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The Ramsey numbers for cycles versus wheels of odd order

Yaojun Chen [a,b,*], T.C. Edwin Cheng [c], Zhengke Miao [d], C.T. Ng [c]

[a] Department of Mathematics, Nanjing University, Nanjing 210093, PR China

[b] State Key Laboratory for Novel Software Technology, Nanjing University, Nanjing 210093, PR China

[c] Department of Logistics and Maritime Studies, The Hong Kong Polytechnic University, Hung Kom, Kowloon, Hong Kong, PR China

[d] School of Mathematical Sciences, Xuzhou Normal University, Xuzhou 221116, PR China

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ABSTRACT

For two given graphs G_1 and G_2, the Ramsey number $R(G_1, G_2)$ is the smallest integer n such that for any graph G of order n, either G contains G_1 or the complement of G contains G_2. Let C_n denote a cycle of order n and W_m a wheel of order $m+1$. It is conjectured by Surahmat, E.T. Baskoro and I. Tomescu that $R(C_n, W_m) = 2n-1$ for even $m \geq 4$, $n \geq m$ and $(n, m) \neq (4, 4)$. In this paper, we confirm the conjecture for $n \geq 3m/2 + 1$.

1. Introduction

Let $G = (V(G), E(G))$ be a graph. The *minimum degree*, *connectivity* and *independence number* of G are denoted by $\delta(G)$, $\kappa(G)$ and $\alpha(G)$, respectively. The number of components of G is $\omega(G)$. We use C_n and mK_n to denote a cycle of order n and the union of m vertex disjoint complete graphs K_n, respectively. A *Wheel* $W_m = K_1 + C_m$ is a graph of $m+1$ vertices. The lengths of the longest and shortest cycles of G are denoted by $c(G)$ and $g(G)$, respectively. A graph G is *Hamilton-connected* if it contains a Hamiltonian path between any two distinct vertices. A graph on n vertices is *pancyclic* if it contains cycles of every length l, $3 \leq l \leq n$ and *weakly pancyclic* if it contains cycles of every length l, $g(G) \leq l \leq c(G)$. All graphs considered in this paper are finite simple graphs without loops. If G has an edge-induced subgraph G_i, we say G contains G_i. For two given graphs G_1 and G_2, the *Ramsey number* $R(G_1, G_2)$ is the smallest integer n such that for any graph G of order n, either G contains G_1 or $\overline{G}$ contains G_2, where $\overline{G}$ is the complement of G.

Some Ramsey numbers concerning cycles versus wheels have been obtained; see [1] for details. In [2], Surahmat et al. considered the Ramsey number $R(C_n, W_m)$ in the case when $n \geq m$ and established the following.

Theorem 1 (*Surahmat et al. [2]*). *$R(C_n, W_m) = 2n - 1$ for even m and $n \geq 5m/2 - 1$.*

In the same paper, they made the following conjecture.

Conjecture 1 (*Surahmat et al. [2]*). *$R(C_n, W_m) = 3n - 2$ for odd m, $n \geq m \geq 3$ and $(n, m) \neq (3, 3)$.*

For Conjecture 1, Surahmat et al. [3] obtained the following partial result.

Theorem 2 (*Surahmat et al. [3]*). *$R(C_n, W_m) = 3n - 2$ for odd m and $n > (5m - 9)/2$.*

* Corresponding author at: Department of Mathematics, Nanjing University, Nanjing 210093, PR China.
E-mail address: yaojunc@nju.edu.cn (Y. Chen).

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In the paper, they posed the following.

Conjecture 2 (*Surahmat et al. [3]*). *$R(C_n, W_m) = 2n - 1$ for even m, $n \geq m \geq 4$ and $(n, m) \neq (4, 4)$.*

Recently, by using the technique of dominating cycles, Zhang et al. [4] established the following result.

Theorem 3 (*Zhang et al. [4]*). *$R(C_n, W_m) = 3n - 2$ for odd m, $n \geq m$ and $n \geq 20$.*

By Theorem 3, we see that Conjecture 1 is solved completely except several small values of n and m. In this paper, we consider the Ramsey number $R(C_n, W_m)$ in the case when m is even. Our main result is the following.

Theorem 4. *$R(C_n, W_m) = 2n - 1$ for even $m \geq 4$ and $n \geq 3m/2 + 1$.*

2. Several lemmas

In order to prove Theorem 4, we need the following lemmas.

Lemma 1 (*Bondy [5]*). *Let G be a graph of order n. If $\delta(G) \geq n/2$, then either G is pancyclic or n is even and $G = K_{n/2,n/2}$.*

Lemma 2 (*Brandt et al. [6]*). *Every non-bipartite graph G of order n with $\delta(G) \geq (n + 2)/3$ is weakly pancyclic with $g(G) = 3$ or 4.*

Lemma 3 (*Dirac [7]*). *Let G be a 2-connected graph of order $n \geq 3$ with $\delta(G) = \delta$. Then $c(G) \geq \min\{2\delta, n\}$.*

Lemma 4 (*Dirac [7]*). *Let G be a graph of order $n \geq 3$. If $\delta(G) \geq n/2 + 1$, then G is Hamilton-connected.*

Lemma 5 (*Surahmat et al. [3]*). *Let G be a graph of order $2n - 1$ without C_n. Suppose $m \geq 4$ is even and $n \geq 3m/2$. If G contains no W_m, then $\delta(G) \geq n - m/2$.*

3. Proof of main result

Proof of Theorem 4. Let G be a graph of order $2n - 1$. Suppose to the contrary neither G contains a C_n nor $\overline{G}$ contains a W_m.

If G is bipartite, then $\alpha(G) \geq |G|/2 \geq n$, which implies that $\overline{G}$ contains a K_n. Since $n \geq 3m/2 + 1 \geq m + 1$, we see that $\overline{G}$ contains a K_{m+1} and hence $\overline{G}$ has a W_m, a contradiction. Thus, we may assume that G is non-bipartite. By Lemma 5, $\delta(G) \geq n - m/2$. Since $n \geq 3m/2 + 1$, we have $\delta(G) \geq n - m/2 \geq n - (n - 1)/3 = [(2n - 1) + 2]/3$. By Lemma 2, G is weakly pancyclic of girth 3 or 4. If $\kappa(G) \geq 2$, then $c(G) \geq \min\{2n - m, 2n - 1\} > n$ by Lemma 3, which implies that G contains a C_n, a contradiction. Thus we have $\kappa(G) \leq 1$.

If G is disconnected, then since $\delta(G) \geq n - m/2 \geq (2n + 1)/3$, we have $\omega(G) = 2$ for otherwise $2n - 1 = |G| \geq 3[\delta(G) + 1] \geq 2n + 4$. If $\kappa(G) = 1$ and s is a cut-vertex of G, then since $\delta(G) \geq n - m/2 \geq (2n + 1)/3$, we have $\omega(G - s) = 2$ for otherwise $2n - 1 = |G| \geq 3[\delta(G - s) + 1] + 1 \geq 3\delta(G) + 1 \geq 2n + 2$. Thus, we may assume without loss of generality that G_1 and G_2 be the two components of G or $G - s$ for some cut-vertex s with $|G_1| \geq |G_2|$. It is easy to see that $|G_1| \leq (2n - 1) - (n - m/2 + 1) = n + m/2 - 2$ in both cases. If $|G_1| \geq n$, then since $\delta(G_1) \geq n - m/2 - 1$, we have $\delta(G_1) > |G_1|/2$ since $n \geq 3m/2 + 1$. By Lemma 1, G contains a C_n, a contradiction. Thus we have $|G_1| = |G_2| = n - 1$. This implies that G is connected and $\kappa(G) = 1$. Since $\delta(G) \geq n - m/2 > 3$, by the symmetry of G_1 and G_2, we may assume that x, y are two neighbors of s in G_1. Note that $\delta(G_1) \geq n - m/2 - 1 \geq (2n - 2)/3$ and $|G_1| = n - 1$, we see that $\delta(G_1) \geq |G_1|/2 + 1$ since $m \geq 4$ and $n \geq 3m/2 + 1$. By Lemma 4, G_1 has a Hamiltonian (x, y)-path P. Thus, P together with s form a C_n in G, a final contradiction.

By the arguments above, we have $R(C_n, W_m) \leq 2n - 1$. On the other hand, since the graph $2K_{n-1}$ has no C_n and its complement graph contains no wheels, we have $R(C_n, W_m) \geq 2n - 1$, and hence $R(C_n, W_m) = 2n - 1$.

The proof of Theorem 4 is completed. ■

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