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Non-nesting actions of Polish groups on real trees

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We prove that if a Polish group G with a comeagre conjugacy class has a non-nesting action on an $\mathbb{R}$-tree, then every element of G fixes a point.

0. Introduction

Non-nesting actions by homeomorphisms on $\mathbb{R}$-trees frequently arise in geometric group theory. For instance, they occur in Bowditch's study of cut points of the boundary at infinity of a hyperbolic group [1], or in the Drutu–Sapir study of tree-graded spaces [4], and their relations with isometric actions were studied in [8]. Non-nesting property is a topological substitute for an isometric action. It asks that no interval of the $\mathbb{R}$-tree is sent properly into itself by an element of the group.

In this paper, we are concerned with a Polish group G having a comeagre conjugacy class. The group S_∞ of all permutations of $\mathbb{N}$ and more generally the automorphism group of any ω-stable ω-categorical structure (see [5]) provide typical model-theoretic examples. Among other examples, we mention the automorphism group of the random graph and the groups $Aut(\mathbb{Q}, <)$, $Homeo(2^{\mathbb{N}})$ and $Homeo_+(\mathbb{R})$. The latter ones appear in [7,12] as important cases of extreme amenability and automatic continuity of homomorphisms. The property of having a comeagre conjugacy class plays an essential role in these respects.

The following theorem is the main result of the paper:

> Consider a group G with a non-nesting action on an $\mathbb{R}$-tree T. If G is a Polish group with a comeagre conjugacy class, then every element of G fixes a point in T.

This theorem generalizes the main result of the paper of Macpherson and Thomas [9] (where the authors study actions of Polish groups on simplicial trees) and extends Section 8 of the paper of Rosendal [11] (concerning isometric actions on Λ-trees). It is worth noting that some related problems have been studied before (see [1–3,8]). Our motivation is partially based on these investigations.

1. Non-nesting actions on $\mathbb{R}$-trees

Definition 1.1. An $\mathbb{R}$-tree is a metric space T such that for any $x \neq y \in T$, there is a unique topologically embedded arc joining x to y, and this arc is isometric to some interval of $\mathbb{R}$.

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Equivalently, as a topological space, T is a metrizable, uniquely arc-connected, locally arc-connected topological space [10]. We define $[x, y]$ as the arc joining x to y if $x \neq y$, and $[x, y] = \{x\}$ if $x = y$. We say that $[x, y]$ is a *segment*.

A subset $S \subseteq T$ is *convex* if $(\forall x, y \in S)[x, y] \subseteq S$. A convex subset is also called a *subtree*. Given $x, y, z \in T$, there is a unique element $c \in [x, y] \cap [y, z] \cap [z, x]$, called the *median* of x, y, z. When $c \notin \{x, y, z\}$, the subtree $[x, y] \cup [x, z] \cup [y, z]$ is called a *tripod*. A *line* is a convex subset containing no tripod and maximal for inclusion.

Given two disjoint closed subtrees $A, B \subseteq T$, there exists a unique pair of points $a \in A$, $b \in B$ such that for all $x \in A$, $y \in B$, $[x, y] \supseteq [a, b]$. The segment $[a, b]$ is called the *bridge* between A and B. If $x \notin A$, the *projection* of x on A is the point $a \in A$ such that $[x, a]$ is the bridge between $\{x\}$ and A.

The betweenness relation B of T is the ternary relation $B(x; y, z)$ defined by $x \in (y, z)$. A *weak homeomorphism* of the $\mathbb{R}$-tree T is a bijection $g : T \to T$ which preserves the betweenness relation. Any homeomorphism of T is clearly a weak homeomorphism. All actions on T are via weak homeomorphisms.

Remark 1.2. If $g : T \to T$ is a weak homeomorphism, then its restriction to each segment, to each line, and to each finite union of segments is a homeomorphism onto its image (for the topology induced by the metric). This is because the metric topology agrees with the topology induced by the order on a line or a segment. Conversely, any bijection $g : T \to T$ which maps each segment homeomorphically onto its image is a weak homeomorphism as it maps $[x, y]$ to the unique embedded arc joining $g(x)$ to $g(y)$.

Remark 1.3. If $S \subseteq T$ is a subtree, then S is closed (for the topology induced by the metric) if and only if $S \cap I$ is closed in I for every segment I. In particular, a weak homeomorphism preserves the set of closed subtrees.

Definition 1.4. An action of G on T by weak homeomorphisms is *non-nesting* if there is no segment $I \subseteq T$, and no $g \in G$ such that $g(I) \subsetneq I$.

From now on, we assume that G has a non-nesting action on an $\mathbb{R}$-tree T. We say that $g \in G$ is *elliptic* if it has a fixed point, and *loxodromic* otherwise.

Lemma 1.5 (*[8, Theorem 3]*)**.** *Let G be a group with a non-nesting action on an $\mathbb{R}$-tree T.*

- *If g is elliptic, its set of fix points T^g is a closed convex subset.*
- *If g is loxodromic, there exists a unique line L_g preserved by g; moreover, g acts on L_g by an order preserving transformation, which is a translation up to topological conjugacy.*

In [8], g is assumed to be a homeomorphism, but the argument still applies, except to prove that T^g is closed. This fact follows from Remark 1.3.

When g is loxodromic, L_g is called the *axis* of g. The action of g on L_g defines a natural ordering on L_g such that for all $x \in L_g$, $x < g(x)$.

The proof of the following lemma is standard (by arguments from [14], Section 3.1) and can be found in [6].

Lemma 1.6. *If g is loxodromic, then for any $p \in T$, $[p, g(p)]$ meets L_g and $[p, g(p)] \cap L_g = [q, g(q)]$ for some $q \in L_g$.*

Proposition 1.7. *Let G be a group with a non-nesting action on an $\mathbb{R}$-tree T. Then*

(1) *If g is elliptic and $x \notin T^g$, then $[x, g(x)] \cap T^g = \{a\}$ where a is the projection of x on T^g.*
(2) *If $g, h \in G$ are elliptic and $T^g \cap T^h = \emptyset$, then gh is loxodromic, its axis contains the bridge between T^g and T^h, and $T^g \cap L_{gh}$ (resp. $T^h \cap L_{gh}$) contains exactly one point. In particular, if g, h and gh are elliptic, then $T^g \cap T^h \cap T^{gh} \neq \emptyset$.*
(3) *Let $h, h' \in G$ be loxodromic elements, and $a \in L_h$ be such that for some $a' \in T$, $[a', (h')^2(a')] \subseteq [a, h(a)]$. Then h and h′ are not conjugate.*

These facts are classical for isometries of an $\mathbb{R}$-tree. Assertion (3) is some substitute for the fact that the translation length of an isometry is a conjugacy invariant.

Proof. To prove Assertion (1), consider $x \notin T^g$, and $I = [x, a]$ the bridge between $\{x\}$ and T^g. If $g(I) \cap I = \{a\}$, we are done. Assume otherwise that $g(I) \cap I = [a, b]$ for some $b \neq a$. Since $g(b) \neq b$, either $g.[a, b] \subsetneq [a, b]$ or $g.[a, b] \supsetneq [a, b]$, in contradiction with the non-nesting assumption.

To see (2), consider $I = [a, b]$ the bridge between T^g and T^h with $a \in T^g$, $b \in T^h$, and let $J = h^{-1}(I) \cup I$. By Assertion (1), $I \cap h^{-1}(I) = \{b\}$ (resp. $I \cap g(I) = \{a\}$, $I \cap h(I) = \{b\}$),) so $h^{-1}(a), b, a$ (resp. $b, a, g(b), a, b, h(a)$ hence $a = g(a), g(b), gh(a)$) are aligned in this order. In particular $h^{-1}(a), b, a, g(b), gh(a)$ are aligned in this order so $h^{-1}(I), I, g(I), gh(I)$ are four consecutive non-degenerate subsegments of the segment $[h^{-1}(a), gh(a)]$. This implies that $gh(J) \cap J = \{a\}$. If gh was elliptic, $J = [h^{-1}(a), gh(h^{-1}(a)]$ would contain a point fixed by gh, and this fix point would have to lie in $gh(J) \cap J$, but this is impossible since $gh(a) \neq a$. We claim that $J \subseteq L_{gh}$. Otherwise, the segment $J_0 = J \cap L_{gh}$ is a proper subsegment of J, and $gh(J_0) \cap J_0 = \emptyset$, contradicting Lemma 1.6. Since $J \cap T^h = \{b\}$ and since T^h is convex, $L_{gh} \cap T^h = \{b\}$. Similarly, $(I \cup g(I)) \cap T^g = \{a\}$ implies that $L_{gh} \cap T^g = \{a\}$.

Statement (3) is easy: let $I = [a, h(a)] \subseteq L_h$, and let $I' = [a', (h')^2(a')] \subseteq I$. By Lemma 1.6, changing I' to some subsegment, we may assume that $I' \subseteq L_{h'}$ so that I' is a fundamental domain for the action of $(h')^2$ on $L_{h'}$ by Lemma 1.5. If $h' = h^g$, $g^{-1}(L_h) = L_{h'}$ and $g^{-1}(I)$ is a fundamental domain for the action of h' on $L_{h'}$. Replacing g by some $g(h')^i$ ($i \in \mathbb{Z}$), if necessary we obtain $g^{-1}(I) \subsetneq I' \subseteq I$, a contradiction with the non-nesting assumption. □

2. Polish groups with comeagre conjugacy classes

A Polish group is a topological group whose topology is *Polish* (a Polish space is a separable completely metrizable topological space). A subset of a Polish space is *comeagre* if it contains an intersection of a countable family of dense open sets.

Macpherson and Thomas have proved in [9] that if a Polish group has a comeagre conjugacy class then every element of the group fixes a point under any action on a $\mathbb{Z}$-tree without inversions. Ch.Rosendal has generalized this theorem to the case when the group acts on an Λ-tree by isometries (see Section 8 in [11]). In this section we consider the case of non-nesting actions.

Theorem 2.1. *Consider a group G with a non-nesting action on an $\mathbb{R}$-tree T. If G is a Polish group with a comeagre conjugacy class, then every element of G is elliptic.*

Remark 2.2. We don't assume any relation between the action of G and its topology as a Polish group: the action of g is not assumed to depend continuously on g.

Remark 2.3. Using Proposition 1.7(2), one can extend the proof of Serre's Lemma [13, Prop 6.5.2], and show that every finitely generated subgroup of G fixes a point in T. It follows that G fixes a point or an end of T.

We start with the following lemma.

Lemma 2.4. *Under the circumstances of Theorem 2.1, assume that $h_1, h_2 \in G$ are conjugate and loxodromic, and that $g = h_2h_1$ is conjugate to h_1^6 or h_1^{-6}. Then $L_{h_1} \cap L_{h_2} = \emptyset$.*

Moreover, denoting by $[a, b]$ the bridge between L_{h_1} and L_{h_2} with $a \in L_{h_1}$, $b \in L_{h_2}$ then

$$[h_1^{-1}(a), a] \cup [a, b] \cup [b, h_2(b)] \subseteq L_g$$

and $h_1^{-1}(a) < a < b < h_2(b)$ for the ordering of L_g defined after Lemma 1.5.

Proof. Assuming the contrary, consider $t \in L_{h_1} \cap L_{h_2}$ and $p = h_1^{-1}(t)$. Since $[p, g(p)] \subseteq [h_1^{-1}(t), t] \cup [t, h_2(t)]$, may find $q \in L_g$ such that $[q, g(q)] \subseteq [h_1^{-1}(t), t] \cup [t, h_2(t)]$.

Consider g_0 such that $g_0^6 = g$, and g_0 conjugate to h_1 or h_1^{-1}. Let $I = [q, g_0^2(q)]$. Since $L_{g_0} = L_g$, $I \subseteq L_{g_0}$ and $I \cup g_0^2(I) \cup g_0^4(I) = [q, g_0^6(q)] \subseteq [h_1^{-1}(t), t] \cup [t, h_2(t)]$. Either I or $g_0^4(I)$ is contained in $[h_1^{-1}(t), t]$ or in $[t, h_2(t)]$, say $I \subseteq [h_1^{-1}(t), t]$ for instance. Since $t \in L_{h_1}$, this contradicts Proposition 1.7(3).

To see the final statement note that L_g intersects $[h_1^{-1}h_2^{-1}(a), a]$ and $[b, h_2h_1(b)]$, hence contains the bridge between these segments, i.e. $[a, b]$. It follows that L_g contains $[h_1^{-1}h_2^{-1}(a), a] \supseteq [h_1^{-1}(a), a]$ and $[b, h_2h_1(b)] \supseteq [b, h_2(b)]$. The lemma follows. □

Proof of Theorem 2.1. Let X be a conjugacy class of G which is comeagre in G. Then $X \cap X^{-1} \neq \emptyset$, but since X is a conjugacy class $X = X^{-1}$. Note that

(*) For every sequence $g_1, \dots, g_m \in G$ there exist $h_0, h_1, \dots, h_m \in X$ such that for every $1 \le i \le m$, $g_i = h_0h_i$.

Indeed, let $g_1, \dots, g_m \in G$. Since X and g_iX^{-1} are comeagre in G, all g_iX^{-1} and X have a common element $h_0 \in X$. Now there are $h_1, \dots, h_m \in X$ such that for any $1 \le i \le m$, $g_i = h_0h_i$.

First assume that X consists of loxodromic elements, and argue towards a contradiction. Take $h \in X$ and consider $g = h^6$. By (*) above find $h_0, h_1, h_2 \in X$ such that $g = h_0h_1$ and $g^{-1} = h_0h_2$.

Applying Lemma 2.4 to h_0, h_1 and to h_0, h_2, we get that $L_{h_0} \cap L_{h_1} = \emptyset$ and $L_{h_0} \cap L_{h_2} = \emptyset$. Let $b \in L_{h_0}$ and $a \in L_{h_1}$ define the bridge between L_{h_0} and L_{h_1}, and let $b' \in L_{h_0}$ and $a' \in L_{h_2}$ define the bridge between L_{h_0} and L_{h_2}. Since $L_g = L_{g^{-1}}$, by Lemma 2.4 we see that the segments $[a, b] \cup [b, h_0(b)]$ and $[a', b'] \cup [b', h_0(b')]$ belong to L_g. Since L_g does not contain a tripod, $b = b'$. Then $b < h_0(b)$ both with respect to the order defined by g and by g^{-1}. This is a contradiction, so X consists of elliptic elements.

Assume that some $g \in G$ is loxodromic, and argue towards a contradiction. Write $g = h' \cdot h$ for some $h, h' \in X$. Then $T^h \cap T^{h'} = \emptyset$ and denote by I the bridge between T^h and $T^{h'}$. By Proposition 1.7(2) $I \subseteq L_g$.

By (*) there exist $h_0, h_1, h_2, h_3 \in X$ such that $h = h_0h_1$, $h' = h_0h_2$, and $g = h_0h_3$. By Proposition 1.7(2) there are $a_1 \in T^{h_0} \cap T^h$ and $b_1 \in T^{h_0} \cap T^{h'}$. Then $I \subseteq [a_1, b_1] \subseteq T^{h_0}$. On the other hand, by Proposition 1.7(2) applied to h_0 and h_3, the intersection $T^{h_0} \cap L_g$ is a singleton. Since I is contained in this intersection, this is a contradiction. □

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