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Generalized Cayley graphs associated to commutative rings

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ABSTRACT

Let R be a commutative ring with identity element. For a natural number n, we associate a simple graph, denoted by Γ_R^n, with $R^n\backslash\{0\}$ as the vertex set and two distinct vertices X and Y in R^n being adjacent if and only if there exists an $n \times n$ lower triangular matrix A over R whose entries on the main diagonal are non-zero and such that $AX^T = Y^T$ or $AY^T = X^T$, where, for a matrix B, B^T is the matrix transpose of B. When we consider the ring R as a semigroup with respect to multiplication, then Γ_R^1 is the usual undirected Cayley graph (over a semigroup). Hence Γ_R^n is a generalization of Cayley graph. In this paper we study some basic properties of Γ_R^n. We also determine all isomorphic classes of finite commutative rings whose generalized Cayley graph has genus at most three.

1. Introduction

The definition of *Cayley graph* was introduced by Arthur Cayley in 1878 to explain the concept of abstract groups which are described by a set of generators. In the last 50 years, the theory of Cayley graphs has been grown into a substantial branch in algebraic graph theory. Cayley graphs have found many useful applications in solving and understanding a variety of problems of scientific interest (see the survey [8] and the monograph [6]). For a semigroup H and a subset S of H, the Cayley graph $\mathrm{Cay}(H, S)$ of H relative to S is defined as the graph with vertex set H and edge set $E(H, S)$ consisting of those ordered pairs (x, y) such that $sx = y$ for some $s \in S$ (cf. [7]). In the undirected Cayley graph $\overline{\mathrm{Cay}}(H, S)$ we assume that x is adjacent to y if and only if (x, y) or (y, x) is an element of $E(H, S)$, defined above.

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Let R be a commutative ring with identity element. In [12], Sharma and Bhatwadekar defined the *comaximal graph* on R, denoted by $\Gamma(R)$, with all elements of R being the vertices of $\Gamma(R)$, where two distinct vertices a and b are adjacent if and only if $aR + bR = R$. In [9,13], the authors considered a subgraph $\Gamma_2(R)$ of $\Gamma(R)$ consisting of non-unit elements of R, and studied several properties of the comaximal graph. Also the comaximal graph of a non-commutative ring was defined and studied in [14]. Recently in [1], the comaximal graph of a finite bounded lattice was introduced and studied. Note that the subgraph of the undirected Cayley graph $\overline{\text{Cay}}(R\backslash\{0\}, R\backslash\{0\})$ consisting of non-unit vertices is a subgraph of the complement of the comaximal graph $\Gamma_2(R)$. Moreover, the two subgraphs of the undirected Cayley graph $\overline{\text{Cay}}(R\backslash\{0\}, R\backslash\{0\})$ and the comaximal graph $\Gamma(R)$ consisting of unit elements of R, are isomorphic.

In this paper, for a natural number n and a commutative ring R with identity element, we associate a simple graph, denoted by Γ_R^n, with $R^n\backslash\{0\}$ as the vertex set and two distinct vertices X and Y in R^n being adjacent if and only if there exists an $n \times n$ lower triangular matrix A over R whose entries on the main diagonal are non-zero such that $AX^T = Y^T$ or $AY^T = X^T$. (We use ^{T} to denote matrix transpose.) In the case that $n = 1$, the resulting graph is the undirected graph $\overline{\text{Cay}}(H, S)$, where $H = S = R\backslash\{0\}$. Therefore, we assume that $n > 1$, in the rest of the paper. In Section 2, we study some basic properties of Γ_R^n. In Section 3, we investigate the genus number of the generalized Cayley graph Γ_R^n.

Now, we recall some definitions of graph theory which are necessary in this paper. Let G be a graph. We say that G is a *connected* graph if there is a path between each pair of distinct vertices of G. For two vertices x and y, we define $\mathrm{d}(x, y)$ to be the length of the shortest path between x and y (we let $\mathrm{d}(x, y) = \infty$ if there is no such path). The *diameter* of a graph G is $\mathrm{diam}(G) = \sup\{\mathrm{d}(a, b) : a \text{ and } b \text{ are distinct vertices of } G\}$. The *girth* of G is the length of the shortest cycle in G, denoted by $\mathrm{gr}(G)$ (we let $\mathrm{gr}(G) = \infty$ if G has no cycles). The graph G is *complete* if each pair of distinct vertices is joined by an edge. We use K_n to denote the complete graph with n vertices. The *degree* of a vertex a is the number of the edges of the graph G incident with a. A *clique* of a graph is a complete subgraph of it and the number of vertices in a largest clique of G is called the *clique number* of G and is denoted by $\omega(G)$. For a positive integer r, an *r-partite graph* is one whose vertex set can be partitioned into r subsets, so that no edge has both ends in any one subset. A *complete r-partite graph* is one in which each vertex is joined to every vertex that is not in the same part. The complete bipartite graph (2-partite graph) with part sizes m and n is denoted by $K_{m,n}$. Let G_1 and G_2 be subgraphs of G. We say that G_1 and G_2 are *disjoint* if they have no vertex and no edge in common. The *union* of two disjoint graphs G_1 and G_2, which is denoted by $G_1 \cup G_2$, is a graph with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. The *genus* of a graph is the minimal integer t such that the graph can be drawn without crossing itself on a sphere with t handles (that is an oriented surface of genus t). Thus a *planar graph* has genus zero, because it can be drawn on a sphere without self-crossing. A genus one graph is called a *toroidal graph*. In other words, a graph G is toroidal if it can be embedded on the torus, this means that, the graph's vertices can be placed on a torus such that no edges cross. Usually, it is assumed that G is also non-planar. A *subdivision* of a graph is any graph that can be obtained from the original graph by replacing edges by paths. A remarkable simple characterization of the planar graphs was given by Kuratowski in 1930. Kuratowski's Theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$ (cf. [2, p. 153]).

2. Basic properties

In this section, we observe the connectivity, diameter and girth of the graph Γ_R^n, where n is a positive integer with $n > 1$. We also investigate the clique number of this graph. Recall that an element x is a *unit* in R if there exists an element y in R such that xy is the identity element. We denote the set of zero-divisors and unit elements of R by $Z(R)$ and $U(R)$, respectively. Also the Jacobson radical of R, the set $R\backslash\{0\}$ and the ideal generated by an element a in R are denoted by $J(R)$, R^* and aR, respectively.

We begin with the following lemma.

Lemma 2.1. *Let $X = (x_1, x_2, \ldots, x_n)$ be a vertex whose first component is a unit and $Y = (y_1, y_2, \ldots, y_n)$ be a vertex whose first component is non-zero. Then X and Y are adjacent in Γ_R^n.*

Proof. Consider the $n \times n$ lower triangular matrix A whose entries satisfy the following equations:

$$a_{11} = {x_1}^{-1} y_1,$$
$$a_{ii} = 1, \text{ for } i > 1,$$
$$a_{i1} = {x_1}^{-1}(-x_i + y_i), \text{ for } i > 1,$$
$$a_{ij} = 0, \text{ for } i \neq j \text{ and } j > 1.$$

Then all entries in the main diagonal of A are non-zero. It is easy to see that $AX^T = Y^T$, and so the result follows. □

The following corollaries follow from Lemma 2.1.

Corollary 2.2. *The induced subgraph of all vertices whose first components are units is a complete graph.*

Corollary 2.3. *Suppose that $p = |\mathrm{U}(R)|\, |R|^{n-1}$ and $q = |R^* \setminus \mathrm{U}(R)|\, |R|^{n-1}$. Then the graph Γ_R^n contains the bipartite graph $\mathrm{K}_{p,q}$. Moreover, Γ_R^n contains a copy of $\mathrm{K}_{p-i,q+i}$ for i with $1 \leqslant i \leqslant p-1$.*

Proof. Let V_1 be the set of all vertices whose first components are units and let V_2 be the set of all vertices whose first components are non-unit elements of R^*. By Lemma 2.1, each vertex in V_1 is adjacent to each vertex in V_2, and so the first statement holds. The last statement is clear by deleting some vertices in V_1 and adding them to V_2. □

In the rest of the paper, for i with $1 \leqslant i \leqslant n$, we use the notation $\mathcal{C}_i$ to denote the set of all vertices whose first non-zero components are in the ith place. Also E_i is a vertex whose ith component is 1 and the other components are zero.

By using a method similar to the one we used in the proof of Lemma 2.1, one can obtain the following Lemma.

Lemma 2.4. *Assume that $X \in \mathcal{C}_i$ such that its ith component is unit. Then X is adjacent to Y for all $Y \in \mathcal{C}_i$.*

Recall that a graph on $n \geqslant 1$ vertices such that $n-1$ of the vertices have degree one, all of which are adjacent only to the remaining vertex a, is called a *star graph* with center a. Also, a *refinement* of a graph H is a graph G such that the vertex sets of G and H are the same and every edge in H is an edge in G.

In the following theorem, we study the connectivity of Γ_R^n, where R is an integral domain.

Theorem 2.5. *If R is an integral domain, then Γ_R^n is disconnected. Moreover, Γ_R^n has n components and every component is a refinement of a star graph.*

Proof. Suppose that X_i is an arbitrary element in $\mathcal{C}_i$. It is not hard to see that if $i \neq j$, then X_i and X_j are not adjacent, where $X_j \in \mathcal{C}_j$. In addition, there is no path between them. Since X_i can be considered as the ith column of some lower triangular matrix A with non-zero entries on the main diagonal, E_i is adjacent to X_i, for each i. By Lemma 2.4, it is clear that every $\mathcal{C}_i$ is a component of Γ_R^n. Also each $\mathcal{C}_i$ is a refinement of a star graph with center E_i. □

The next corollary follows from Lemma 2.4 and Theorem 2.5.

Corollary 2.6. *Let $\mathbb{F}$ be a field. Then $\Gamma_{\mathbb{F}}^n$ is a union of n complete graphs.*

Lemma 2.7. *Suppose that R is not an integral domain and $X = (x_1, x_2, \ldots, x_n)$, $Y = (y_1, y_2, \ldots, y_n) \notin \mathcal{C}_1$. Then there is a vertex in $\mathcal{C}_1$ which is adjacent to both X and Y.*

Proof. Let x and z be non-zero elements in R such that $xz = 0$. Let $Z = (z, 1, \ldots, 1)$ and consider the lower triangular matrix A whose entries satisfy the following equations:

$a_{11} = x$.
For $i \geqslant 2$ and $x_i \neq 0$ we put $a_{i1} = 0, \ldots, a_{i(i-1)} = 0, a_{ii} = x_i$.
For $i \geqslant 2$ and $x_i = 0$ we put $a_{i1} = -1, a_{i2} = 0, \ldots, a_{i(i-1)} = 0, a_{ii} = z$.

So we have that $AZ^T = X^T$.

Similarly there exists a lower triangular matrix B such that all entries on the main diagonal are non-zero and $BZ^T = Y^T$. $\square$

In the next theorem, we investigate the connectivity of Γ_R^n, where R is not an integral domain.

Theorem 2.8. *If R is not an integral domain, then the graph Γ_R^n is connected and* $\mathrm{diam}(\Gamma_R^n) \in \{2, 3\}$.

Proof. Let $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$ be arbitrary non-adjacent vertices. We have the following cases:

(a) If x_1 is unit and $y_1 \neq 0$, then, by Lemma 2.1, $\mathrm{d}(X, Y) = 1$.
(b) If $x_1 = y_1 = 0$, then, by Lemma 2.7, $\mathrm{d}(X, Y) \leqslant 2$.
(c) If $x_1 = 0$ and y_1 is unit, then, by Lemma 2.7, there exists a vertex T in $\mathcal{C}_1$ which is adjacent to X. Now, by Lemma 2.1, T is adjacent to Y. So $\mathrm{d}(X, Y) \leqslant 2$.
(d) If $x_1 = 0$ and y_1 is non-zero and non-unit, then again, by Lemma 2.7, there exists a vertex T in $\mathcal{C}_1$ which is adjacent to X. Hence we have the path $X - T - E_1 - Y$. Therefore $\mathrm{d}(X, Y) \leqslant 3$.
(e) If x_1 and y_1 are non-zero and non-unit elements, then, by Lemma 2.1, we have the path $X - E_1 - Y$. So $\mathrm{d}(X, Y) \leqslant 2$.

Hence, by considering the above situations, we have that $\mathrm{diam}(\Gamma_R^n) \leqslant 3$. On the other hand, if $x_1 = 0$ and y_1 is unit, then X and Y are not adjacent. So the graph Γ_R^n is never complete. Hence the result holds. $\square$

Remark 2.9. Let $X = (x_1, x_2, \ldots, x_n)$ be a vertex with $x_1 \neq 0$. Then X is adjacent to $Z = (x_1, 1, \ldots, 1)$. To prove this, it is sufficient to consider the lower triangular matrix A whose entries satisfy the following equations:

$a_{11} = 1$.
For $i \geqslant 2$ and $x_i \neq 0$ put $a_{i1} = 0, \ldots, a_{i(i-1)} = 0, a_{ii} = x_i$.
For $i \geqslant 2$ and $x_i = 0$ put $a_{i1} = -1, a_{i2} = 0, \ldots, a_{i(i-1)} = 0, a_{ii} = x_1$.

So we have $AZ^T = X^T$.

In the following theorem we determine the rings R for which, the diameter of Γ_R^n is exactly 2 or 3.

Theorem 2.10. *Assume that R is not an integral domain. If $R = \mathrm{Z}(R) \cup \mathrm{U}(R)$, then* $\mathrm{diam}(\Gamma_R^n) = 2$. *Otherwise,* $\mathrm{diam}(\Gamma_R^n) = 3$.

Proof. Let $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$ be two non-adjacent vertices. By the proof of Theorem 2.8, $\mathrm{d}(X, Y) \leqslant 2$ unless we have that $x_1 = 0$ and that y_1 is non-zero and non-unit. In this situation, we have $\mathrm{d}(X, Y) \leqslant 3$. Now if $R = \mathrm{Z}(R) \cup \mathrm{U}(R)$, then y_1 is a zero-divisor. So, by the proof of Lemma 2.7, $Z = (y_1, 1, \ldots, 1)$ is adjacent to X. Now, by Remark 2.9, Y and Z are adjacent. Hence $\mathrm{d}(X, Y) = 2$. Thus whenever $R = \mathrm{Z}(R) \cup \mathrm{U}(R)$, we have that $\mathrm{diam}(\Gamma_R^n) = 2$.

Now, suppose that $R \neq \mathrm{Z}(R) \cup \mathrm{U}(R)$ and that y_1 is not a zero-divisor in R. We let $x_2 = 1$ and $y_2 = y_1$. Then we claim that $\mathrm{d}(X, Y) = 3$. Assume to the contrary that $\mathrm{d}(X, Y) = 2$. Then there exists a vertex Z which is adjacent to both X and Y. So there are $n \times n$ lower triangular matrices A and B over R with entries on the main diagonal are non-zero, and we have the following situations.

(i) $AX^T = Z^T$ and $BZ^T = Y^T$. So $a_{11}0 = z_1$ and $b_{11}z_1 = y_1$. Consequently $y_1 = 0$, which is impossible.
(ii) $AX^T = Z^T$ and $BY^T = Z^T$. So $a_{11}0 = z_1$ and $b_{11}y_1 = z_1$. Consequently $b_{11}y_1 = 0$, which is impossible.
(iii) $AZ^T = X^T$ and $BZ^T = Y^T$. So $a_{11}z_1 = 0$ and $b_{11}z_1 = y_1$. Consequently $a_{11}y_1 = 0$, which is impossible.
(iv) $AZ^T = X^T$ and $BY^T = Z^T$. So $a_{21}z_1 + a_{22}z_2 = x_2$ and $b_{21}y_1 + b_{22}y_2 = z_2$.

Thus $a_{21}b_{11}y_1 + a_{22}(b_{21}y_1 + b_{22}y_2) = x_2$. Since y_1 is non-unit, we have that $y_1 \in \mathfrak{m}$, for some maximal ideal $\mathfrak{m}$. Now, since $x_2 = 1$ and $y_2 = y_1$, we have that $1 \in \mathfrak{m}$, which is impossible.

Therefore we have $\mathrm{d}(X, Y) = 3$. Thus, by Theorem 2.8, we have that $\mathrm{diam}(\Gamma_R^n) = 3$, whenever $R \neq \mathrm{Z}(R) \cup \mathrm{U}(R)$. □

We recall that the dimension of R, denoted by $\dim(R)$, is the supremum of the length of all chains of prime ideals in R.

Corollary 2.11. *If R is a ring with $\dim(R) = 0$ (in particular an Artinian ring), which is not an integral domain, then $\mathrm{diam}(\Gamma_R^n) = 2$.*

Proof. By [5, Theorem 91], we have $R = \mathrm{Z}(R) \cup \mathrm{U}(R)$, and so the result follows from Theorem 2.10. □

Example 2.12. Let $\mathbb{R}$ and $\mathbb{C}$ be the sets of all real and complex numbers, respectively. It is easy to see that $\Gamma_{\mathbb{R}}^n \cong \Gamma_{\mathbb{C}}^n$.

In the following proposition, we study the rings whose generalized Cayley graphs are isomorphic.

Proposition 2.13. *Let R and R' be two rings. Then the following statements hold:*

(a) If $\Gamma_R^n \cong \Gamma_{R'}^n$ and R is an integral domain, then R' is also an integral domain.
(b) If $\Gamma_R^n \cong \Gamma_{R'}^n$ and R is a field, then R' is also a field.

Proof. (a) Assume to the contrary that R' is not an integral domain. So, by Theorem 2.8, $\Gamma_{R'}^n$ is connected. On the other hand, by Theorem 2.5, Γ_R^n is disconnected, which is a contradiction.

(b) If R is a field, then, by Part (a), R' is an integral domain. By Corollary 2.6, every connected component of Γ_R^n is complete. Thus every connected component of $\Gamma_{R'}^n$ is complete. Now, suppose to the contrary that R' is not a field and that $\mathfrak{m}$ is a non-zero maximal ideal in R'. Also assume that m is a non-zero element in $\mathfrak{m}$. Since the vertices $X = (m^2, 1, 1, \ldots, 1)$ and $Y = (m, m, 1, \ldots, 1)$ are adjacent in $\Gamma_{R'}^n$, there exists a lower triangular matrix A over R' whose entries on the main diagonal are non-zero, and so $AX^T = Y^T$ or $AY^T = X^T$. If $AX^T = Y^T$, then $a_{11}m = 1$ which is impossible. Also, if $AY^T = X^T$, then $(a_{21} + a_{22})m = 1$ which is again impossible. Hence the vertices X and Y are not adjacent which is the required contradiction. □

The next corollary follows immediately from Proposition 2.13 in conjunction with Moore's Theorem which says that every two finite fields of the same cardinality are isomorphic.

Corollary 2.14. *If R is a commutative ring such that $\Gamma_R^n \cong \Gamma_{R'}^n$, for some finite field R', then $R \cong R'$.*

In the following example, we show that Proposition 2.13 does not hold for $n = 1$.

Example 2.15. Let F be a field. It is well known that $R = F[[x]]$ is a local ring with unique maximal ideal xR. Also one can easily see that every non-zero element in R is of the form $x^k u$, where u is a unit in R and $k \geqslant 0$. So, for every non-zero elements a and b in R, we have $a \in bR$ or $b \in aR$. Thus $\overline{\mathrm{Cay}}(R^*, R^*)$ is a complete graph. Now, let $\mathbb{R}$ and $\mathbb{Q}$ be the sets of all real and rational numbers, respectively and

$S = \mathbb{Q}[[x]]$. It is easy to see that $\overline{\mathrm{Cay}}(\mathbb{R}^*, \mathbb{R}^*) \cong \overline{\mathrm{Cay}}(S^*, S^*)$. Hence Proposition 2.13 does not hold in the case that $n = 1$.

Lemma 2.16. *Let R be a ring such that, for every non-zero elements a and b in R, $a \in bR$ or $b \in aR$. Then R is a local ring.*

Proof. Consider the element x in R with $x \neq 0, 1, -1$. So $x + 1 \neq 0$. Therefore, by our hypothesis, $x \in (x+1)R$ or $x + 1 \in xR$. If $x \in (x+1)R$, then, by considering $t = x + 1$, we have that $t - 1 \in tR$. So there exists an element k' in R such that $t - 1 = k't$. Thus $t = x + 1$ is unit. Also, if $x + 1 \in xR$, then again one can easily see that x is unit. Therefore, for every element x in R, x is unit or $x+1$ is unit. Suppose that z is a non-unit element in R. Therefore az is not unit, for every $a \in R$, and consequently $1+az$ is unit. So $z \in J(R)$. Hence, the union of all maximal ideals is $J(R)$ which implies that R is local. □

The next result follows from Lemma 2.16.

Corollary 2.17. *Let R be a field and R' be a ring. If $\overline{\mathrm{Cay}}(R^*, R^*) \cong \overline{\mathrm{Cay}}(R'^*, R'^*)$, then R' is local.*

In the next result we study the girth of Γ_R^n.

Proposition 2.18. $\mathrm{gr}(\Gamma_R^n) = 3$ *if and only if $n \geqslant 3$ or $R \neq \mathbb{Z}_2$.*

Proof. If $n \geqslant 3$, then, by Lemma 2.1, the vertices $(1, 1, 1, 0, \ldots, 0)$, $(1, 1, 0, \ldots, 0)$ and $(1, 0, \ldots, 0)$ form a triangle. If $n = 2$ and $R \neq \mathbb{Z}_2$, then the vertices $(1, 0)$, $(1, 1)$ and $(1, a)$ form a triangle, where $a \neq 0, 1$. Otherwise, we have that $n = 2$ and $R = \mathbb{Z}_2$. In this situation $\Gamma_R^n \cong \mathrm{K}_2 \cup \mathrm{K}_1$, and so its girth is infinity.

The converse statement is clear. □

In the following proposition we investigate the clique number of Γ_R^n.

Proposition 2.19. *We have the following statements.*

(a) If R is a field, then $\omega(\Gamma_R^n) = |\mathrm{U}(R)|\,|R|^{n-1}$.
(b) If R is not a field, then $\omega(\Gamma_R^n) \geqslant |\mathrm{U}(R)|\,|R|^{n-1} + 1$.

Proof. (a) If R is a field, then, every non-zero element of R is unit, and so, by Lemmas 2.1 and 2.4, our claim holds.

(b) If R is not a field, then one can choose a non-zero and non-unit element z in R. Let $X = (z, 1, 1, \ldots, 1)$. Let $\mathcal{C}$ be the set of all vertices whose first component is unit. Now, by Lemma 2.1, it is easy to see that the set $\mathcal{C} \cup \{X\}$ forms a clique, and so the result follows. □

Example 2.20. If $R = \mathbb{Z}_2$, then clearly, by Proposition 2.19(a), $\omega(\Gamma_R^n) = 2^{n-1}$.

Example 2.21. If $R = \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\omega(\Gamma_R^2) = 8$ and Γ_R^2 has 68 edges.

Proof. Put

$$X_1 = ((1,1),(1,0)),\quad X_2 = ((1,1),(0,1)),\quad X_3 = ((1,1),(0,0)),\quad X_4 = ((1,1),(1,1)),$$

$$X_5 = ((0,1),(1,0)),\quad X_6 = ((0,1),(0,1)),\quad X_7 = ((0,1),(0,0)),\quad X_8 = ((0,1),(1,1)).$$

By Lemma 2.1, the vertices X_1, X_2, X_3 and X_4 form the graph K_4. Also, by Lemma 2.1, these four vertices are adjacent to all vertices X_i, for $5 \leqslant i \leqslant 8$. The following equations show that the vertices X_5, X_6, X_7, X_8 form the graph K_4 too. Put

$$A =: \begin{pmatrix} (1,1) & (0,0) \\ (1,1) & (1,1) \end{pmatrix}, \quad B =: \begin{pmatrix} (1,1) & (0,0) \\ (1,0) & (0,1) \end{pmatrix}, \quad C =: \begin{pmatrix} (1,1) & (0,0) \\ (1,1) & (0,1) \end{pmatrix}.$$

Now, we have the following equalities.

$$AX_6^T = X_7^T,\ AX_5^T = X_8^T,\ BX_5^T = X_7^T,\ BX_8^T = X_6^T,\ CX_5^T = X_6^T,\ CX_8^T = X_7^T.$$

It is not hard to see that there is no vertex which is adjacent to all vertices $X_1, \ldots, X_8$. Also one can easily check that the cardinality of all cliques is not greater than eight. Thus $\omega(\Gamma_R^2) = 8$.

Now we calculate the number of edges in Γ_R^2. Put

$$X_9 = ((1,0),(1,0)), \quad X_{10} = ((1,0),(0,1)), \quad X_{11} = ((1,0),(0,0)), \quad X_{12} = ((1,0),(1,1)),$$
$$X_{13} = ((0,0),(1,0)), \quad X_{14} = ((0,0),(0,1)), \quad X_{15} = ((0,0),(1,1)).$$

By a similar argument as in the first paragraph of this proof, the vertices X_9, X_{10}, X_{11} and X_{12} form the graph K_4. By Lemma 2.1, the vertices X_1, X_2, X_3 and X_4 are adjacent to each of the vertices X_9, X_{10}, X_{11} and X_{12}. One can also easily see that X_{13} is adjacent to $X_5, X_8, X_9, X_{10}, X_{11}$ and X_{12}. Similarly, X_{14} is adjacent to $X_5, X_6, X_7, X_8, X_{10}$ and X_{12}. Also X_{15} is adjacent to $X_5, X_8, X_{10}, X_{12}, X_{13}$ and X_{14}.

Therefore Γ_R^2 has 68 edges. □

Recall that a *Hamilton cycle* in a graph G is a cycle that contains every vertex of G. Moreover G is called *Hamiltonian* if it contains a Hamilton cycle.

We end this section with the following theorem which study the Hamiltonian generalized Cayley graphs.

Theorem 2.22. *Assume that R is a finite ring such that Γ_R^n is connected for some $n \geqslant 1$. Then Γ_R^n is Hamiltonian, whenever $|\mathrm{Z}^*(R)| \leqslant |\mathrm{U}(R)|$.*

Proof. Since Γ_R^n is connected, we have that R is not an integral domain and therefore $|\mathrm{Z}^*(R)| \geqslant 1$. Suppose that $\mathrm{Z}^*(R) = \{z_1, z_2, \ldots, z_p\}$ and $\mathrm{U}(R) = \{u_1, u_2, \ldots, u_q\}$, where $p \leqslant q$. First we consider the case that $n = 1$. Then we have the following Hamilton cycle in the graph $\overline{\mathrm{Cay}}(R^*, R^*)$.

$$u_1 - z_1 - u_2 - z_2 - \cdots - u_p - z_p - u_{p_{i+1}} - \cdots - u_q - u_1.$$

Now, suppose that $n \geqslant 2$. For each $1 \leqslant i \leqslant n$, let $p_i = p|R|^{n-i}$ and $q_i = q|R|^{n-i}$. Let $\mathcal{Z}_i = \{Z_j^i \mid 1 \leqslant j \leqslant p_i\}$ be the subset of $\mathcal{C}_i$ such that the ith components of its elements are non-zero zero divisors and $\mathcal{U}_i = \{U_j^i \mid 1 \leqslant j \leqslant q_i\}$ be the subset of $\mathcal{C}_i$ such that the ith components of its elements are units. Obviously, for each $1 \leqslant i \leqslant n$, we have the path $\mathcal{P}_i$ in $\mathcal{C}_i$,

$$\mathcal{P}_i:\ U_1^i - Z_1^i - U_2^i - Z_2^i - \cdots - U_{p_i}^i - Z_{p_i}^i - U_{p_{i+1}}^i - \cdots - U_{q_i}^i,$$

where $Z_1^i = (x_1, x_2, \ldots, x_n)$ and $Z_2^i = (y_1, y_2, \ldots, y_n)$ such that $x_i = z_1,\ x_{i+1} = \cdots = x_n = u_1$ and $y_i = z_1,\ y_{i+1} = \cdots = y_n = u_2$.

Now, we construct a Hamiltonian cycle in Γ_R^n. First we consider the paths $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ and delete the edges $Z_1^i - U_2^i$ and $Z_2^i - U_2^i$, for $i = 1, \ldots, n-1$. Now, by considering the following edges, one can easily see that Γ_R^n contains a Hamilton cycle.

$$Z_1^1 - U_1^2, Z_1^2 - U_1^3, \ldots, Z_1^{n-1} - U_1^n,$$
$$U_2^1 - U_{q_1}^1, U_2^2 - U_{q_2}^2, \ldots, U_2^{n-1} - U_{q_{n-1}}^{n-1},$$
$$Z_2^1 - U_2^2, Z_2^2 - U_2^3, \ldots, Z_2^{n-2} - U_2^{n-1},$$
$$U_1^1 - U_2^1, Z_2^{n-1} - U_{q_n}^n. \quad \square$$

Corollary 2.23. *Suppose that Γ_R^n is connected, for some $n \geqslant 1$, and that R is finite. If $\mathrm{Z}(R)$ is an ideal of R, then Γ_R^n is Hamiltonian. In particular, if R is local, then Γ_R^n is Hamiltonian.*

3. On the genus and crosscap numbers of Γ_R^n

It is well known that any compact surface is either homeomorphic to a sphere, or to a connected sum of g tori, or to a connected sum of k projective planes (see [10, Theorem 5.1]). We denote by S_g the surface formed by a connected sum of g tori, and by N_k the one formed by a connected sum of k projective planes. The number g is called the genus of the surface S_g and k is called the crosscap of N_k. When considering the orientability, the surfaces S_g and sphere are among the orientable class and the surfaces N_k are among the non-orientable one.

A simple graph which can be embedded in S_g but not in S_{g-1} is called a graph of *genus* g. Similarly, if it can be embedded in N_k but not in N_{k-1}, then we call it a graph of *crosscap* k. The notations $\gamma(G)$ and $\overline{\gamma}(G)$ are denoted for the genus and crosscap of a graph G, respectively. It is easy to see that $\gamma(H) \leqslant \gamma(G)$ and $\overline{\gamma}(H) \leqslant \overline{\gamma}(G)$, for all subgraph H of G. Also a graph G is called planar if $\gamma(G) = 0$, and it is called toroidal if $\gamma(G) = 1$.

Recall that, for a rational number q, $\lceil q \rceil$ is the first integer number greater or equal than q. In the following lemma we bring some well-known formulas for genus of a graph (see [16,15]).

Lemma 3.1. *The following statements hold:*

(a) For $n \geqslant 3$, we have $\gamma(\mathrm{K}_n) = \lceil \frac{1}{12}(n-3)(n-4) \rceil$.
(b) For $m, n \geqslant 2$, we have $\gamma(\mathrm{K}_{m,n}) = \lceil \frac{1}{4}(m-2)(n-2) \rceil$.

According to Lemma 3.1, we have $\gamma(\mathrm{K}_n) = 0$, for $1 \leqslant n \leqslant 4$, and $\gamma(\mathrm{K}_n) = 1$, for $5 \leqslant n \leqslant 7$, and, for other values of n, $\gamma(\mathrm{K}_n) \geqslant 2$.

The following lemma, which is from [17], is needed in the rest of the paper.

Lemma 3.2. *Let G be a simple graph with n vertices ($n \geqslant 4$) and m edges. Then $\gamma(G) \geqslant \lceil \frac{1}{6}(m-3n)+1 \rceil$.*

In the following theorem we determine all isomorphic classes of finite commutative rings R whose Γ_R^n has genus at most three.

Theorem 3.3. *The following statements hold:*

(a) $\gamma(\Gamma_R^n) = 0$ if and only if $R = \mathbb{Z}_2$ and $n = 2$ or 3.
(b) $\gamma(\Gamma_R^n) = 1$ if and only if $R = \mathbb{Z}_3$ and $n = 2$.
(c) $\gamma(\Gamma_R^n) = 2$ if and only if $R = \mathbb{Z}_2$ and $n = 4$.
(d) There is no ring R with $\gamma(\Gamma_R^n) = 3$.

Proof. We consider the following cases:

Case 1. $n \geqslant 4$. If $R \neq \mathbb{Z}_2$, then, by Proposition 2.19, $\omega(\Gamma_R^n) \geqslant 28$, and so, by Lemma 3.1, $\gamma(\Gamma_R^n) \geqslant 50$. If $R = \mathbb{Z}_2$, then $\omega(\Gamma_R^4) = 8$ and $\omega(\Gamma_R^5) = 16$. Hence, by Lemma 3.1, $\gamma(\Gamma_R^4) = 2$ and $\gamma(\Gamma_R^5) = 13$.

Case 2. n=3. If $|R| \geqslant 3$, then, by Proposition 2.19, $\omega(\Gamma_R^n) \geqslant 10$, and so, by Lemma 3.1, $\gamma(\Gamma_R^n) \geqslant 4$. If $|R| = 2$, then $\omega(\Gamma_R^n) = 4$. Since $\Gamma_R^n = K_4 \cup K_2 \cup K_1$, we have that $\gamma(\Gamma_R^4) = 0$.

Case 3. n=2. If $|R| \geqslant 9$, then, by Proposition 2.19, $\omega(\Gamma_R^n) \geqslant 10$, and so, by Lemma 3.1, $\gamma(\Gamma_R^n) \geqslant 4$. If $|R| = 8$, then, by [4, p. 687], R is one of the following rings.

$$R_1 = \mathbf{F}_8, \quad R_2 = \frac{\mathbb{Z}_2[x,y]}{\langle x,y\rangle^2}, \quad R_3 = \frac{\mathbb{Z}_2[x]}{\langle x^3\rangle}, \quad R_4 = \frac{\mathbb{Z}_4[x]}{\langle 2x, x^2\rangle}, \quad R_5 = \frac{\mathbb{Z}_4[x]}{\langle 2x, x^2-2\rangle},$$

$$R_6 = \mathbb{Z}_2 \times \mathbb{Z}_4, \quad R_7 = \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{\langle x^2\rangle}, \quad R_8 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.$$

For R_1, by Proposition 2.19, we have $\omega(\Gamma_{R_1}^n) \geqslant 56$, and so, by Lemma 3.1, $\gamma(\Gamma_R^n) \geqslant 230$. For other rings of order 8 we have $|R^* \setminus \mathrm{U}(R)| \geqslant 2$. Thus, by Corollary 2.3, Γ_R^n contains a $\mathrm{K}_{8,16}$. Hence, by Lemma 3.1, $\gamma(\Gamma_R^n) \geqslant 21$.

If $|R| = 7$, then R is a field, and so, by Proposition 2.19, $\omega(\Gamma_R^n) = 42$. Hence, by Lemma 3.1, $\gamma(\Gamma_R^n) \geqslant 124$.

If $|R| = 6$, then R is a field or $R = \mathbb{Z}_6$. If R is a field, then, by Proposition 2.19, $\omega(\Gamma_R^n) \geqslant 30$, and so, by Lemma 3.1, $\gamma(\Gamma_R^n) \geqslant 59$. If $R = \mathbb{Z}_6$, then, by Proposition 2.19, $\omega(\Gamma_R^n) \geqslant 13$, and so, by Lemma 3.1, $\gamma(\Gamma_R^n) \geqslant 8$.

If $|R| = 5$, then R is a field. So, by Proposition 2.19, $\omega(\Gamma_R^n) \geqslant 20$. Thus, by Lemma 3.1, $\gamma(\Gamma_R^n) \geqslant 23$.

If R is local with $|R| = 4$, then, by [4, p. 687], R is a field or R is one of the rings $\mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$. If $R = \mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$, then $|\mathrm{U}(R)| = 2$. So, by Corollary 2.3, Γ_R^n contains a $\mathrm{K}_{6,6}$. Thus, by Lemma 3.1, $\gamma(\Gamma_R^n) \geqslant 4$. If R is a field and $|R| = 4$, then, by Proposition 2.19, we have $\omega(\Gamma_R^n) = 12$, and so, by Lemma 3.1, $\gamma(\Gamma_R^n) \geqslant 6$. If R is not local and $|R| = 4$, then $R = \mathbb{Z}_2 \times \mathbb{Z}_2$. Now, by Example 2.21, Γ_R^n has 68 edges. Hence, by Lemma 3.2, $\gamma(\Gamma_R^n) \geqslant 5$.

If $|R| = 3$, then $\Gamma_R^n = \mathrm{K}_6 \cup \mathrm{K}_2$. Hence $\gamma(\Gamma_R^n) = 1$.

If $|R| = 2$, then obviously $\gamma(\Gamma_R^n) = 0$.

Now by considering the above cases, the results hold. □

The following two results about the crosscap formulae of a complete graph and a complete bipartite graph are very useful in the proof of next theorem (see [3] or [11]).

Lemma 3.4. *The following statements hold:*

(a) $$\overline{\gamma}(\mathrm{K}_n) = \begin{cases} \lceil \frac{1}{6}(n-3)(n-4) \rceil & \text{if } n \geqslant 3 \text{ and } n \neq 7, \\ 3 & \text{if } n = 7. \end{cases}$$

(b) $\overline{\gamma}(\mathrm{K}_{m,n}) = \lceil \frac{1}{2}(m-2)(n-2) \rceil$.

By slight modifications in the proof of Theorem 3.3 in conjunction with Lemma 3.4, one can prove the following theorem.

Theorem 3.5. *The following statements hold:*

(a) $\overline{\gamma}(\Gamma_R^n) = 0$ *if and only if* $R = \mathbb{Z}_2$ *and* $n = 2$ *or* 3.
(b) $\overline{\gamma}(\Gamma_R^n) = 1$ *if and only if* $R = \mathbb{Z}_3$ *and* $n = 2$.
(c) $\overline{\gamma}(\Gamma_R^n) = 4$ *if and only if* $R = \mathbb{Z}_2$ *and* $n = 4$.
(d) There is no ring R *with* $\overline{\gamma}(\Gamma_R^n) = 2$ *or* 3.

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