R) \cap P_{q+1}$ we obtain $\gamma \geqslant q - t + 1$ so in either case r_i contains at least $q - t + 1$ points of K distinct from R.

If $n = s - m$, this implies that

$$n \leqslant h - m(q - t + 1)$$

and (1)

$$s \leqslant h - m(q - t)$$

Now, for any line r_i, $1 \leqslant i \leqslant s$, let $y_i = | r_i \cap \bigcup_{j=1}^{q-1} P_j |$ and $z_i = | r_i \cap P_0 |$ and let w_i be the number of q-secants through points of $r_i \cap \bigcup_{j=1}^{q-1} P_j$, and $\beta_i = w_i / y_i$. Counting q-secants through external points of r_i as before yields:

$$\beta_i \leqslant 1 + \frac{z_i + t - 1}{y_i}, \qquad m + 1 \leqslant i \leqslant s,$$

and (2)

$$\beta_i \leqslant 1 + \frac{z_i + t - q}{y_i}, \qquad 1 \leqslant i \leqslant m.$$

Let $x_j = | P_j |$, $1 \leqslant j \leqslant q + 1$ and $\alpha = | K \cap P_0 |$ and $x_{ij} = |(r_i - R) \cap P_j |$, $1 \leqslant i \leqslant q + 1 + s$. We have

$$1 + \sum_{i=1}^{q+1+s} \sum_{j=1}^{q+1} x_{ij} = \sum_{j=1}^{q+1} x_j = q^2 + h - \alpha,$$

$$\sum_{j=1}^{q-1} x_{ij} = y_i, \qquad \sum_{i=1}^{s} z_i = \alpha, \qquad \sum_{i=1}^{s} y_i = h - \alpha - m.$$

We now count pairs (P, s), where $P \in s$ is a point of K and s is a q-secant:

$$(q^2 + q + t)\, q = \sum_{j=1}^{q+1} j x_j = q \sum_{j=1}^{q+1} x_j + x_{q+1} - \sum_{j=1}^{q-1} (q - j)\, x_j$$

$$= q(q^2 + h - \alpha) + x_{q+1} - \sum_{j=1}^{q-1} (q - j)\, x_j .$$

Now, since $x_{q+1} \leqslant q^2 + m$, this yields

$$tq + \sum_{j=1}^{q-1} (q - j)\, x_j \leqslant m + q(h - \alpha). \tag{3}$$

We now have

$$\sum_{j=1}^{q-1} (q - j)\, x_j = \sum_{j=1}^{q-1} \sum_{i=1}^{q+s+1} (q - j)\, x_{ij}$$

$$\geqslant \sum_{j=1}^{q-1} \sum_{i=1}^{s} (q - j)\, x_{ij}$$

$$= \sum_{i=1}^{s} \left(q \sum_{j=1}^{q-1} x_{ij} - \sum_{j=1}^{q-1} j x_{ij} \right)$$

$$= \sum_{i=1}^{s} (q y_i - \beta_i y_i)$$

$$
\begin{aligned}
&= \sum_{i=1}^{m} (q - \beta_i)\, y_i + \sum_{i=m+1}^{s} (q - \beta_i)\, y_i \\
&\geqslant \sum_{i=1}^{m} \left(q - 1 - \frac{z_i + t - q}{y_i}\right) y_i \\
&\quad + \sum_{i=m+1}^{s} \left(q - 1 - \frac{z_i + t - 1}{y_i}\right) y_i \qquad \text{(by 2)} \\
&= (q-1) \sum_{i=1}^{s} y_i - \sum_{i=1}^{s} z_i - st + mq + n \\
&= (q-1)(h - \alpha - m) - \alpha - st + mq + n
\end{aligned}
$$

and, using (3),

$$tq - h - st + n \leqslant 0. \tag{4}$$

Now, $m \geqslant 1$ since if $m = 0$, $n = s$ and this last inequality yields $t(q - s) \leqslant h - s$ which is impossible since $t \geqslant 1$. Combining (1) and (4) we have

$$
\begin{aligned}
0 &\geqslant tq - h - t(h - m(q - t)) + n \\
&= tq - h(t + 1) + mt(q - t) + n \geqslant (q - t + 1)(t - 1) + 3 + n
\end{aligned}
$$

because $h \leqslant q - 2$ and $m \geqslant 1$. But this last expression is strictly positive, which is a contradiction, and hence the theorem is proved.

3. Some Examples

In this section we give some specific examples for the case $q = 3$; in particular we exhibit complete $\{k; 3\}$-arcs for every possible value of k in the desarguesian plane of order 9, PG(2, 9).

Results of Cossu [2] and Barlotti [1] show that in this plane such arcs can exist only if $k \leqslant 19$, and an elementary counting argument shows that for completeness $k > 10$. Computer programs were written for a CDC 7600 computer using backtrack algorithms to effect exhaustive searches for complete arcs with $k = 11$, 18 and 19. Our conclusion is that no such arcs exist. On the other hand, theorem 1 guarantees the existence of complete arcs with $k = 12$ and 15.

Let PG(2, 9) be coordinatized homogeneously over $GF(9) = \{i + j\alpha \mid i, j \in GF(3), \alpha^2 = 2\alpha + 1\}$ and let π_0 be the Baer subplane coordinatized by GF(3). If l is the line $z = 0$, define $A = \pi_0 - l - \{(0, 0, 1)\}$, $B = \{(1, 2\alpha, 0),$

$(1, 2\alpha + 1, 0)$, $(1, 2\alpha + 2, 0)\}$, $C = \{(0, \alpha, 1)$, $(\alpha, \alpha, 1)$, $(\alpha + 2, 2\alpha + 1, 1)$, $(2\alpha + 2, 0, 1)\}$, and $D = \{(0, \alpha, 1), (\alpha, \alpha, 1), (2\alpha, \alpha, 1), (2\alpha, 0, 1)\}$. It is then easy to verify that the following are all complete $\{k; 3\}$-arcs:

(i) $k = 12 : A \cup B \cup \{0, 0, 1)\}$,

(ii) $k = 13 : A \cup C \cup \{(1, \alpha + 2, 0)\}$,

(iii) $k = 14 : A \cup \{(0, \alpha, 1), (\alpha, \alpha, 1), (2\alpha + 1, 0, 1), (2\alpha + 1, \alpha + 2, 1), (1, \alpha, 0), (1, \alpha + 1, 0)\}$,

(iv) $k = 15 : A \cup B \cup D$,

(v) $k = 16 : A \cup B \cup D \cup \{(1, 2\alpha + 2, 1), (\alpha, 0, 1)\} - \{(1, 0, 1)\}$,

(vi) $k = 17 : \{(0, 1, 1), (0, 2, 1), (1, 0, 1), (1, 2, 1), (2, 0, 1), (0, \alpha, 1), (2, \alpha, 1), (2, 2\alpha + 2, 1), (\alpha, 0, 1), (\alpha + 1, 1, 1), (\alpha + 1, \alpha, 1), (\alpha + 1, 2\alpha, 1), (2\alpha, 2, 1), (2\alpha + 2, 1, 1), (1, \alpha + 1, 0), (1, \alpha + 2, 0), (1, 2\alpha, 0)\}$.

We have thus shown:

THEOREM 3. *In* PG(2, 9) *a complete* $\{k; 3\}$*-arc exists if and only if* $12 \leqslant k \leqslant 17$.

REFERENCES

1. A. BARLOTTI, Sui $\{k; n\}$-archi di un piano lineare finito, *Boll. Un. Mat. Ital.* **11** (1956), 553–556.
2. A. COSSU, Su alcune proprietà dei $\{k; n\}$-archi di un piano proiettivo sopra un corpo finito, *Rend. Mat. e Appl.* **20** (1961), 271–277.
3. R. H. F. DENNISTON, Some maximal arcs in finite projective planes, *J. Combinatorial Theory* **6** (1969), 317–319.
4. JU. N. HAIMULIN, Some properties of $\{k; n\}_q$-arcs in Galois Planes, *Soviet Math. Dokl.* **7** (1966), 1100–1103.
5. J. A. THAS, Construction of maximal arcs and partial geometries, *Geometriae Dedicata* **3** (1974), 61–64.
6. J. A. THAS, Some results concerning $\{(q + 1)(n - 1); n\}$-arcs and $\{(q + 1)(n - 1) + 1; n\}$-arcs in finite projective planes of order q, *J. Combinatorial Theory* **19** (1975), 228–232.