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Last syzygies of 1-generic spaces

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Abstract

We consider determinantal varieties $X(\gamma)$ of expected codimension defined by the maximal minors of a matrix $M(\gamma)$ of linear forms representing a linear map γ. Eisenbud and Popescu have conjectured that 1-generic linear maps γ have the property that the syzygy ideals $I(s)$ of all last syzygies s of $X(\gamma)$ coincide with $I_{X(\gamma)}$. We prove a geometric version of this conjecture: for 1-generic linear maps γ the syzygy varieties $\mathrm{Syz}(s) = V(I(s))$ of all last syzygies have the same support as $X(\gamma)$.

Introduction

In this note we study syzygies of determinantal varieties which are cut out by the maximal minors of a matrix of linear forms M that represents a linear map $\gamma : A \otimes B \to C$. Eisenbud and Popescu have studied these syzygies in [4]. There they define the syzygy ideal $I(s)$ of a syzygy s and prove the following:

Theorem (Eisenbud, Popescu). *Let $\gamma : A \otimes B \to C$ be a linear map such that the associated determinantal variety $X(\gamma_C) \subset \mathbb{P}(C)$ is of expected codimension. If $I(s) = I_{X(\gamma_C)}$ holds for all last syzygies $s \in E_{a-b}$ of $X(\gamma_C)$, then γ is 1-generic.*

Conversely they conjecture:

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Conjecture (Eisenbud, Popescu). *Let $\gamma : A \otimes B \to C$ be a* 1*-generic linear map. Then the equality $I(s) = I_{X(\gamma_C)}$ holds for all last syzygies $s \in E_{a-b}$ of the determinantal variety $X(\gamma_C) \subset \mathbb{P}(C)$ associated to γ.*

They can prove this conjecture in the case of $\dim B = 2$. Here we consider a more geometric invariant, namely the syzygy variety $\mathrm{Syz}(s)$ of a syzygy s which is the vanishing locus of the syzygy ideal $I(s)$. With this we obtain a geometric version of the Eisenbud–Popescu Conjecture:

Theorem 3.2. *Let $\gamma : A \otimes B \to C$ be a* 1*-generic linear map. Then*

$$\operatorname{supp} \mathrm{Syz}(s) = \operatorname{supp} X(\gamma_C)$$

holds for all last syzygies $s \in E_{a-b}$ of the determinantal variety $X(\gamma_C) \subset \mathbb{P}(C)$ associated to γ.

Also we obtain a partial strengthening of their theorem by

Theorem 3.3. *Let $\gamma : A \otimes B \to C$ be a linear map, such that the associated determinantal variety $X(\gamma_C)$ has expected codimension $a - b + 1$ and also satisfies $a > 2b - 2$. If for every last syzygy $s \in E_{a-b}$ of $X(\gamma_C)$*

$$\operatorname{supp} \mathrm{Syz}(s) = \operatorname{supp} X(\gamma_C)$$

holds, then γ is 1*-generic.*

Our methods also show that in the situation of our Theorem 3.2 both $\mathrm{Syz}(s)$ and $X(\gamma_C)$ have the same smooth locus. To obtain the conjecture of Eisenbud and Popescu one would have to show, that $\mathrm{Syz}(s)$ has no embedded components in the singular locus of $X(\gamma_C)$ and that the syzygy ideal $I(s)$ is always saturated.

The main ingredient of our proof is an observation of Green [6] about exterior minors of 1-generic maps. This allows us to evaluate syzygies explicitly at certain points.

The paper has three sections. In the first the definition and properties of 1-generic maps are reviewed. The second section we collect what we need to know about syzygies and syzygy varieties. The last section contains the proofs of our theorems.

1. 1-generic linear maps

Let A, B and C be finite dimensional vector spaces of dimensions a, b and c together with a linear map $\gamma : A \otimes B \to C$. γ can be interpreted as a triple tensor $\gamma \in A^* \otimes B^* \otimes C$ or after choosing bases as an $(a \times b)$-matrix of linear forms on $\mathbb{P}(C)$. Here we adhere to the Grothendieck convention of interpreting elements of $\mathbb{P}(C)$ as linear forms on C or equivalently the elements of C as linear forms on $\mathbb{P}(C)$.

Definition 1.1. A nonzero linear map $\mathbb{C} \to A$ is called a *generalized row index* of γ since it induces a map $\mathbb{C} \otimes B \to C$ which can be interpreted, up to a constant factor, as a $1 \times b$ row vector of linear-forms.

If $\mathbb{C} \to A$ is such a generalized row index, the image of $\mathbb{C}$ in A under this map is a line. We will call these images *generalized rows*. The generalized rows form a projective space $\mathbb{P}(A^*)$ which we call the *row space* of γ. Similarly $\mathbb{P}(B^*)$ is the *column space* of γ.

On the row space $\mathbb{P}(A^*)$ the linear map γ induces a map of vector bundles

$$\gamma_A : \mathcal{O}_{\mathbb{P}(A^*)}(-1) \otimes B \to C$$

by composing γ with the first map of the twisted Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}(A^*)}(-1) \otimes B \to A \otimes B \to \mathbb{T}_{\mathbb{P}(A^*)}(-1) \otimes B \to 0$$

on $\mathbb{P}(A^*)$. Similarly we have

$$\gamma_B : A \otimes \mathcal{O}_{\mathbb{P}(B^*)}(-1) \to C$$

on the column space $\mathbb{P}(B^*)$. From now on we will restrict our discussion to the row space $\mathbb{P}(A^*)$, leaving the analogous constructions for the column space $\mathbb{P}(B^*)$ to the reader.

Given a generalized row $\alpha \in \mathbb{P}(A^*)$ the restriction of γ_A to α

$$\gamma_\alpha : B \to C$$

is a map of vector spaces.

Definition 1.2. The *rank* of a generalized row α is defined as $\operatorname{rank} \alpha := \operatorname{rank} \gamma_\alpha$.

Example 1.3. Consider vector spaces A, B and C of dimension 2, 3 and 4 with basis a_i, b_i and c_i. The linear map $\gamma : A \otimes B \to C$ with

$$\begin{aligned} &\gamma(a_1 \otimes b_1) = c_1, \qquad &\gamma(a_1 \otimes b_2) = c_2, \qquad &\gamma(a_1 \otimes b_3) = c_3, \\ &\gamma(a_2 \otimes b_1) = c_2, \qquad &\gamma(a_2 \otimes b_2) = c_3, \qquad &\gamma(a_2 \otimes b_3) = c_4, \end{aligned}$$

can be represented by the matrix

$$\begin{pmatrix} c_1 & c_2 & c_3 \\ c_2 & c_3 & c_4 \end{pmatrix}.$$

In this basis we see two rows of rank 3. Generalized rows are linear combinations of those two. The map $\gamma_A : \mathcal{O}_{\mathbb{P}(A^*)}(-1) \otimes B \to C$ can be represented by the matrix

$$\begin{pmatrix} a_1 & a_2 & 0 & 0 \\ 0 & a_1 & a_2 & 0 \\ 0 & 0 & a_1 & a_2 \end{pmatrix}.$$

Since this matrix has full rank everywhere on the row-space $\mathbb{P}(A^*)$ we see that all generalized rows of γ have the same rank 3.

Definition 1.4. A linear map $\gamma : A \otimes B \to C$ is called 1-*generic*, if all generalized rows have rank b and all generalized columns have rank a.

Example 1.5. The (2×3)-matrix considered above is 1-generic.

In this paper we will use two properties of 1-generic linear maps. The first one concerns the following:

Definition 1.6. Let $\gamma : A \otimes B \to C$ be a linear map and

$$\gamma_C : A \otimes \mathcal{O}_{\mathbb{P}(C)}(-1) \to B^*$$

the third induced morphism of vector bundles. We call the locus $X(\gamma_C)$ where γ_C does not have rank b the *determinantal variety associated to* γ. The scheme structure of $X(\gamma_C)$ is given by the image $I_{X(\gamma_C)}$ of the natural map

$$\bigwedge^b A \otimes \bigwedge^b B \to \mathcal{O}_{\mathbb{P}(C)}(b)$$

induced by γ_C. If the codimension of $X(\gamma_C)$ in $\mathbb{P}(C)$ is $a - b + 1$ we say that $X(\gamma_C)$ is *of expected codimension*.

Proposition 1.7 (Eisenbud). *Let* $\gamma : A \otimes B \to C$ *be a* 1-*generic linear map, then* $X(\gamma_C) \subset \mathbb{P}(C)$ *is of expected codimension.*

Proof. [2, Corollary 3.3]. □

Green has observed, that the exterior minors of a 1-generic linear map also behave nicely:

Definition 1.8. Consider the natural map

$$\bigwedge^n A \otimes S_n B \hookrightarrow \bigwedge^n (A \otimes B) \longrightarrow \bigwedge^n C, \qquad e_n: \bigwedge^n A \otimes S_n B \to \bigwedge^n C$$

obtained by taking the nth exterior power of γ. Then the elements in the image of e_n are called degree n exterior minors of γ.

Proposition 1.9 (Green). *If* γ *is* 1-*generic, then* e_a *is injective.*

Proof. [6, Proposition 1.2]. □

2. Syzygies and syzygy varieties

In this section we recall some facts about the syzygies of determinantal varieties.

Theorem 2.1 (Eagon–Northcott). *Let $\gamma : A \otimes B \to C$ be a linear map and $X(\gamma_C) \subset \mathbb{P}(C)$ be the associated determinantal variety. If $X(\gamma_C) \subset \mathbb{P}(C)$ is of expected codimension then there exists a minimal free resolution $I_{X(\gamma_C)} \leftarrow \mathcal{E}_\bullet$ with terms $\mathcal{E}_i := E_i \otimes \mathcal{O}(-i-b)$, where*

$$E_i := \bigwedge^{b+i} A \otimes \bigwedge^{b} B \otimes S_i B.$$

Proof. See for example [3, Theorem A2.10]. □

Definition 2.2. In the situation of Theorem 2.1 we call E_i the *space of ith syzygies* and E_{a-b} the *space of last syzygies*.

Lemma 2.3. *In the situation of Theorem* 2.1 *we have*

$$E_i = H^0\big(\Omega^i_{\mathbb{P}(C)} \otimes I_{X(\gamma_C)}(i+b)\big) \subset \bigwedge^{i} C \otimes H^0\big(I_{X(\gamma_C)}(b)\big).$$

In particular an ith syzygy of $X(\gamma_C)$ can be interpreted as a twisted i-form that vanishes on $X(\gamma_C)$.

Proof. By Koszul cohomology [5] we have

$$E_i = \ker\Big(\bigwedge^{i} C \otimes H^0\big(I_{X(\gamma_C)}(b)\big) \to \bigwedge^{i-1} C \otimes H^0\big(I_{X(\gamma_C)}(b+1)\big)\Big)$$

since $H^0(I_{X(\gamma_C)}(b-1)) = 0$. This kernel can easily be identified with

$$H^0\big(\Omega^i_{\mathbb{P}(C)} \otimes I_{X(\gamma_C)}(i+b)\big)$$

by considering exterior powers of the Euler sequence [1, Section 4]. □

Definition 2.4 (*Ehbauer*). Let $s \in E_i$ be an ith syzygy of $X(\gamma_C)$. Then the *syzygy scheme* Syz(s) of s is the vanishing locus of the corresponding twisted i-form. The scheme structure of Syz(s) is given by the syzygy ideal

$$I(s) := s \wedge \bigwedge^{i} C^* \subset H^0\big(I_{X(\gamma_C)}(b)\big).$$

Remark 2.5. Syzygy ideals are not necessarily reduced or even saturated. Consider for example the variety X of 4 general points in $\mathbb{P}^3$. The minimal free resolution of X is given by an Eagon–Northcott-Complex. Let $s \in E_1$ be a general first syzygy. As can be checked with a computer algebra program $I(s)$ is generated by 4 quadrics and $\mathrm{Syz}(s)$ is the union of X with one additional point. The saturation of $I(s)$ turns out to be generated by 5 quadrics.

3. Main results

Lets now consider the last syzygies of $X(\gamma_C)$. The representation of a last syzygy of $X(\gamma_C)$ as an element of $\bigwedge^{a-b} C \otimes H^0(I_{X(\gamma_C)}(b))$ can be given explicitly:

Lemma 3.1 (Eisenbud, Popescu). *The inclusion*

$$E_{a-b} \hookrightarrow \bigwedge^{a-b} C \otimes H^0\big(I_{X(\gamma_C)}(b)\big)$$

is given by the composition

$$\begin{array}{ccc}
E_{a-b} = \bigwedge^a A \otimes \bigwedge^b B \otimes S_{a-b} B & & \\
\Big\downarrow \hookrightarrow & & \\
\bigwedge^b A \otimes \bigwedge^b B \otimes \bigwedge^{a-b} A \otimes S_{a-b} B & & \\
\Big\downarrow \mathrm{id} \otimes e_{a-b} & & \\
\bigwedge^b A \otimes \bigwedge^b B \otimes \bigwedge^{a-b} C & = & H^0(I_{X(\gamma_C)}(b)) \otimes \bigwedge^{a-b} C.
\end{array}$$

Proof. [4, Theorem 2.1 and proof of Theorem 3.1]. □

With this we can prove our first theorem:

Theorem 3.2. *Let $\gamma : A \otimes B \to C$ be a 1-generic linear map. Then*

$$\operatorname{supp} \mathrm{Syz}(s) = \operatorname{supp} X(\gamma_C)$$

holds for all last syzygies $s \in E_{a-b}$ of the determinantal variety $X(\gamma_C) \subset \mathbb{P}(C)$ associated to γ.

Proof. Let $x \in \mathbb{P}(C)$ a point not contained in $X(\gamma_C)$ and $s \in E_{a-b}$ any last syzygy. We have to prove that s does not vanish in x.

Since $x \notin X(\gamma_C)$ the map γ_C has full rank in x. Therefore we can choose bases of A, B and C such that γ_C can be represented by a matrix of linear forms

$$M = \begin{pmatrix} c_{11} & \dots & c_{1b} \\ \vdots & & \vdots \\ c_{a1} & \dots & c_{ab} \end{pmatrix}$$

such that

$$M(x)=\begin{pmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \\ \hline 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix}.$$

Now by the Lemma 3.1, the representation of a last syzygy s in this basis is

$$s=\sum_{|\beta|=b} f_\beta \otimes g_{\bar{\beta}}$$

where f_β is the $(b \times b)$-minor involving the rows $\beta_1,\dots,\beta_b$ of M and $g_{\bar{\beta}}$ is a degree $a-b$ exterior minor of the remaining $(a-b)\times b$ matrix. At x all minors of M except $f_{1,2,\dots,b}(x)=1$ vanish, and therefore $s(x)=g_{b+1,\dots,a}$. Since $g_{b+1,\dots,a}$ is a degree $a-b$ exterior minor of a 1-generic $(a-b)\times b$ matrix it is nonzero by Proposition 1.9. □

We can also prove a partial converse of this theorem, strengthening the theorem of Eisenbud and Popescu in the case where $a>2b-2$.

Theorem 3.3. *Let $\gamma: A\otimes B \to C$ be a linear map, such that the associated determinantal variety $X(\gamma_C)$ has expected codimension $a-b+1$ and also satisfies $a>2b-2$. If for every last syzygy $s \in E_{a-b}$ of $X(\gamma_C)$*

$$\operatorname{supp}\operatorname{Syz}(s)=\operatorname{supp} X(\gamma_C)$$

holds, then γ is 1-generic.

Proof. Suppose γ is not 1-generic. Then there exists a generalized row α of rank at most $b-1$. We can therefore choose bases of A, B and C such that M has the form

$$M=\begin{pmatrix} 0 & c_{12} & \dots & c_{1b} \\ c_{21} & c_{22} & \dots & c_{2b} \\ \vdots & \vdots & & \vdots \\ c_{a1} & c_{a2} & \dots & c_{ab} \end{pmatrix}.$$

Since $\operatorname{codim} X(\gamma_C)=a-b+1>b-1$ by the assumptions of the theorem, the vanishing locus of the first row $L=\{x\in\mathbb{P}(C) \mid c_{12}(x)=\dots=c_{1b}(x)=0\}$ can not lie completely inside $X(\gamma_C)$. We can therefore find a point $x\in L$ outside of $X(\gamma_C)$. There $M(x)$ has full rank and after a coordinate change in A which does not involve the first row, we can assume that $M(x)$ has the form

$$M(x) = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \\ 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{pmatrix}.$$

Now consider the syzygy $s = (b_1)^{a-b}$ where b_1 is the basis element of B corresponding to the first column. When we evaluate s at x we obtain $s(x) = f_{a-b+1,\dots,a}(x) \otimes g_{1,\dots,a-b,s} = g_{1,\dots,a-b,s}$ since $f_{a-b+1,\dots,a}(x) = 1$ is the only nonzero maximal minor of $M(x)$.

The exterior minor $g_{1,\dots,a-b,s}$ of the upper $(a-b) \times b$ submatrix corresponding to $s = (b_1)^{a-b}$ is the wedge product of the first $a-b$ linear forms in the first column of M. This wedge product vanishes since the first of these linear forms is identically zero. So s is a syzygy whose syzygy variety has support outside of $X(\gamma_C)$. □

Our methods also allow us to describe the smooth locus of all last syzygy varieties:

Theorem 3.4. *Let $\gamma : A \otimes B \to C$ be a 1-generic linear map. Then*

$$\operatorname{reg} \operatorname{Syz}(s) = \operatorname{reg} X(\gamma_C)$$

for all last syzygies $s \in E_{a-b}$ of the determinantal variety $X(\gamma_C) \subset \mathbb{P}(C)$ associated to γ.

Proof. Let $s \in E_{a-b}$ be any last syzygy of $X(\gamma_C)$. Since $I(s) \subset I_{X(\gamma_C)}$ by definition and $\operatorname{supp} X(\gamma_C) = \operatorname{supp} \operatorname{Syz}(s)$ by Theorem 3.2, we know that the smooth locus of $\operatorname{Syz}(s) = V(I(s))$ is contained in the smooth locus of $X(\gamma_C)$.

For the converse, let $x \in \mathbb{P}(C)$ be a point contained in the smooth locus of $X(\gamma_C)$. We have to prove, that the tangent space of $\operatorname{Syz}(s)$ in x is the same as the tangent space of $X(\gamma_C)$ in x.

Since x is in the smooth locus of $X(\gamma_C)$ the morphism γ_C has rank $b-1$ in x. Therefore we can choose bases of A, B and C such that γ_C can be represented by a matrix of linear forms

$$M = \begin{pmatrix} c_{11} & \dots & c_{1b} \\ \vdots & & \vdots \\ c_{a1} & \dots & c_{ab} \end{pmatrix}$$

such that

$$M(x)=\begin{pmatrix} 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \\ \hline 0 & \dots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 \end{pmatrix}.$$

Now suppose $x+\varepsilon y$ is a tangent vector of $X(\gamma_C)$ at x. Then all maximal minors of $M(x+\varepsilon y)$ have to vanish, in particular those that contain the first $b-1$ rows and the ith row ($i \geqslant b$):

$$0=\det\begin{pmatrix} 1+\varepsilon c_{11}(y) & \dots & \varepsilon c_{1,b-1}(y) & \varepsilon c_{1b}(y) \\ \vdots & \ddots & \vdots & \vdots \\ \varepsilon c_{b-1,1}(y) & \dots & 1+\varepsilon c_{b-1,b-1}(y) & \varepsilon c_{b-1,b}(y) \\ \varepsilon c_{i,1}(y) & \dots & \varepsilon c_{i,1b-1}(y) & \varepsilon c_{ib}(y) \end{pmatrix}=\varepsilon c_{ib}(y).$$

All other minors vanish since every term of the corresponding determinant involves at least ε^2. So $x+\varepsilon y$ is tangent to $X(\gamma_C)$ if and only if

$$c_{bb}(y)=\cdots=c_{ab}(y)=0.$$

Now assume that $x+\varepsilon y$ is not a tangent vector of $X(\gamma_C)$. Then we can assume after another base change of C, that $M(x+\varepsilon y)$ has the form

$$M(x+\varepsilon y)=\begin{pmatrix} 1+\varepsilon c_{11}(y) & \dots & \varepsilon c_{1,b-1}(y) & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \varepsilon c_{b-1,1}(y) & \dots & 1+\varepsilon c_{b-1,b-1}(y) & 0 \\ \varepsilon c_{b,1}(y) & \dots & \varepsilon c_{b,b-1}(y) & \varepsilon \\ \varepsilon c_{b+1,1}(y) & \dots & \varepsilon c_{b+1,b-1}(y) & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \varepsilon c_{a1}(y) & \dots & \varepsilon c_{a,b-1}(y) & 0 \end{pmatrix}.$$

As before the representation of a last syzygy s in this basis is

$$s=\sum_{|\beta|=b} f_\beta \otimes g_{\bar{\beta}}$$

where f_β is the $(b\times b)$-minor involving the rows $\beta_1,\dots,\beta_b$ of M and $g_{\bar{\beta}}$ is a degree $a-b$ exterior minor of the remaining $a-b\times b$ matrix. At $x+\varepsilon y$ all minors of M except $f_{1,2,\dots,b}(x)=\varepsilon$ vanish, and $s(x)=\varepsilon g_{b+1,\dots,a}$. Since $g_{b+1,\dots,a}$ is again a degree $a-b$ exterior minor of a 1-generic $(a-b)\times b$ matrix it is nonzero by Proposition 1.9. Therefore $x+\varepsilon y$ is not a tangent vector of $\mathrm{Syz}(s)$. This shows that the tangent space of

Syz(s) at x is contained in the tangent space of $X(\gamma_C)$ at x. Since on the other hand Syz(s) contains $X(\gamma_C)$ as scheme both tangent spaces have to coincide. □

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