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A Note on a Problem of Halin's

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In [2], Halin poses the following problem: Let A be an arbitrary graph, and suppose that a graph G contains, for each positive integer n, n disjoint subgraphs each isomorphic (or homeomorphic) to A. Does it follow that G necessarily contains infinitely many disjoint subgraphs each isomorphic (or homeomorphic) to A?

In [1], Halin answers this question (for homeomorphic embeddings) affirmatively if A is an arbitrary tree in which each vertex has valency at most 3. The purpose of this note is to show by an example that the corresponding result does not hold for isomorphic embeddings. (A slight modification to [3, Example 1] gives another example that yields the same conclusion; see also [4].)

Let A be the infinite tree shown in Fig. 1. Clearly A is a tree of the required type. We can represent A more succinctly by means of the graph

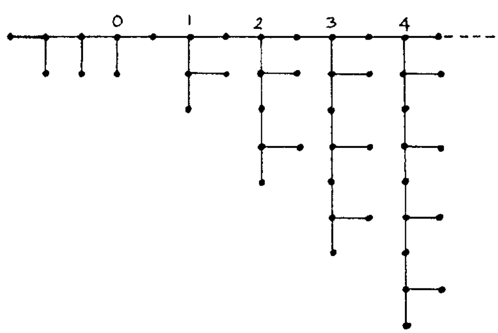

FIG. 1. The Graph A.

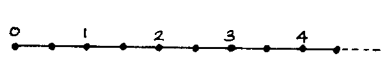

FIG. 2. The Graph A'.

A' in Fig. 2, where A is obtained from A' by appending, at the labeled vertices of A', the configurations shown in Fig. 3. With the same convention, let G be the graph represented by the graph G' shown in Fig. 4, G being formed from G' by appending the configurations of Fig. 3 at all the appropriately labeled vertices of G'.

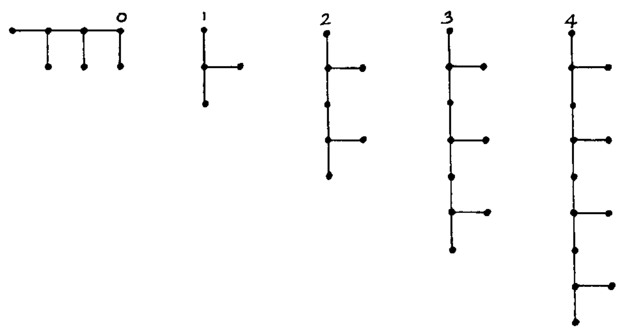

FIGURE 3

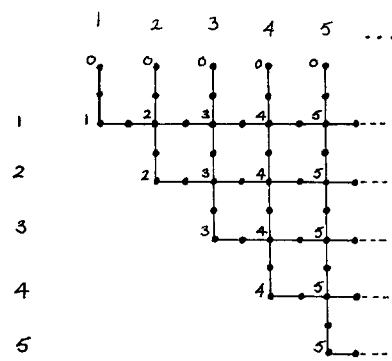

FIG. 4. The Graph G'.

Note first that the only place in G at which we find three consecutive vertices, all with valency at least 3, is at a vertex labeled 0. It is thus easy to see that, in any embedding of A in G, the vertex of A labeled 0 must correspond to one of the vertices of G labeled 0. In a similar way, it is easy to see that a vertex of A labeled i (> 0) must correspond to a vertex of G labeled c, for some $c \geqslant i$. Thus an embedding of A in G, whose 0-vertex lies in the jth column, is confined to the first j rows of G (as labeled in Fig. 4): For if one of its vertices, labeled i, corresponds to the vertex in the rth row and cth column of G, then clearly $i \geqslant r + (c - j)$; the inequality $c \geqslant i$ now gives $r \leqslant j$, as stated.

It follows that there cannot be infinitely many disjoint copies of A within G. For if there were, and the "leftmost" copy of A started at vertex 0 in the jth column of G, then no vertical line in G could cross more than j copies of A, which is absurd.

On the other hand, for each $n \geqslant 1$, there are n disjoint copies of A. One way of finding these would be to let A_i $(1 \leqslant i \leqslant n)$ start at vertex 0 in column $n + i - 1$, run vertically downwards to row $n + 1 - i$, and then run to the right *ad infinitum* along that row.

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