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On Artinian Rings of Finite Representation Type

KUNIO YAMAGATA

Department of Mathematics, University of Tsukuba, Sakuramura Ibaraki, 300-31, Japan

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In [5], Roiter has solved the Brauer–Thrall conjecture for finite-dimensional algebras over fields, which states that, if the lengths of the finitely generated indecomposable modules are bounded, then there is only a finite number of finitely generated indecomposable modules. Recently Auslander [1] has proved it for Artinian rings. In this paper, we show how we construct all indecomposable modules from simple modules over an Artinian ring of finite representation type. That is, if A is an Artinian ring of finite representation type, then every indecomposable A-module appears as a direct summand of (a) the radical of a projective indecomposable A-module or (b) the middle term of an almost split sequence [2], which is successively obtained from a simple A-module. Simultaneously, we give a module-theoretical, self-contained, and simple proof for the conjecture though Auslander's proof is categorical.

Throughout this paper A will be a right Artinian ring with identity and all modules will be finitely generated right A-modules. Let M be an indecomposable module and N a module. Following Auslander [1], a homomorphism $f: N \to M$ is said to be *almost splittable* if (a) it is not a splittable epimorphism and (b) for any homomorphism $g: X \to M$ which is not a splittable epimorphism, there is a homomorphism $h: X \to N$ such that $g = fh$. In the following, an almost splittable homomorphism $f: N \to M$ will be called *almost split extension over* M provided that (a) if M is projective, then N is the unique maximal submodule of M and f is the inclusion, or (b) if M is not projective, then f is an epimorphism and Ker f is indecomposable, in which case $0 \to \operatorname{Ker} f \to N \xrightarrow{f} M \to 0$ is called *almost split sequence* in the sense of [2]. It is known [1, 2] that an almost split extension is uniquely determined up to isomorphism and that if the ring A is an Artin algebra or is of finite representation type, then there is an almost split extension over any indecomposable A-module. But it is an open question whether almost split extensions always exist for arbitrary right Artinian rings.

In the following, $[M]$ denotes the isomorphism class of a given module M.

For an indecomposable module M, we define a set $\mathbf{E}_n(M)$ $(n \geqslant 0)$ of finitely many isomorphism classes of indecomposable modules as follows:

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(i) $\mathbf{E}_0(M) = \{[M]\}$;

(ii) $[X] \in \mathbf{E}_{n+1}(M)$ if and only if X is a direct summand of an almost split extension over some module whose isomorphism class belongs to $\mathbf{E}_n(M)$.

Then our main theorem shows that

THEOREM. *Let A be a right Artinian ring and $\{[S_i] \mid 1 \leqslant i \leqslant n\}$ the set of all isomorphism classes of simple right A-modules. If A is of finite representation type, then there is an integer m such that*

$$\mathbf{E}_{m+1}(S) \subset \bigcup_{\substack{1 \leqslant i \leqslant n \\ 0 \leqslant j \leqslant m}} \mathbf{E}_j(S_i)$$

for every simple right A-module S. And if A is an Artin algebra, the converse holds. Further, in these cases,

$$\bigcup_{\substack{1 \leqslant i \leqslant n \\ 0 \leqslant j \leqslant m}} \mathbf{E}_j(S_i)$$

is the set of all isomorphism classes of indecomposable right A-modules.

We recall the definitions from [1]. Let $\{M_i \mid i \in I\}$ be a family of finitely generated indecomposable modules. Then the family is called *noetherian* (resp., *conoetherian*) if for any sequence of *nonisomorphisms*

$$M_{i_1} \xrightarrow{f_{i_1}} M_{i_2} \xrightarrow{f_{i_2}} M_{i_3} \to \cdots$$

$$(\text{resp., } \cdots \to M_{i_3} \xrightarrow{f_{i_2}} M_{i_2} \xrightarrow{f_{i_1}} M_{i_1}),$$

there is an integer n such that $f_{i_n} \cdots f_{i_2} f_{i_1} = 0$ (resp., $f_{i_1} f_{i_2} \cdots f_{i_n} = 0$) (here we do not assume that $M_{i_j} \neq M_{i_k}$ for $j \neq k$). If the family of all finitely generated indecomposable right A-modules is noetherian (resp., conoetherian), we say that the ring A *satisfies the noetherian* (resp., *conoetherian*) *condition* for finitely generated indecomposable right A-modules.

The following lemma is well known and we omit the proof (cf. [3, Lemma 12]).

LEMMA 1. *Let $\{M_i \mid i \in I\}$ be a set of indecomposable modules such that for some integer m, length $(M_i) \leqslant m$ for all $i \in I$. Then there is an integer n such that $f_n \cdots f_2 f_1 = 0$ for any nonisomorphisms $f_j\colon M_{i_j} \to M_{i_{j+1}}$.*

LEMMA 2. *Let $f\colon N \to M$ be an almost split extension over an indecomposable right A-module M. Assume that $N = \sum \oplus_{i \in I} N_i$ is a direct sum decomposition into indecomposable submodules N_i's, and $\kappa_i\colon N_i \to N$ is a canonical injection for $i \in I$. Then every $f\kappa_i$ is not an isomorphism.*

Proof. This follows immediately from the definition.

THEOREM 1. *Let A be a right Artinian ring and $\{[S_i] \mid 1 \leqslant i \leqslant n\}$ the set of all isomorphism classes of simple right A-modules. We assume that there is an almost split extension over any finitely generated indecomposable right A-module. Then the following are equivalent.*

(i) *A is of finite representation type.*

(ii) *A satisfies the conoetherian condition for finitely generated indecomposable right A-modules.*

(iii) *There is an integer m such that*

$$\mathbf{E}_{m+1}(S) \subset \bigcup_{\substack{1 \leqslant i \leqslant n \\ 0 \leqslant j \leqslant m}} \mathbf{E}_j(S_i)$$

for every simple right A-module S.

Further, in these cases

$$\bigcup_{\substack{1 \leqslant i \leqslant n \\ 0 \leqslant j \leqslant m}} \mathbf{E}_j(S_i)$$

is the set of all isomorphism classes of finitely generated indecomposable right A-modules.

Remark 1. The equivalence (i) $\Leftrightarrow$ (ii) is proved by Auslander [1]. However, we shall give a module-theoretical proof for it.

Proof. (i) $\Rightarrow$ (ii) is followed by Lemma 1.

(ii) $\Rightarrow$ (iii). For each S_i, we define a finite set $\mathbf{H}_l(S_i)$ of nonisomorphisms $f_l \colon M_l \to M_{l-1}$ with $[M_l] \in \mathbf{E}_l(S_i)$ and $[M_{l-1}] \in \mathbf{E}_{l-1}(S_i)$ as follows: if M_l is a direct summand of an almost split extension N_l over M_{l-1} with a canonical homomorphism $u_l \colon N_l \to M_{l-1}$, then we put $f_l = u_l \kappa_l$, where κ_l is a canonical injection of M_l to N_l, and otherwise, we put $f_l = 0$. Here M_0 denotes the S_i, and in either case, f_l is not an isomorphism by Lemma 2.

Let $A_{i,n}$ be a finite set $\{f_1 \cdots f_n \mid f_j \in \mathbf{H}_j(S_i),\ f_1 \cdots f_n \neq 0\}$ and F_i a family of functions $\{\theta_{i,n}\}$, where $\theta_{i,n}$ is a function of $A_{i,n}$ to the power set of $A_{i,n+1}$ such that $\theta_{i,n}(a_n) = \{a_n f_{n+1} \mid f_{n+1} \in \mathbf{H}_{n+1}(S_i),\ a_n f_{n+1} \neq 0\}$ for each $a_n \in A_{i,n}$. Then, applying König's Graph Theorem to the graph $(\{A_{i,n}\}, F_i)$ (cf. [4, Lemma 10]), we obtain an integer m_i for each S_i such that $f_1 \cdots f_{m_i} = 0$ for all $f_j \in \mathbf{H}_j(S_i)$ $(1 \leqslant j \leqslant m_i)$. We put $m = \operatorname{Max}\{m_i \mid 1 \leqslant i \leqslant n\}$.

Now, suppose that

$$[M] \notin \bigcup_{\substack{1 \leqslant i \leqslant n \\ 0 \leqslant j \leqslant m}} \mathbf{E}_j(S_i) = \mathbf{E}.$$

Since all isomorphism classes of simple modules are contained in $\mathbf{E}$, M is not simple. Hence there is a nonzero homomorphism $g: M \to S$ such that it is not a splittable epimorphism and S is a nonprojective simple module. By the definition, there are homomorphisms $g_1': M \to N_1$ and $f_1': N_1 \to S$ such that $g = f_1'g_1'$, where f_1' is an almost split extension over S. This means that there are homomorphisms $g_1: M \to M_1$ and $f_1: M_1 \to S$ such that $[M_1] \in \mathbf{E}_1(S)$, $f_1 g_1 \neq 0$ and $f_1 \in \mathbf{H}_1(S)$ by Lemma 2. Next, consider g_1 instead of g in the above. Then, by the same method, it is seen that there are $g_2: M \to M_2$ and $f_2: M_2 \to M_1$ such that $[M_2] \in \mathbf{E}_2(S)$, $f_1 f_2 g_2 \neq 0$, and $f_2 \in \mathbf{H}_2(S)$. Repeating this way we have homomorphisms $f_i \in \mathbf{H}_i(S)$ $(1 \leqslant i \leqslant m)$ such that $f_1 \cdots f_{m-1} f_m \neq 0$, which contradicts the choice of m. This completes the proof (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (i). Assume that there is an indecomposable module M such that

$$[M] \notin \bigcup_{\substack{1 \leqslant i \leqslant n \\ 0 \leqslant j \leqslant m}} \mathbf{E}_j(S_i).$$

Then M is not simple and therefore there is a nonprojective simple module S which is a proper homomorphic image of M. By the same reason as in the proof (ii) $\Rightarrow$ (iii), we have nonisomorphisms $f_i: M_i \to M_{i-1}$ such that $f_i \in \mathbf{H}_i(S)$ and $f_1 \cdots f_{m+1} \neq 0$, where M_0 denotes S. Furthermore, by the definition of almost split extension and the assumption of M, we also have a homomorphism f_{m+2}: $M_{m+2} \to M_{m+1}$ such that $f_1 \cdots f_{m+1} f_{m+2} \neq 0$ and $[M_{m+2}] \in \mathbf{E}_{m+2}(S)$. Here, we can choose a nonisomorphism f_{m+2} by Lemma 2. Repeating this method, we obtain a series of nonisomorphisms

$$\cdots \to M_{m+2} \xrightarrow{f_{m+2}} M_{m+1} \xrightarrow{f_{m+1}} M_m \to \cdots \to M_2 \xrightarrow{f_2} M_1 \xrightarrow{f_1} S$$

such that each

$$[M_k] \in \mathbf{E} = \bigcup_{\substack{1 \leqslant i \leqslant n \\ 0 \leqslant j \leqslant m}} \mathbf{E}_j(S_i) \qquad \text{and} \qquad f_1 \cdots f_{k-1} f_k \neq 0$$

for any $k > 0$. This, however, contradicts Lemma 1, because $\mathbf{E}$ is a finite set and so there is an integer l such that length $(M_k) \leqslant l$ for all $k > 0$.

The following Lemmas 3 and 4 have been essentially proved by Auslander. But for the sake of completeness we shall give the direct and module-theoretical proofs.

LEMMA 3 [1]. *Let a right Artinian ring A satisfy the noetherian condition for indecomposable modules, and let E: $0 \to L \to N \to M \to 0$ be a nonsplittable exact sequence, where M and L are indecomposable. Then there are an indecomposable module L_1 and a homomorphism $f: L \to L_1$ such that*

(i) $f \cdot E \neq 0$ *in* $\mathrm{Ext}_A^1(M, L_1)$,

(ii) *if* $g \cdot f \cdot E \neq 0$ *for a homomorphism* $g: L_1 \to X_1$, *then* g *is a splittable monomorphism.*

Proof. Suppose that, for any homomorphism $f: L \to L'$ such that $f \cdot E \neq 0$ and L' is indecomposable, there is a nonisomorphism $g: L' \to L''$ such that $g \cdot f \cdot E \neq 0$ and L'' is indecomposable. Then for the identity 1_L, there is a nonisomorphism $f_2: L_2 \to L_3$ with $f_2 \cdot f_1 \cdot E \neq 0$ (where $f_1 = 1_L$ and $L = L_1 = L_2$), in particular $f_2 f_1 \neq 0$. In this way, we obtain a sequence of nonisomorphisms

$$L_1 \xrightarrow{f_1} L_2 \xrightarrow{f_2} L_3 \to \cdots$$

such that $f_n \cdots f_2 f_1 \neq 0$ for any $n \geqslant 1$, where all L's are indecomposable. This contradicts the noetherian condition. Hence there is a homomorphism $f: L \to L_1$ such that (0) L_1 is indecomposable, (i) $f \cdot E \neq 0$, and (ii′) $g \cdot f \cdot E = 0$ for every nonisomorphism $g: L_1 \to X$ with X indecomposable. Now we show that f satisfies the property (ii). Let $g: L_1 \to X$ be a nonzero homomorphism such that $g \cdot f \cdot E \neq 0$. For an indecomposable decomposition $X = \bigoplus_{i=1}^m X_i$, let κ_i: $X_i \to X$ and ρ_i: $X \to X_i$ be a canonical injection and projection, respectively, and put $g_i = \kappa_i \rho_i g$. Then $g = \sum_{i=1}^m g_i$ and $0 \neq g \cdot f \cdot E = \sum_{i=1}^m (g_i \cdot f \cdot E)$. Hence there is some g_i, say g_1, with $g_1 \cdot f \cdot E \neq 0$. Then it is easily proved that $g_1 \cdot f \cdot E \neq 0$ shows that $(\rho_1 g) \cdot f \cdot E \neq 0$ holds. Therefore, by the first result (ii′), $\rho_1 g$ must be an isomorphism and so g is a splittable monomorphism.

LEMMA 4 [1]. *Assume that a right Artinian ring A satisfies the noetherian condition for indecomposable modules. Then for any indecomposable A-module M, there is an almost split extension over M.*

Proof. If M is projective, it has a unique maximal submodule MJ, where J is the Jacobson radical of A. Hence it is clear that the inclusion $f: MJ \hookrightarrow M$ is almost splittable.

Let M be not projective. Then there is an indecomposable module L' such that $\mathrm{Ext}_A^1(M, L') \neq 0$. Let $E' \in \mathrm{Ext}_A^1(M, L')$ be a nonsplittable sequence E': $0 \to L' \to N \to M \to 0$. By Lemma 3, there is a homomorphism $a: L' \to L$ which satisfies the conditions (i) and (ii) in Lemma 3. We denote $a \cdot E'$ by E: $0 \to L \xrightarrow{u} N \xrightarrow{v} M \to 0$. Let $f: X \to M$ be not a splittable epimorphism. Then we can show that there is a homomorphism $g: X \to N$ with $f = vg$. For this, consider the following commutative diagram

$$\begin{array}{ccccccc} E: 0 \to & L & \xrightarrow{u} & N & \xrightarrow{v} & M \to 0 \\ & \downarrow s & & \downarrow \binom{1_N}{0} & & \| \\ s \cdot E: 0 \to & \mathrm{Ker}\,\phi & \to & N \oplus X & \xrightarrow[\phi = (v, f)]{} & M \to 0, \end{array}$$

where $\binom{1_N}{0}: N \to N \oplus X$ and $\phi = (v, f): N \oplus X \to M$ (here we represent homomorphisms by matrices according to the decompositions of given modules). If we can show that $s \cdot E \neq 0$, then s must be a splittable monomorphism by Lemma 3, say t: $\operatorname{Ker}\phi \to L$ and $ts = 1_L$. Since $E = 1_L \cdot E = t \cdot s \cdot E$, there is a homomorphism $(w, g): N \oplus X \to N$ such that $(v, f) = v(w, g)$, and so $f = vg$. Thus we have only to prove that $s \cdot E \neq 0$. Now we suppose the contrary. Then there is a $\psi = \binom{v'}{f'}: M \to N \oplus X$ with $1_M = \phi\psi$. Since the endomorphism ring of M is local and $1_M = vv' + ff'$, either vv' or ff' must be an automorphism. On the other hand, clearly vv' is not an automorphism, because $v: N \to M$ is not splittable. Moreover, since f is not a splittable epimorphism by assumption, ff' is not an automorphism, either. Thus we have a contradiction and conclude the lemma.

Now, by Theorem 1 and Lemmas 1 and 4, it is easy to prove module theoretically the Brauer-Thrall conjecture solved by Roiter and Auslander.

THEOREM 2 [1, 5]. *For a right Artinian ring A the following are equivalent.*

(i) *A is of finite representation type.*

(ii) *A satisfies the noetherian and conoetherian conditions for finitely generated indecomposable right A-modules.*

In particular, if there is an integer n such that $\text{length}(M) \leqslant n$ *for all finitely generated indecomposable right A-modules M, then there is only a finite number of finitely generated indecomposable right A-modules.*

Remark 2. It is proved in [1] that *for an Artinian ring A of finite representation type, the endomorphism ring of any finitely generated right A-module is also Artinian.* We can directly prove this result, too.

Let M be a direct sum of all nonisomorphic indecomposable right A-modules M_l $(1 \leqslant l \leqslant n)$ with the canonical injections κ_l: $M_l \to M$ and projections ρ_l: $M \to M_l$ and $B = \operatorname{End}_A(M)$ the endomorphism ring of M. Let W be the Jacobson radical of B and $\kappa_l \rho_l = e_l$. Then $1_B = \sum_{l=1}^n e_l$ and W $(= \sum_l \oplus e_l W)$ consists of all such endomorphisms b of M as each $\rho_\beta b \kappa_\alpha$ is not an isomorphism. Then, since B is semiprimary, to show that B is right Artinian, it suffices to show that each $e_\alpha W$ is finitely generated as a right B-module. Let $f: N \to M_\alpha$ be almost split extension over M_α. Since A is of finite representation type, N can be embedded in a finite direct sum $M^{(m)}$ of m copies of M as a direct summand. Let $\kappa: N \to M^{(m)}$ and $\rho: M^{(m)} \to N$ be homomorphisms with $\rho\kappa = 1_N$. Then, since $\rho_\alpha g \kappa_\beta$: $M_\beta \to M_\alpha$ is not an isomorphism for any $g \in W$, there is a homomorphism h_β: $M_\beta \to N$ such that $\rho_\alpha g \kappa_\beta = f h_\beta$ and so $\rho_\alpha g \kappa_\beta = f \rho \kappa h_\beta = f\rho(\sum_{i=1}^m \psi_i \phi_i)\, \kappa h_\beta = \sum_{i=1}^m (f\rho\psi_i)(\phi_i \kappa h_\beta)$, where ψ_i and ϕ_j are canonical ith injection $M \to M^{(m)}$ and jth projection $M^{(m)} \to M$, respectively. Therefore $e_\alpha g e_\beta =$

$\sum_{i=1}^{m}(\kappa_\alpha f\rho\psi_i)(\phi_i\kappa h_\beta\rho_\beta)$. Clearly $\kappa_\alpha f\rho\psi_i$ and $\phi_i\kappa h_\beta\rho_\beta$ belong to $e_\alpha W$ and B, respectively. Put $\kappa_\alpha f\rho\psi_i = w_i$ and $\phi_i\kappa h_\beta\rho_\beta = b_{\beta,i}$. Then we have the

$$e_\alpha g = \sum_\beta e_\alpha g e_\beta = \sum_{\substack{1\leqslant i\leqslant m\\ 1\leqslant\beta\leqslant n}} w_i b_{\beta,i}\,.$$

Since each w_i does not depend on $g \in W$, this equation shows that $e_\alpha W = \sum_{1\leqslant i\leqslant m} w_i B$ and hence W is finitely generated as a right B-module.

Considering the dual of the almost split sequence and the projective module, we can also show that B is left Artinian.

Remark 3. We will conclude this paper with noting a method of construction of an almost split sequence of a given indecomposable module. If we know how to decompose a given module into a direct sum of indecomposable modules, this method and Theorem 1 will suggest a way of constructing all indecomposable modules from simple modules over an Artin algebra of finite representation type.

Let A be an Artin algebra over a center C and M a nonprojective indecomposable A-module. Then we know from [2, Theorem 4.2] that there is an almost split sequence E for M and it is characterized by the property that $\mathrm{Ext}_A^1(M, DTr(M)) \ni E \neq 0$ and $g \cdot E = 0$ for all nonisomorphisms $g\colon DTr(M) \to DTr(M)$, where $Tr(M) = \mathrm{Coker}(\mathrm{Hom}_A(\rho, A))$ for a minimal projective presentation $P_1 \xrightarrow{\rho} P_0 \to M \to 0$, and $D = \mathrm{Hom}_C(\ \ , E(C/\mathrm{rad}\ C))$ a duality functor with an injective hull $E(C/\mathrm{rad}\ C)$ of $C/\mathrm{rad}\ C$.

Now let $0 \to K \xrightarrow{u} P \xrightarrow{v} M \to 0$ be a projective cover of M. For any maximal submodule K_0 of K, we have the following commutative diagram by the definition of almost split sequence:

$$\begin{array}{ccccccc} E_0 : 0 \longrightarrow & K/K_0 & \xrightarrow{\bar u} & P/K_0 & \xrightarrow{\bar v} & M \to 0 \\ & \downarrow & & \downarrow & & \| \\ E : 0 \to & D\,\mathrm{Tr}(M) & \longrightarrow & X & \longrightarrow & M \to 0, \end{array}$$

where the top row is canonically indued from the projective cover of M and the bottom row denotes the almost split sequence E. This means that there is a homomorphism $f_1\colon K/K_0 \to DTr(M)$ with $f_1 \cdot E_0 \neq 0$, and we put $E_1 = f_1 \cdot E_0$. If we have already known E_{i-1} $(i > 1)$ and still there is a nonisomorphism f_i with $0 \neq f_i \cdot E_{i-1} \in \mathrm{Ext}_A^1(M, DTr(M))$, then we consider $E_i = f_i \cdot E_{i-1}$. Then, by repeating this method, finally there must be homomorphisms $f_s, \ldots, f_2, f_1$ such that $E_s = f_s \cdot E_{s-1} = \cdots = f_s \cdots f_2 \cdot f_1 \cdot E_0 \neq 0$, $g \cdot E_s = 0$ for all nonisomorphisms $g\colon DTr(M) \to DTr(M)$ and $1 \leqslant s \leqslant m$, where $m > 0$ and $\mathrm{rad}(\mathrm{End}(DTr(M)))^m = 0$. This shows that E_s is an almost split sequence E.

Therefore an almost split sequence E is obtained by a pushout for the diagram

$$\begin{array}{ccc} K/K_0 & \overset{\bar{u}}{\hookrightarrow} & P/K_0 \\ {\scriptstyle f_1}\downarrow & & \\ D\,\mathrm{Tr}(M) & & \\ {\scriptstyle f}\downarrow & & \\ D\,\mathrm{Tr}(M) & & \end{array}$$

where f is either an identity or a multiplication of at most $m - 1$ many non-isomorphisms from $DTr(M)$ to itself.

In particular, if A is an algebra over a field F, then clearly $\dim_F \mathrm{Hom}_A(K/K_0, DTr(M))$ and $\dim_F(\mathrm{End}(DTr(M))$ are finite. Let $\mathrm{Hom}_A(K/K_0, DTr(M)) = Fg_1 \oplus \cdots \oplus Fg_l$ and $\mathrm{End}(DTr(M)) = Fh_1 \oplus \cdots \oplus Fh_n$. Then, since $(\sum_{i=1}^{l} a_i g_i)\, E_0 = \sum \{g_i E_0 \mid a_i \neq 0\}$ for $a_i \in F$, we can always choose f_1 from the basis $g_1, ..., g_l$ and, by the same reason, f_i $(1 < i \leqslant s < m)$ from $h_1, ..., h_n$, in the above.

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